ON FUZZY S-CONTINUOUS FUNCTIONS

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Abstract. We introduce the concepts of fuzzy s-continuous functions. And we investigate several properties of the fuzzy s-continuous function. In particular, we study the relation between fuzzy continuous functions and fuzzy s-continuous functions.

1. INTRODUCTION

Fuzzy topological spaces were first introduced in the literature by Chang [1] who studied a number of the basic concepts including fuzzy continuous maps and compactness. And fuzzy topological spaces are a very natural generalization of topological spaces. In 1983, A.S. Mashhour et al.[3] introduced supra topological spaces and studied s-continuous functions and s^*-continuous functions. In 1987, M.E. Abd El-Monsef et al.[2] introduced the fuzzy supra topological spaces and studied fuzzy supra-continuous functions and characterized a number of basic concepts. Also fuzzy supra topological spaces are generalizations of supra topological spaces. In this paper, we introduce fuzzy s-continuous function and establish a number of characterizations. Let X be a set and let $I = [0, 1]$. Let I^X denote the set of all mapping $a\colon X \to I$.

A member of I^X is called a fuzzy subset of X. And unions and intersections of fuzzy sets are denoted by $\vee$ and $\wedge$ respectively and defined by

$$\vee\, a_i = \sup\{a_i(x) \mid i \in J \text{ and } x \in X\},$$
$$\wedge\, a_i = \inf\{a_i(x) \mid i \in J \text{ and } x \in X\}.$$

Definition 1.1[1]. A fuzzy topology T on X is a collection of subsets of I^X such that

(1) $0, 1 \in T$,
(2) if $a, b \in T$, then $a \wedge b \in T$,
(3) if $a_i \in T$ for all $i \in J$, then $\vee a_i \in T$.

(X, T) is called a fuzzy topological space. Members of T are called fuzzy open sets in (X, T) and complement of a fuzzy open set is called a fuzzy closed set.

Definition 1.2. Let f be a mapping from a set X into a set Y. Let a and b be the fuzzy sets of X and Y, respectively. Then $f(a)$ is a fuzzy set in Y, defined by

$$f(a)(y) = \begin{cases} \sup_{z \in f^{-1}(y)} a(z), & \text{if } f^{-1}(y) \neq \emptyset,\ y \in Y \\ 0, & \text{otherwise,} \end{cases}$$

and $f^{-1}(b)$ is a fuzzy set in X, defined by $f^{-1}(b)(x) = b(f(x))$, $x \in X$.

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DEFINITION 1.3[3]. A subfamily T^* of I^X is said to be a fuzzy supra-topology on X if

(1) $1 \in T^*$,
(2) if $a_i \in T^*$ for all $i \in J$, then $\vee a_i \in T^*$.

(X, T^*) is called a fuzzy supra topological space. The elements of T^* are called fuzzy supra open sets in (X, T^*). And a fuzzy set a is supra closed iff $co(a) = 1 - a$ is a fuzzy supra open set. And the fuzzy supra topological spaces T^* is denoted by fsts.

DEFINITION 1.4[3]. The supra closure of a fuzzy set a is denoted by $scl(a)$, and given by

$$scl(a) = \wedge\{s \mid s \text{ is a fuzzy supra closed set and } a \leq s\}.$$

The supra interior of a fuzzy set a is denoted by $si(a)$ and given by

$$si(a) = \vee\{t \mid t \text{ is a fuzzy supra open set and } t \leq a\}.$$

2. Fuzzy s-continuous function

DEFINITION 2.1[2]. Let (X, T) be a fuzzy topological space and T^* be a fuzzy supra-topology on X. We call T^* a fuzzy supra-topology associated with T if $T \subset T^*$.

DEFINITION 2.2[2]. Let $f: (X, T^*) \to (Y, S^*)$ be a mapping between two fuzzy supratopological spaces. f is a fuzzy supracontinuous function if $f^{-1}(S^*) \subseteq T^*$.

DEFINITION 2.3. Let (X, T) and (Y, S) be fuzzy topological spaces and T^* be an associated fuzzy supra-topology with T. A function $f: X \to Y$ is a fuzzy s-continuous function if the inverse image of each fuzzy open set in Y is T^*-fuzzy supra open in X.

