ON FUZZY ALMOST S-CONTINUOUS FUNCTIONS

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Abstract. In this note, the notion of fuzzy almost s-continuity is introduced and some results related to this notion are obtained.

1. Introduction

In [1], B. Ghosh introduced and investigated the notions of fuzzy semi-$T_2$ spaces and fuzzy semi-connected spaces. In particular, he studied these spaces under fuzzy semi-continuity. T. Noiri, B. Ahmad and M. Khan [2] introduced and studied the notion of almost s-continuous functions. The purpose of this paper is to introduce the notion of fuzzy almost s-continuous functions and to study fuzzy semi-$T_2$ spaces and fuzzy semi-connected spaces under fuzzy almost s-continuity.

2. Preliminaries

Throughout this paper $X$ and $Y$ will denote fuzzy topological spaces. For definitions and notations which are not explained in this paper, we refer to [1]. For any $\alpha \in (0, 1]$ and any $x \in X$, a fuzzy point $x_\alpha$ in $X$ is a fuzzy set in $X$ defined by

$$x_\alpha(y) = \begin{cases} \alpha & \text{for } y = x \\ 0 & \text{for } y \neq x. \end{cases}$$

A fuzzy point $x_\alpha$ is said to belong to a fuzzy set $A$ in $X$ if $\alpha \leq A(x)$. In this case we shall use the notation $x_\alpha \in A$ [1]. A fuzzy open set [resp. fuzzy semi-open set] $U$ in $X$ is called a fuzzy open neighborhood [resp. fuzzy semi-open neighborhood] of a fuzzy point $x_\alpha$ in $X$ if $x_\alpha \in U$. Two
fuzzy sets A and B in X are said to be q-coincident (denoted by \(A_qB\)) if there exists \(x \in X\) such that \(A(x) + B(x) > 1\). When two fuzzy sets A and B in X are not q-coincident, we shall write \(A \not\sim q B\) [1]. For a fuzzy point \(x_\alpha\) in X, a fuzzy set A in X is called a q-neighborhood [resp. semi q-neighborhood] of \(x_\alpha\) if there exists a fuzzy open set [resp. fuzzy semi-open set] \(U\) in X such that \(x_\alpha U \leq A\) [1]. A fuzzy set is called a fuzzy semi-regular if it is both fuzzy semi-open and fuzzy semi-closed. The intersection of all fuzzy semi-closed set containing a fuzzy set A is called the fuzzy semi-closure of A and is denoted by \(sCl(A)\). The union of all fuzzy semi-open sets contained in a fuzzy set B is called the fuzzy semi-interior of B and is denoted by \(sInt(B)\).

3. The characterization of fuzzy almost s-continuity

**Lemma 3.1.** If A is fuzzy semi-open in X, then \(sCl(A)\) is fuzzy semi-regular in X.

**Proof.** By definition, \(sCl(A)\) is fuzzy semi-closed, we are left to show that \(sCl(A)\) is fuzzy semi-open. Since A is fuzzy semi-open, there exists a fuzzy open set \(O\) in X such that \(O \leq A \leq Cl(O)\). This implies that \(O \leq sCl(O) \leq sCl(A) \leq sCl(Cl(O)) = Cl(O)\) and hence \(sCl(A)\) is fuzzy semi-open. □ □

**Definition 3.2.** A function \(f : X \to Y\) is said to be fuzzy almost s-continuous if for each fuzzy point \(x_\alpha \in X\) and each fuzzy semi-open set \(V\) in Y with \(f(x_\alpha) \in V\), there exists a fuzzy open set \(O\) in X with \(x_\alpha \in O\) such that \(f(O) \leq sCl(V)\).

**Lemma 3.3.** A function \(f : X \to Y\) is fuzzy almost s-continuous if and only if for any fuzzy semi-regular set \(A\) in Y, \(f^{-1}(A)\) is both fuzzy open and fuzzy closed in X.

**Proof.** Suppose that \(f : X \to Y\) is fuzzy almost s-continuous and that \(A\) is fuzzy semi-regular in Y. If \(f^{-1}(A) = O_X\), then clearly \(f^{-1}(A)\) is both fuzzy open and fuzzy closed in X. Let \(x_\alpha\) be a fuzzy point in \(f^{-1}(A)\). Then \(f(x_\alpha) \in A\). By hypothesis, there exists a fuzzy open set \(O_{x_\alpha}\) in X with \(x_\alpha \in O_{x_\alpha}\) such that \(f(O_{x_\alpha}) \leq sCl(A) = A\), and hence we obtain \(f^{-1}(A) = \cup \{x_\alpha \mid x_\alpha \in f^{-1}(A)\} \leq \cup \{O_{x_\alpha} \mid x_\alpha \in A\} \leq f^{-1}(A)\).

This shows that \(f^{-1}(A)\) is fuzzy open in X. Now, since \(1 - A\) is
fuzzy semi-regular in $Y$, $1 - f^{-1}(A) = f^{-1}(1 - A)$ is fuzzy open in $X$. Consequently, $f^{-1}(A)$ is fuzzy closed in $X$.

