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THE COMPLETENESS OF CONVERGENT SEQUENCES SPACE OF FUZZY NUMBERS

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Abstract. In this paper we define a new fuzzy metric $\tilde{\theta}$ of fuzzy number sequences, and prove that the space of convergent sequences of fuzzy numbers is a fuzzy complete metric space in the fuzzy metric $\tilde{\theta}$.

1. Introduction

D. Dubois and H. Prade introduced the notions of fuzzy numbers and defined its basic operations [2]. R. Goetschel, A. Kaufmann, M. Gupta and G. Zhang [3-7] have done much work about fuzzy numbers.

Let $\mathbb{R}$ be the set of all real numbers and $F^*(\mathbb{R})$ all fuzzy subsets defined on $\mathbb{R}$. G. Zhang [5-7] defined the fuzzy number $\tilde{a} \in F^*(\mathbb{R})$ as follows :

(1) $\tilde{a}$ is normal, i.e., there exists $x \in \mathbb{R}$ such that $\tilde{a}(x) = 1$,
(2) for every $\lambda \in (0, 1]$, $a_\lambda = \{x \mid \tilde{a}(x) \geq \lambda\}$ is a closed interval, denoted by $[a_\lambda^-, a_\lambda^+]$.

Now, let us denote the set of all fuzzy numbers defined by G. Zhang as $F(\mathbb{R})$. In this paper, we will use Zhang's fuzzy distance $\tilde{\rho}$ of fuzzy numbers [5-7] as follows :

$$\tilde{\rho}(\tilde{a}, \tilde{b}) = \bigcup_{\lambda \in [0,1]} \lambda \big[|a_1^- - b_1^-|, \sup_{\lambda \leq \eta \leq 1} |a_\eta^- - b_\eta^-| \vee |a_\eta^+ - b_\eta^+| \big]$$

for any $\tilde{a}, \tilde{b} \in F(\mathbb{R})$, where $\vee$ means max.

The purpose of this paper is to prove that the space of convergent sequences of fuzzy numbers is a fuzzy complete metric space in some fuzzy metric $\tilde{\theta}$.

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In section 2, we quote basic definitions and theorems from [1,5,6] which will be needed in the proof of main theorem.

In section 3, we prove main theorem: The space of convergent sequences of fuzzy numbers is a fuzzy complete metric space in a fuzzy metric $\tilde{\theta}$ defined by

$$\tilde{\theta}(A_i,\, A_j) = \bigcup_{\lambda \in [0,1]} \lambda \big[\sup_n |(a_{in})_1^- - (a_{jn})_1^-|,$$
$$\sup_n \sup_{\lambda \le \eta \le 1} |(a_{in})_\eta^- - (a_{jn})_\eta^-| \ \vee |(a_{in})_\eta^+ - (a_{jn})_\eta^+|\big]$$

where $A_i = \{\tilde{a}_{in}\}$ and $A_j = \{\tilde{a}_{jn}\}$ are convergent sequences of fuzzy numbers with respect to the fuzzy metric $\tilde{\rho}$.

2. Definitions and preliminaries

In this section, we quote basic definitions [1,5,6,7] and theorems, proved in [6,7], which will be needed in the proof of main theorem.

Let $F^*(\mathbb{R})$ be the set of all fuzzy subsets defined on $\mathbb{R}$.

Definition 2.1. Let $\tilde{a} \in F^*(\mathbb{R})$. $\tilde{a}$ is called a fuzzy number if $\tilde{a}$ has the following properties:

(1) $\tilde{a}$ is normal, i.e., there exists $x \in \mathbb{R}$ such that $\tilde{a}(x) = 1$.
(2) For every $\lambda \in (0,1], a_\lambda = \{x | \tilde{a}(x) \geqslant \lambda\}$ is a closed interval, denoted by $[a_\lambda^-, a_\lambda^+]$.

Let $F(\mathbb{R})$ be the set of all fuzzy numbers on the real line $\mathbb{R}$.

