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A CHARACTERIZATION OF MCSHANE INTEGRABILITY

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Abstract. In this paper we prove that for functions from [0,1] into a totally ordered AL-space, Mcshane integrability and absolute Mcshane integrability are equivalent.

1. Introduction

In 1990 Gordon[4] introduced the Mcshane integral of Banach-valued functions. This integral is a generalized Riemann integral of functions which have values in a Banach space. For real-valued functions the Mcshane integral and the Lebesgue integral are equivalent. Gordon[4] and Fremlin and Mendoza[2] have developed the properties of this integral. A Bochner integrable function is Mcshane integrable [4], and a Mcshane integrable function is Pettis integrable [2]. Many authors have studied the Bochner integral and the Pettis integral ([1],[3],[5],[6]).

In this paper we prove that for functions from [0,1] into a totally ordered AL-space, Mcshane integrability and absolute Mcshane integrability are equivalent.

2. Preliminaries

Unless otherwise stated, we always assume that X is a real Banach space with dual X^* and $([0,1],\Sigma,\mu)$ is the Lebesgue measure space.

Gorden [4] introduced the Mcshane integral of Banach-valued functions.

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Definition 2.1. A Mcshane partition of [0,1] is a finite collection $\mathcal{P} = \{([a_i, b_i], t_i) : 1 \leq i \leq n\}$ such that $\{[a_i, b_i] : 1 \leq i \leq n\}$ is a non-overlapping family of subintervals of [0,1] covering [0,1] and $t_i \in [0, 1]$ for each $i \leq n$. A gauge on [0,1] is a function $\delta : [0, 1] \to (0, \infty)$. A Mcshane partition $\mathcal{P} = \{([a_i, b_i], t_i) : 1 \leq i \leq n\}$ is subordinate to a gauge δ if $[a_i, b_i] \subset (t_i - \delta(t_i), t_i + \delta(t_i))$ for every $i \leq n$. If $f : [0, 1] \to X$ and if $\mathcal{P} = \{([a_i, b_i], t_i) : 1 \leq i \leq n\}$ is a Mcshane partition of $[0, 1]$, we will denote $f(\mathcal{P})$ for $\sum_{i=1}^{n} f(t_i)(b_i - a_i)$. A function $f : [0, 1] \to X$ is Mcshane integrable on $[0, 1]$, with Mcshane integral ω, if for every $\varepsilon > 0$ there exists a gauge $\delta : [0, 1] \to (0, \infty)$ such that $\|\omega - f(\mathcal{P})\| < \varepsilon$ for every Mcshane partition $\mathcal{P} = \{([a_i, b_i], t_i) : 1 \leq i \leq n\}$ of $[0, 1]$ subordinate to δ.

Gorden [4] obtained the following theorem which is useful to prove our result.

Theorem 2.2 [4]. *The function $f : [0, 1] \to X$ is Mcshane integrable on $[0, 1]$ if and only if for each $\varepsilon > 0$ there exists a gauge δ on $[0, 1]$ such that $\|f(\mathcal{P}_1) - f(\mathcal{P}_2)\| < \varepsilon$ whenever $\mathcal{P}_1$ and $\mathcal{P}_2$ are Mcshane partitions of $[0, 1]$ subordinate to δ.*

Definition 2.3. A function $f : [0, 1] \to X$ is absolutely Mcshane integrable on $[0, 1]$ if for each $\varepsilon > 0$ there exists a gauge δ on $[0, 1]$ such that

$$\sum_{i=1}^{k} \sum_{j=1}^{h} \|f(t'_i) - f(t''_j)\| \mu([a'_i, b'_i] \cap [a''_j, b''_j]) < \varepsilon$$

whenever $\mathcal{P}' = \{([a'_i, b'_i], t'_i) : 1 \leq i \leq k\}$ and $\mathcal{P}'' = \{([a''_j, b''_j], t''_j) : 1 \leq j \leq h\}$ are Mcshane partitions of $[0, 1]$ subordinate to δ.

3. Main Result

In this section, we give a characterization of Mcshane integrablility in terms of absolute Mcshane integrability.

Lemma 3.1. *$f : [0, 1] \to X$ is Mcshane integrable if and only if for each $\varepsilon > 0$ there exists a gauge δ on $[0, 1]$ such that*

$$\| \sum_{i=1}^{k} \sum_{j=1}^{h} [f(t'_i) - f(t''_j)] \mu([a'_i, b'_i] \cap [a''_j, b''_j]) \| < \varepsilon$$

whenever $\mathcal{P}' = \{([a_i', b_i'], t_i') : 1 \le i \le k\}$ *and* $\mathcal{P}'' = \{([a_j'', b_j''], t_j'') : 1 \le j \le h\}$ *are Mcshane partitions of* $[0,1]$ *subordinate to* δ.