Conversely, assume that the given condition holds. Let $x_\alpha$ be a fuzzy point in $X$ and let $V$ be a fuzzy semi-open set in $Y$ with $f(x_\alpha) \in V$. By Lemma 3.1, $sCl(V)$ is fuzzy semi-regular. By hypothesis, $f^{-1}(sCl(V))$ is fuzzy open in $X$ with $x_\alpha \in f^{-1}(sCl(V))$. Since $f(f^{-1}(sCl(V))) \leq sCl(V)$, we conclude that $f$ is fuzzy almost s-continuous. $\square$ $\square$

4. Fuzzy semi-$T_2$ spaces and fuzzy semi-connected spaces

**Definition 4.1.** ([1]) A fuzzy topological space $X$ is fuzzy $T_2$ [resp. fuzzy semi-$T_2$] if for every pair of distinct fuzzy points $x_\alpha$ and $y_\beta$, the following conditions are satisfied:

1. If $x \neq y$, then there exist two fuzzy open sets [resp. fuzzy semi-open sets] $U$ and $V$ such that $x_\alpha \in U$, $y_\beta \in V$ and $U \cap V = \emptyset$.

2. If $x = y$ and $\alpha < \beta$, then $x_\alpha$ has a fuzzy open neighborhood [resp. fuzzy semi-open neighborhood] $U$ and $y_\beta$ has a fuzzy neighborhood [resp. semi q-neighborhood] $V$ such that $V_q V$.

Obviously, every fuzzy $T_2$ space is fuzzy semi-$T_2$.

**Lemma 4.2.** A fuzzy topological space $X$ is fuzzy semi-$T_2$ if and only if for every pair of distinct fuzzy points $x_\alpha$ and $y_\beta$, the following conditions are satisfied:

1. If $x \neq y$, then there exist two fuzzy semi-open sets $U'$ and $V'$ such that $x_\alpha \in U'$, $y_\beta \in V'$ and $sCl(U')_q sCl(V')$.

2. If $x = y$ and $\alpha < \beta$, then there exist two fuzzy semi-open sets $U'$ and $V'$ such that $x_\alpha \in U'$, $y_\beta \in V'$ and $sCl(U')_q sCl(V')$.

**Proof.** ($\Rightarrow$) Clear.

($\Leftarrow$) Assume $x \neq y$. By hypothesis, there exist fuzzy semi-open sets $U$ and $V$ such that $x_\alpha \in U$, $y_\beta \in V$ and $U_q V$. Let $U' = sInt(1 - V)$. Clearly, $U'$ is fuzzy semi-open in $X$. Since $U_q V$, we have $x_\alpha \in U = sInt(U) \leq sInt(1 - V) = U'$. Now, let $V' = 1 - sCl(U')$. Then $V'$ is a fuzzy semi-open set in $X$. Since $sCl(U') + V = sCl(sInt(1 - V)) + V \leq (1 - V) + V = 1 \leq 1$, we obtain $y_\beta \in V \leq 1 - sCl(U') = V'$. By Lemma 3.1, $sCl(V') = V'$. Since $sCl(U') + sCl(V') = sCl(U') + V' = sCl(U') + (1 - sCl(U')) = 1 \leq 1$, we have $sCl(U')_q sCl(V')$. 

Assume that \(x = y\) and \(\alpha < \beta\). By hypothesis, \(x_\alpha\) has a fuzzy semi-open neighborhood \(U\) and \(y_\beta\) has a semi \(q\)-neighborhood \(V\) such that \(U_qV\). Choose a fuzzy semi-open set \(W\) in \(X\) such that \(y_\beta W \leq V\). Let \(V' = s\text{Int}(1-U)\). Then \(V'\) is fuzzy semi-open in \(X\). Since \(U_qW\) and \(y_\beta W\), we have \(\beta + V'(y) = \beta + s\text{Int}(1-U)(y) \geq \beta + W(y) > 1\). Thus \(y_\beta V'\). Now, let \(U' = 1 - s\text{Cl}(V')\). Clearly, \(U'\) is a fuzzy semi-open set in \(X\). Since \(s\text{Cl}(V') + U = s\text{Cl}(s\text{Int}(1-U)) + U \leq (1-U) + U = 1 \leq 1\), we have \(x_\alpha \in U \leq 1 - s\text{Cl}(V') = U'\). By Lemma 3.1, \(s\text{Cl}(U') = U'\). Since \(s\text{Cl}(V') + s\text{Cl}(U') = s\text{Cl}(V') + U' = s\text{Cl}(V') + (1 - s\text{Cl}(V')) = 1 \leq 1\), we obtain \(s\text{Cl}(U'), q\text{sCl}(V')\). □ □

**Theorem 4.3.** Let \(f : X \rightarrow Y\) be injective and fuzzy almost \(s\)-continuous. If \(Y\) is fuzzy semi-\(T_2\), then \(X\) is fuzzy \(T_2\).