By the decomposition theorem of fuzzy sets

$$\tilde{a} = \bigcup_{\lambda \in [0,1]} \lambda \big[a_\lambda^-, a_\lambda^+\big]$$

for every $\tilde{a} \in F(\mathbb{R})$. If we define $\tilde{a}(x)$ by

$$\tilde{a}(x) = \begin{cases} 1 & \text{for } x = k \\ 0 & \text{for } x \neq k \quad (k \in \mathbb{R}), \end{cases}$$

then $k \in F(\mathbb{R})$ and $k = \bigcup_{\lambda \in [0,1]} \lambda[k,k]$.

Definition 2.2. Let $\tilde{a}, \tilde{b}, \tilde{c} \in F(\mathbb{R})$. We define as follows:

(1) $\tilde{c} = \tilde{a} + \tilde{b}$ if $c_\lambda^- = a_\lambda^- + b_\lambda^-$ and $c_\lambda^+ = a_\lambda^+ + b_\lambda^+$ for every $\lambda \in (0,1]$.
(2) $\tilde{c} = \tilde{a} - \tilde{b}$ if $c_\lambda^- = a_\lambda^- - b_\lambda^+$ and $c_\lambda^+ = a_\lambda^+ - b_\lambda^-$ for every $\lambda \in (0,1]$.
(3) For every $k \in \mathbb{R}$ and $\tilde{a} \in F(\mathbb{R})$,

$$\begin{aligned} k\tilde{a} &= \bigcup_{\lambda \in [0,1]} \lambda\big[ka_\lambda^-, ka_\lambda^+\big] \quad \text{if } \ k \geqslant 0, \\ &= \bigcup_{\lambda \in [0,1]} \lambda\big[ka_\lambda^+, ka_\lambda^-\big] \quad \text{if } \ k < 0. \end{aligned}$$

(4) $\tilde{a} \leqslant \tilde{b}$ if $a_\lambda^- \leqslant b_\lambda^-$ and $a_\lambda^+ \leqslant b_\lambda^+$ for every $\lambda \in (0,1]$.
(5) $\tilde{a} < \tilde{b}$ if $\tilde{a} \leqslant \tilde{b}$ and there exists $\lambda \in (0,1]$ such that $a_\lambda^- < b_\lambda^-$ or $a_\lambda^+ < b_\lambda^+$.
(6) $\tilde{a} = \tilde{b}$ if $\tilde{a} \leqslant \tilde{b}$ and $\tilde{b} \leqslant \tilde{a}$.

Definition 2.3. Let $A \subset F(\mathbb{R})$.

(1) If there exists $\tilde{M} \in F(\mathbb{R})$ such that $\tilde{a} \leqslant \tilde{M}$ for every $\tilde{a} \in A$, then A is said to have an upper bound $\tilde{M}$.
(2) If there exists $\tilde{m} \in F(\mathbb{R})$ such that $\tilde{m} \leqslant \tilde{a}$ for every $\tilde{a} \in A$, then A is said to have a lower bound $\tilde{m}$.
(3) A is said to be bounded if A has both upper and lower bounds.
(4) A sequence $\{\tilde{a}_n\} \subset F(\mathbb{R})$ is said to be bounded if the set $\{\tilde{a}_n | n \in \mathbb{N}\}$ is bounded.

Definition 2.4. A fuzzy distance $\tilde{\rho}$ of two fuzzy numbers $\tilde{a}$, $\tilde{b} \in F(\mathbb{R})$ is a function $\tilde{\rho} \ : \ F(\mathbb{R}) \times F(\mathbb{R}) \longrightarrow F(\mathbb{R})$ with the properties :

(1) $\tilde{\rho}(\tilde{a}, \tilde{b}) \geqslant 0$, $\tilde{\rho}(\tilde{a}, \tilde{b}) = 0$ iff $\tilde{a} = \tilde{b}$.
(2) $\tilde{\rho}(\tilde{a}, \tilde{b}) = \tilde{\rho}(\tilde{b}, \tilde{a})$.
(3) Whenever $\tilde{c} \in F(\mathbb{R})$, we have $\tilde{\rho}(\tilde{a}, \tilde{b}) \leqslant \tilde{\rho}(\tilde{a}, \tilde{c}) + \tilde{\rho}(\tilde{c}, \tilde{b})$.