THEOREM 2.4. *Let (X, T) and (Y, S) be fts. Let f be a function from X into Y. Let T^* be an associated fuzzy supra-topology with T. Then the followings are equivalent :*

(1) *f is fuzzy s-continuous.*
(2) *The inverse image of each fuzzy closed set in Y is T^*-fuzzy supra closed.*
(3) *$scl(f^{-1}(a)) \leq f^{-1}(cl(a))$ for every fuzzy set a in Y.*
(4) *$f(scl(a)) \leq cl(f(a))$ for every fuzzy set a in X.*
(5) *$f^{-1}(int(b)) \leq si(f^{-1}(b))$ for every fuzzy set b in Y.*
(6) *For each fuzzy set a in X and each fuzzy neighborhood b of $f(a)$, there is a fuzzy supra neighborhood c of a such that $f(c) \leq b$.*

Proof. (1) ⇒ (2). Let a be fuzzy closed set in Y. Since f is a fuzzy s-continuous, $f^{-1}(1 - a) = 1 - f^{-1}(a)$ is fuzzy supra open in X. Therefore $f^{-1}(a)$ is a fuzzy supra closed set in X.

(2) ⇒ (3). Since $cl(a)$ is fuzzy closed for every fuzzy set a in Y, $f^{-1}(cl(a))$ is T^*-fuzzy supra closed. Therefore,

$$f^{-1}(cl(a)) = scl(f^{-1}(cl(a))) \geq scl(f^{-1}(a)).$$

(3) $\Rightarrow$ (4). Let a be fuzzy subset in X and let $f(a) = b$. Then $f^{-1}(scl(b)) \geq scl(f^{-1}(b))$. So $f^{-1}(cl(f(a))) \geq scl(f^{-1}f(a)) \geq scl(a)$, and hence $cl(f(a)) \geq f(scl(a))$.

(4) $\Rightarrow$ (2). Let b be a fuzzy closed set in Y and be $a = f^{-1}(b)$. Then $f(scl(a)) \leq cl(f(a)) = cl(f(f^{-1}(b))) \leq cl(b) = b$. Since $scl(a) \leq f^{-1}(f(scl(a)) \leq f^{-1}(b) = a$, then a is T^*-fuzzy supra closed.

(2) $\Rightarrow$ (1). Obvious.

(1) $\Rightarrow$ (5). Let b be a fuzzy subset in Y. Since $f^{-1}(int(b))$ is T^*-fuzzy supra open set in X, $f^{-1}(int(b)) \leq si(f^{-1}(int(b))) \leq si(f^{-1}(b))$.

(5) $\Rightarrow$ (1). Let a be a fuzzy open set in Y. Since $f^{-1}(a) \leq si(f^{-1}(a)) \leq f^{-1}(a)$, $f^{-1}(a)$ is T^*-fuzzy supra open.

(6) $\Rightarrow$ (1). Let b be any fuzzy open set in Y and let $f^{-1}(b) = a$. Then b is a fuzzy neighborhood of $f(a) = f(f^{-1}(b))$. There exists a fuzzy supra neighborhood c of $a = f^{-1}(b)$ such that $f(c) \leq b$. Thus $c \leq f^{-1}f(c) \leq f^{-1}(b)$. Therefore, $f^{-1}(b)$ is a fuzzy supra neighborhood of $f^{-1}(b)$. And $f^{-1}(b)$ is a fuzzy supra open set in X, by [6, Theorem 2.2].

(1) $\Rightarrow$ (6). Obvious. □ □

Remark. Every fuzzy continuous function is fuzzy s-continuous. But the converse of this implication is not true, as following example shows.

Example 2.1. Let a_1, a_2, and a_3 be fuzzy subsets of $X = I$, defined as

$$a_1(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq 1/2, \\ 2x - 1, & \text{if } 1/2 \leq x \leq 1; \end{cases}$$

$$a_2(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq 1/4, \\ -4x + 2, & \text{if } 1/4 \leq x \leq 1/2, \\ 0, & \text{if } 1/2 \leq x \leq 1; \end{cases}$$

$$a_3(x) = \begin{cases} 1, & \text{if } \leq x \leq 1/2, \\ -2x + 2, & \text{if } 1/2 \leq x \leq 1. \end{cases}$$

Consider the fuzzy space $T_1 = \{0, a_1, a_2, a_1 \vee a_2, 1\}$ and an associated supra fuzzy space ${T_1}^* = \{0, a_1, a_2, a_3, a_1 \vee a_2, a_1 \vee a_3, 1\}$. Let $g\colon X \to X$ be defined by $g(x) = (1/2)x$. Clearly, we have $g^{-1}(0) = 0,\ g^{-1}(1) = 1,\ g^{-1}(a_1 \vee a_2) = a_3,\ g^{-1}(a_2) = a_3$, and $g^{-1}(a_1) = 0$. $co(a_1) = a_3$ is a fuzzy supra open in (X, T_1^*) but it is not fuzzy open in (X, T_1). Hence the fuzzy mapping g is fuzzy s-continuous but not fuzzy continuous.