Proof. Let $\mathcal{P}' = \{([a_i', b_i'], t_i') : 1 \le i \le k\}$ and $\mathcal{P}'' = \{([a_j'', b_j''], t_j'') : 1 \le j \le h\}$ be any Mcshane partitions of $[0,1]$. Then

$$\begin{aligned} f(\mathcal{P}') &= \sum_{i=1}^{k} f(t_i')(b_i' - a_i') \\ &= \sum_{i=1}^{k}\sum_{j=1}^{h} f(t_i')\mu([a_i', b_i'] \cap [a_j'', b_j'']) \end{aligned}$$

and

$$\begin{aligned} f(\mathcal{P}'') &= \sum_{j=1}^{h} f(t_j'')(b_j'' - a_j'') \\ &= \sum_{i=1}^{k}\sum_{j=1}^{h} f(t_j'')\mu([a_i', b_i'] \cap [a_j'', b_j'']) \end{aligned}$$

Hence $\|f(\mathcal{P}') - f(\mathcal{P}'')\| = \|\sum_{i=1}^{k}\sum_{j=1}^{h}[f(t_i') - f(t_j'')]\mu([a_i', b_i'] \cap [a_j'', b_j''])\|$ Therefore by Theorem 2.2, $f : [0,1] \to X$ is Mcshane integrable if and only if for each $\varepsilon > 0$ there exists a gauge δ on $[0,1]$ such that $\|\sum_{i=1}^{k}\sum_{j=1}^{h}[f(t_i') - f(t_j'')]\mu([a_i', b_i'] \cap [a_j'', b_j''])\| < \varepsilon$ whenever $\mathcal{P}' = \{([a_i', b_i'], t_i') : 1 \le i \le k\}$ and $\mathcal{P}'' = \{([a_j'', b_j''], t_j'') : 1 \le j \le h\}$ are Mcshane partitions of $[0,1]$ subordinate to δ. □

The following is a main result of this paper.

THEOREM 3.2. *Let X be a totally ordered AL-space. Then $f : [0,1] \to X$ is Mcshane integrable if and only if f is absolutely Mcshane integrable.*

Proof. Suppose that $f : [0,1] \to X$ is absolutely Mcshane integrable. Let $\varepsilon > 0$ be given. Then there exists a gauge δ on $[0,1]$ such that $\sum_{i=1}^{k}\sum_{j=1}^{h}\|[f(t_i') - f(t_j'')\|\mu([a_i', b_i'] \cap [a_j'', b_j''])\| < \varepsilon$ whenever $\mathcal{P}' = \{([a_i', b_i'], t_i') : 1 \le i \le k\}$ and $\mathcal{P}'' = \{([a_j'', b_j''], t_j'') : 1 \le j \le h\}$ are Mcshane partitions of $[0,1]$ subordinate to δ.
Therefore $\|\sum_{i=1}^{k}\sum_{j=1}^{h}[f(t_i') - f(t_j'')]\mu([a_i', b_i'] \cap [a_j'', b_j''])\| < \varepsilon$ whenever

$\mathcal{P}' = \{([a_i', b_i'], t_i'); 1 \leq i \leq k\}$ and $\mathcal{P}'' = \{([a_j'', b_j''], t_j''); 1 \leq j \leq h\}$ are Mcshane partitions of $[0,1]$ subordinate to δ. By Lemma 3.1, f is Mcshane integrable.

For the converse, suppose that $f : [0,1] \to X$ is Mcshane integrable. Let $\varepsilon > 0$ be given. Then there exists a gauge δ on [0,1] such that $\| \sum_{i=1}^{k} f(t_i)(b_i - a_i) - \int_0^1 f d\mu \| < \frac{\varepsilon}{2}$ whenever $\mathcal{P} = \{([a_i, b_i], t_i) : 1 \leq i \leq k\}$ is a Mcshane partition of $[0,1]$ subordinate to δ.
Let $\mathcal{P}' = \{([a_i', b_i'], t_i') : 1 \leq i \leq k\}$ and $\mathcal{P}'' = \{([a_j'', b_j''], t_j'') : 1 \leq j \leq h\}$ be any Mcshane partitions of $[0,1]$ subordinate to δ. Define $t_{ij}' = t_i'$ and $t_{ij}'' = t_j''$ if $f(t_i') \geq f(t_j'')$ and define $t_{ij}' = t_j''$ and $t_{ij}'' = t_i'$ if $f(t_i') < f(t_j'')$. Then $f(t_{ij}') - f(t_{ij}'') \in X_+$ and $\|f(t_{ij}') - f(t_{ij}'')\| = \|f(t_i') - f(t_j'')\|$ for $1 \leq i \leq k, 1 \leq j \leq h$. Moreover $\{([a_i', b_i'] \cap [a_j'', b_j''], t_{ij}') : 1 \leq i \leq k, 1 \leq j \leq h\}$ and $\{([a_i', b_i'] \cap [a_j'', b_j''], t_{ij}'') : 1 \leq i \leq k, 1 \leq j \leq h\}$ are both Mcshane partitions of $[0,1]$ subordinate to δ. Hence

$$\| \sum_{i=1}^{k} \sum_{j=1}^{h} f(t_{ij}') \mu([a_i', b_i'] \cap [a_j'', b_j'']) - \int_0^1 f d\mu \| < \frac{\varepsilon}{2}$$

and

$$\| \sum_{i=1}^{k} \sum_{j=1}^{h} f(t_{ij}'') \mu([a_i', b_i'] \cap [a_j'', b_j'']) - \int_0^1 f d\mu \| < \frac{\varepsilon}{2}.$$