If $\tilde{\rho}$ is a fuzzy distance of fuzzy numbers, we call $(F(\mathbb{R}), \tilde{\rho})$ a fuzzy metric space. We define

$$\tilde{\rho}(\tilde{a}, \tilde{b}) = \bigcup_{\lambda \in [0,1]} \lambda\big[|a_1^- - b_1^-|, \sup_{\lambda \leq \eta \leq 1} |a_\eta^- - b_\eta^-| \vee |a_\eta^+ - b_\eta^+|\big] \qquad (*)$$

for any $\tilde{a}, \tilde{b} \in F(\mathbb{R})$, where $\vee$ means max.

THEOREM 2.1. [5,6,7] *$\tilde{\rho}$ defined by the above equality $(*)$ is a fuzzy distance of fuzzy numbers, that is, $(F(\mathbb{R}), \tilde{\rho})$ is a fuzzy metric space.*

DEFINITION 2.5. Let $\{\tilde{a}_n\} \subset F(\mathbb{R}),\ \tilde{a} \in F(\mathbb{R})$. Then the sequence $\{\tilde{a}_n\}$ is said to converge to $\tilde{a}$ in fuzzy distance $\tilde{\rho}$, denoted by

$$(\tilde{\rho}) \lim_{n\to\infty} \tilde{a}_n = \tilde{a}$$

if for any given $\varepsilon > 0$ there exists an integer $N > 0$ such that $\tilde{\rho}(\tilde{a}_n, \tilde{a}) < \varepsilon$ for $n \geqslant N$.

A sequence $\{\tilde{a}_n\}$ in $F(\mathbb{R})$ is said to be a Cauchy sequence if for every $\varepsilon > 0$ there exists an integer $N > 0$ such that $\tilde{\rho}(\tilde{a}_n, \tilde{a}_m) < \varepsilon$ for $n,\, m > N$. A fuzzy metric space $(F(\mathbb{R}), \tilde{\rho})$ is called the fuzzy complete metric space if every Cauchy sequence in $F(\mathbb{R})$ converges.

THEOREM 2.2. [1,7] *The sequence $\{\tilde{a}_n\}$ in $F(\mathbb{R})$ is convergent in the metric $\tilde{\rho}$ if and only if $\{\tilde{a}_n\}$ is a Cauchy sequence.*

THEOREM 2.3. [7] *The fuzzy metric space $(F(\mathbb{R}), \tilde{\rho})$ is complete.*

3. Main theorem

In this section, we prove that the space of convergent sequences in $F(\mathbb{R})$ is a fuzzy complete metric space with some fuzzy metric $\tilde{\theta}$.
Let $\mathcal{C}$ denote the set of all convergent sequences of fuzzy numbers.

MAIN THEOREM. *$(\mathcal{C}, \tilde{\theta})$ is a fuzzy complete metric space with the fuzzy metric $\tilde{\theta}$ defined by*

$$\begin{aligned}\tilde{\theta}(A_i,\, A_j) = \bigcup_{\lambda\in[0,1]} \lambda\big[&\sup_n |(a_{in})_1^- - (a_{jn})_1^-|, \\ &\sup_n \sup_{\lambda\le\eta\le 1} |(a_{in})_\eta^- - (a_{jn})_\eta^-| \ \vee |(a_{in})_\eta^+ - (a_{jn})_\eta^+|\big]\end{aligned}$$

where $A_i = \{\tilde{a}_{in}\}$ and $A_j = \{\tilde{a}_{jn}\}$ are convergent sequences of fuzzy numbers with respect to the fuzzy metric $\tilde{\rho}$.

Proof. First we shall check that $\tilde{\theta}$ is a metric in $\mathcal{C}$. It is easy to see that (i)$\tilde{\theta}(A_i, A_j) \geqslant 0$, $\tilde{\theta}(A_i, A_j) = 0$ iff $A_i = A_j$, (ii)$\tilde{\theta}(A_i, A_j) = \tilde{\theta}(A_j, A_i)$. Since $\tilde{\rho}$ is a fuzzy metric in $F(\mathbb{R})$, the following triangle inequality holds :

$$\tilde{\rho}(\tilde{a}_{in}, \tilde{a}_{jn}) \leqslant \tilde{\rho}(\tilde{a}_{in}, \tilde{a}_{kn}) + \tilde{\rho}(\tilde{a}_{kn}, \tilde{a}_{jn})$$