Remark. In general, the composition of two fuzzy s-continuous functions need not be fuzzy s-continuous.

Example 2.2. Let $X = I$. Consider the fuzzy sets

$$a(x) = \begin{cases} 1, & \text{if } 0 \leq x < 1/3 \\ 1/2, & \text{if } 1/3 \leq x \leq 2/3 \\ 0, & \text{if } 2/3 < x \leq 1, \end{cases}$$

$$b(x) = 1/2, \text{if } 0 \leq x \leq 1,$$

$$c(x) = 1/3, \text{if } 0 \leq x \leq 1.$$

Let $T_1 = \{0, a, 1\}$ and $T_1{}^* = \{0, a, b, a \vee b, 1\}$. Let $T_2 = \{0, c, 1\}$ and $T_2{}^* = \{0, a, c, a \vee c, 1\}$. Let $f\colon (X, T_1) \to (X, T_1)$ be a fuzzy mapping defined by $f(x) = (x+1)/3$. Let $g\colon (X, T_2) \to (X, T_1)$ be a fuzzy mapping defined by $g(x) = (1/3)x$. Clearly, f and g are fuzzy s-continuous. But $(f \circ g)$ is not fuzzy s-continuous, since a is a fuzzy open set in (X, T_1) but $(f \circ g)^{-1}(a) = b$ is not fuzzy supra open in T_2^*.

THEOREM 2.5. *If a fuzzy mapping $f\colon (X, T_1) \to (Y, T_2)$ is fuzzy s-continuous and $g\colon (Y, T_2) \to (Z, T_3)$ is fuzzy continuous, then $(g \circ f)$ is fuzzy s-continuous.*

Proof. The proof is clear by the definitions of fuzzy s-continuous functions and fuzzy continuous functions. □ □

THEOREM 2.6. *Let (X, T) and (Y, S) be fts, T^* and S^* be two associated fuzzy supra-topologies with T and S, respectively. If $f\colon X \to Y$ is a fuzzy mapping, and one of the followings;*

(1) *$f^{-1}(si(a)) \le int(f^{-1}(a))$ for each fuzzy set a in (Y, S),*
(2) *$cl(f^{-1}(a)) \le f^{-1}(scl(a))$ for each fuzzy set a in (Y, S),*
(3) *$f(cl(b)) \le scl(f(b))$ for each fuzzy set b in (X, T), holds, then f is fuzzy continuous.*

Proof. If the condition (2) is satisfied, let b be a fuzzy closed set in Y, then $cl(f^{-1}(b)) \le f^{-1}(scl(b)) = f^{-1}(b)$. Therefore $f^{-1}(b)$ is a fuzzy closed set in X.

If the condition (3) is satisfied, let b be a fuzzy subset in Y, then $f^{-1}(b)$ is a fuzzy subset in X and $f(cl(f^{-1}(b))) \le scl(f(f^{-1}(b)))$. Thus $cl(f^{-1}(b)) \le f^{-1}(scl(b))$. Therefore, since the condition (2) is satisfied, f is a fuzzy continuous function.

Similarly, we can prove in the case (1). □ □

LEMMA [4]. *Let $g\colon X \to X \times Y$ be the graph of a fuzzy mapping $f\colon X \to Y$. Then, if a is a fuzzy set in X and b is a fuzzy set in Y, $g^{-1}(a \times b) = a \wedge f^{-1}(b)$.*

THEOREM 2.7. *Let $f\colon (X, T) \to (Y, S)$ be a fuzzy mapping and T^* be an associated supra-topology with T. Let $g\colon X \to X \times Y$, given by $g(x) = (x, f(x))$ be its graph mapping. If g is fuzzy s-continuous, then f is fuzzy s-continuous.*

Proof. Suppose that g is a fuzzy s-continuous and a is a fuzzy open set in (Y, S). Then $f^{-1}(a) = 1 \wedge f^{-1}(a) = g^{-1}(1 \times a)$. Therefore, $f^{-1}(a)$ is a fuzzy supra open set in (X, T^*). □ □

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