**Proof.** Let \(x_\alpha\) and \(y_\beta\) be two distinct fuzzy points in \(X\).

First, assume that \(x \neq y\). Since \(f(x) \neq f(y)\), by (1) of Lemma 4.2, there exist two fuzzy semi-open sets \(U\) and \(V\) in \(Y\) such that \(f(x)_\alpha \in U, f(y)_\beta \in V\) and \(s\text{Cl}(U), q\text{sCl}(V)\). This implies that \(x_\alpha \in f^{-1}(s\text{Cl}(U)), y_\beta \in f^{-1}(s\text{Cl}(V))\) and \(f^{-1}(s\text{Cl}(U)), qf^{-1}(s\text{Cl}(V))\). Moreover, by Lemma 3.1 and Lemma 3.3, \(f^{-1}(s\text{Cl}(U))\) and \(f^{-1}(s\text{Cl}(V))\) are fuzzy open in \(X\).

Now, assume that \(x = y\) and \(\alpha < \beta\). Since \(f(x) = f(y)\), by (2) of Lemma 4.2, there exist two fuzzy semi-open sets \(U\) and \(V\) in \(Y\) such that \(f(x)_\alpha \in U, f(y)_\beta \in V\) and \(s\text{Cl}(U), q\text{sCl}(V)\). Thus, we have \(x_\alpha \in f^{-1}(s\text{Cl}(U)), y_\beta f^{-1}(s\text{Cl}(V))\) and \(f^{-1}(s\text{Cl}(U)), qf^{-1}(s\text{Cl}(V))\). Moreover, by Lemma 3.1 and Lemma 3.3, \(f^{-1}(s\text{Cl}(U))\) and \(f^{-1}(s\text{Cl}(V))\) are fuzzy open in \(X\). □ □

**Corollary 4.4.** Let \(f : X \rightarrow Y\) be injective and fuzzy almost \(s\)-continuous. If \(Y\) is fuzzy semi-\(T_2\), so is \(X\).

**Definition 4.5.** (1) Two nonempty fuzzy sets \(A\) and \(B\) in \(X\) are said to be **fuzzy separated** [resp. **fuzzy semi-separated**] if \(A_qCl(B)\) and \(B_qCl(A)\) [resp. \(A_q\text{sCl}(B)\) and \(B_q\text{sCl}(A)\)]. A fuzzy topological space which can not be expressed as the union of two fuzzy separated sets [resp. fuzzy semi-separated sets] is said to be **fuzzy connected** [resp. **fuzzy semi-connected**].
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**Lemma 4.6.** ([1]) Two nonempty fuzzy sets $A$ and $B$ are fuzzy semi-separated if and only if there exist two fuzzy semi-open sets $U$ and $V$ such that $A \leq U, B \leq V, A_qV$ and $B_qU$.

**Lemma 4.7.** A fuzzy topological space $X$ is not fuzzy semi-connected if and only if there exist two fuzzy semi-open sets $U$ and $V$ such that $U_qV$ and $U \cup V = X$.

**Proof.** ($\Leftarrow$) Obvious.

($\Rightarrow$) Assume that $X$ is not fuzzy semi-connected. Then there exist two fuzzy semi-separated sets $A$ and $B$ in $X$ such that $A \cup B = X$. By Lemma 4.6, it is possible to choose two fuzzy semi-open sets $U$ and $V$ such that $A \leq U, B \leq V, A_qV$ and $B_qU$. We wish to show that $A = U$ and $B = V$. Note that $B \leq 1 - A$ and $A \leq 1 - B$. Since $A \cup B = X$, we have that for any $x \in X$, either $A(x) = 1$ or $A(x) = 0$, and $A(x) = 1$ if and only if $B(x) = 0$. Assume that $A(x) = 0$. Then $B(x) = 1$. Since $B_qU$, we obtain $U(x) = 0$, and hence $A = U$. Similarly, we obtain $B = V$. □ □

**Theorem 4.8.** Let $f : X \rightarrow Y$ be surjective and fuzzy almost $s$-continuous. If $X$ is fuzzy connected, then $Y$ is fuzzy semi-connected.

**Proof.** Suppose to the contrary that $Y$ is not fuzzy semi-connected. By Lemma 4.7, there exist two fuzzy semi-open sets $U$ and $V$ in $Y$ such that $U_qV$ and $U \cup V = Y$. This means that $f^{-1}(U)_qf^{-1}(V)$ and $f^{-1}(U) \cup f^{-1}(V) = X$. Moreover, $U$ and $V$ are fuzzy semi-regular in $Y$. By Lemma 3.3, $f^{-1}(U)$ and $f^{-1}(V)$ are fuzzy closed in $X$. This says that $X$ is not fuzzy connected, contrary to the hypothesis. □ □

Since every fuzzy semi-connected set is fuzzy connected, we have

**Corollary 4.9.** Let $f : X \rightarrow Y$ be surjective and fuzzy almost $s$-continuous. If $X$ is fuzzy connected, so is $Y$.

**References**


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