where $A_i = \{\tilde{a}_{in}\}$, $A_j = \{\tilde{a}_{jn}\}$ and $A_k = \{\tilde{a}_{kn}\}$ are convergent sequences with respect to $\tilde{\rho}$ in $F(\mathbb{R})$. Thus, we have

$$|(a_{in})_1^- - (a_{jn})_1^-| \leqslant |(a_{in})_1^- - (a_{kn})_1^-| + |(a_{kn})_1^- - (a_{jn})_1^-|,$$

$$\begin{aligned}
&\sup_{\lambda\leq\eta\leq 1} |(a_{in})_\eta^- - (a_{jn})_\eta^-| \ \vee\ |(a_{in})_\eta^+ - (a_{jn})_\eta^+| \\
&\quad\leqslant \sup_{\lambda\leq\eta\leq 1} |(a_{in})_\eta^- - (a_{kn})_\eta^-| \ \vee\ |(a_{in})_\eta^+ - (a_{kn})_\eta^+| \\
&\qquad + \sup_{\lambda\leq\eta\leq 1} |(a_{kn})_\eta^- - (a_{jn})_\eta^-| \ \vee\ |(a_{kn})_\eta^+ - (a_{jn})_\eta^+|.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\sup_n |(a_{in})_1^- - (a_{jn})_1^-| \\
&\leqslant \sup_n |(a_{in})_1^- - (a_{kn})_1^-| + \sup_n |(a_{kn})_1^- - (a_{jn})_1^-|, \\
&\quad \sup_n \sup_{\lambda\leq\eta\leq 1} |(a_{in})_\eta^- - (a_{jn})_\eta^-| \vee |(a_{in})_\eta^+ - (a_{jn})_\eta^+| \\
&\leqslant \sup_n \sup_{\lambda\leq\eta\leq 1} |(a_{in})_\eta^- - (a_{kn})_\eta^-| \vee |(a_{in})_\eta^+ - (a_{kn})_\eta^+| \\
&\quad + \sup_n \sup_{\lambda\leq\eta\leq 1} |(a_{kn})_\eta^- - (a_{jn})_\eta^-| \vee |(a_{kn})_\eta^+ - (a_{jn})_\eta^+|.
\end{aligned}$$

Hence, (iii) the triangle inequality $\tilde{\theta}(A_i, A_j) \leqslant \tilde{\theta}(A_i, A_k) + \tilde{\theta}(A_k, A_j)$ follows. Consequently, by (i), (ii) and (iii), $\tilde{\theta}$ is a fuzzy metric in $\mathcal{C}$.

To show that $\mathcal{C}$ is complete in the fuzzy metric $\tilde{\theta}$, let $\{A_i\}_{i=1}^\infty$ (where $A_i = \{\tilde{a}_{in}\}_{n=1}^\infty$) be a Cauchy sequence in $\mathcal{C}$. Then, for any $\varepsilon > 0$ there exists an integer N_n such that

$$\begin{aligned}
&\tilde{\rho}(\tilde{a}_{in}, \tilde{a}_{jn}) \\
&\leqslant \bigcup_{\lambda\in[0,1]} \lambda\big[\sup_n |(a_{in})_1^- - (a_{jn})_1^-|, \sup_n \sup_{\lambda\leq\eta\leq 1} |(a_{in})_\eta^- - (a_{jn})_\eta^-| \\
&\qquad \vee |(a_{in})_\eta^+ - (a_{jn})_\eta^+|\big] \\
&= \tilde{\theta}(A_i, A_j) < \frac{\varepsilon}{5}
\end{aligned} \tag{1}$$

for $i, j > N_n$. Thus, $\{\tilde{a}_{in}\}_{i=1}^{\infty}$ is a Cauchy sequence in $F(\mathbb{R})$ for each fixed n. Since $(F(\mathbb{R}), \tilde{\rho})$ is complete from Theorem 2.3, by Theorem 2.2 $(\tilde{\rho}) \lim_{i\to\infty} \tilde{a}_{in} = \tilde{a}_n$ (say) for each n. Hence, we have