Therefore

$$\| \sum_{i=1}^{k} \sum_{j=1}^{h} [f(t_{ij}') - f(t_{ij}'')] \mu([a_i', b_i'] \cap [a_j'', b_j''])\| < \varepsilon.$$

Since X is an AL-space and $f(t_{ij}') - f(t_{ij}'') \in X_+$ for $1 \leq i \leq k, 1 \leq j \leq h$,

$$\begin{aligned}
&\sum_{i=1}^{k} \sum_{j=1}^{h} \|f(t_i') - f(t_j'')\| \mu([a_i', b_i'] \cap [a_j'', b_j'']) \\
&= \sum_{i=1}^{k} \sum_{j=1}^{h} \|f(t_{ij}') - f(t_{ij}'')\| \mu([a_i', b_i'] \cap [a_j'', b_j'']) \\
&= \| \sum_{i=1}^{k} \sum_{j=1}^{h} [f(t_{ij}') - f(t_{ij}'')] \mu([a_i', b_i'] \cap [a_j'', b_j''])\| < \varepsilon.
\end{aligned}$$

Therefore $f : [0,1] \to X$ is absolutely Mcshane integrable. □ □

COROLLARY 3.3. *Let X be a totally ordered AL-space and Y a Banach space. If $f : [0,1] \to X$ is Mcshane integrable and $g : X \to Y$ is Lipschitz continuous, then the composite function $g \circ f : [0,1] \to Y$ is Mcshane integrable.*

Proof. Let $\varepsilon > 0$ be given. Since g is Lipschitz continuous, there exists a $K > 0$ such that $\|g(x') - g(x)\| \le K\|x' - x\|$ for all $x', x \in X$. Since f is Mcshane integrable, by Theorem 3.2, f is absolutely Mcshane integrable. Hence there exists a gauge δ on $[0,1]$ such that

$$\sum_{i=1}^{k}\sum_{j=1}^{h} \|f(t_i') - f(t_{ij}'')\|\mu([a_i', b_i'] \cap [a_j'', b_j'']) < \frac{\varepsilon}{K}$$

whenever $\mathcal{P}' = \{([a_i', b_i'], t_i') : 1 \le i \le k\}$ and $\mathcal{P}'' = \{([a_j'', b_j''], t_j'') : 1 \le j \le h\}$ are Mcshane partitions of $[0,1]$ subordinate to δ.
Hence

$$\begin{aligned}
&\|\sum_{i=1}^{k}\sum_{j=1}^{h}[(g\circ f)(t_i') - (g\circ f)(t_j'')]\mu([a_i', b_i'] \cap [a_i'', b_j''])\| \\
&\le \sum_{i=1}^{k}\sum_{j=1}^{h} \|(g\circ f)(t_i') - (g\circ f)(t_j'')\|\mu([a_i', b_i'] \cap [a_i'', b_j'']) \\
&\le K\sum_{i=1}^{k}\sum_{j=1}^{h} \|f(t_i') - f(t_j'')\|\mu([a_i', b_i'] \cap [a_i'', b_j'']) < \varepsilon
\end{aligned}$$

whenever $\mathcal{P}' = \{([a_i', b_i'], t_i') : 1 \le i \le k\}$ and $\mathcal{P}'' = \{([a_j'', b_j''], t_j'') : 1 \le j \le h\}$ are Mcshane partitions of $[0,1]$ subordinate to δ. Thus $g \circ f$ is Mcshane integrable by Lemma 3.1. □ □

References

1. J.Diestel and J.J.Uhl JR., *Vector measures*, Math. Surveys, No. 15, Amer. Math. Soc., Providence, R.I., 1977.
2. D.H.Fremlin and J.Mendoza, *On the integration of vector-valued functions*, Illinois J. Math., **38** (1994), no. 1, 127-147.

3. R.F. Geitz, *Pettis integration*, Proc. Amer. Math. Soc., **82** (1981), no. 1, 81-86.
4. R.A.Gordon, *The Mcshane integral of Banach-valued functions*, Illinois J. Math., **34** (1990), no. 3, 557-567.
5. R.Huff, *Remarks on Pettis integrability*, Proc. Amer. Math. Soc., **96** (1986), no. 3, 402-404.
6. M. Talagrand, *Pettis integral and measure theory*, Mem. Amer. Math. Soc., No. 307, 1984.

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