$$\begin{aligned}
&\lim_{i\to\infty} \tilde{\rho}(\tilde{a}_{in}, \tilde{a}_n) = 0 \quad \text{for each } n \\
\Longleftrightarrow\ &\lim_{i\to\infty} \bigcup_{\lambda\in[0,1]} \lambda\big[|(a_{in})_1^- - (a_n)_1^-|, \sup_{\lambda\leq\eta\leq 1} |(a_{in})_\eta^- - (a_n)_\eta^-| \vee \\
&\qquad |(a_{in})_\eta^+ - (a_n)_\eta^+|\big] = 0 \ \text{ for each } n \\
\Longleftrightarrow\ &\lim_{i\to\infty} \bigcup_{\lambda\in[0,1]} \lambda\big[\sup_n |(a_{in})_1^- - (a_n)_1^-|, \sup_n \sup_{\lambda\leq\eta\leq 1} |(a_{in})_\eta^- - (a_n)_\eta^-| \vee \\
&\qquad |(a_{in})_\eta^+ - (a_n)_\eta^+|\big] = 0 \\
\Longleftrightarrow\ &\lim_{i\to\infty} \tilde{\theta}(A_i, \{\tilde{a}_n\}) = 0 \\
\Longleftrightarrow\ &(\tilde{\theta}) \lim_{i\to\infty} A_i = \{\tilde{a}_n\}_{n=1}^{\infty}.
\end{aligned}$$

So, the sequence $\{A_i\}_{i=1}^{\infty}$ converges to $\{\tilde{a}_n\}_{n=1}^{\infty}$, i.e., $(\tilde{\theta}) \lim_{i\to\infty} A_i = \{\tilde{a}_n\}_{n=1}^{\infty}$ (This notation means that the sequence $\{A_i\}_{i=1}^{\infty}$ converges to the sequence $\{\tilde{a}_n\}_{n=1}^{\infty}$ in the fuzzy metric $\tilde{\theta}$).

We shall now show that $\{\tilde{a}_n\}_{n=1}^{\infty} \in \mathcal{C}$. In (1), taking the limit as $j \to \infty$, we have $\tilde{\rho}(\tilde{a}_{in}, \tilde{a}_n) < \frac{\varepsilon}{5}$. Since A_i is convergent for each i, A_i is a Cauchy sequence. Thus, for given $\varepsilon > 0$ there exists an integer $N_i > 0$ such that $\tilde{\rho}(\tilde{a}_{in}, \tilde{a}_{ik}) < \frac{\varepsilon}{5}$ for $n, k > N_i$, for fixed i. Similarly, for given $\varepsilon > 0$ there exists $N_j > 0$ such that $\tilde{\rho}(\tilde{a}_{jn}, \tilde{a}_{jk}) < \frac{\varepsilon}{5}$ for $n, k > N_j$, for fixed j. Let we put $N = max\{N_n, N_i, N_j\}$. Then for given $\varepsilon > 0$ there exist $\tilde{a}_{ik}, \tilde{a}_{jk} \in F(\mathbb{R})$ in connection with (1) such that

$$\tilde{\rho}(\tilde{a}_{in}, \tilde{a}_{ik}) < \frac{\varepsilon}{5}, \quad \tilde{\rho}(\tilde{a}_{jn}, \tilde{a}_{jk}) < \frac{\varepsilon}{5}, \quad \tilde{\rho}(\tilde{a}_{in}, \tilde{a}_{jn}) < \frac{\varepsilon}{5}$$

for $i, j, k, n > N$. Hence, we have

$$\begin{aligned}
\tilde{\rho}(\tilde{a}_{ik}, \tilde{a}_{jk}) &\leqslant \tilde{\rho}(\tilde{a}_{in}, \tilde{a}_{jn}) + \tilde{\rho}(\tilde{a}_{in}, \tilde{a}_{ik}) + \tilde{\rho}(\tilde{a}_{jn}, \tilde{a}_{jk}) \\
&\leqslant \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} = \frac{3}{5}\varepsilon
\end{aligned}$$

for $i, j, k > N$. Hence, $\{\tilde{a}_{ik}\}_{i=1}^{\infty}$ is a Cauchy sequence in $F(\mathbb{R})$, by the completeness of $F(\mathbb{R})$, there exists $\tilde{a}_k$(say)$\in F(\mathbb{R})$ such that $\tilde{\rho}(\tilde{a}_{ik}, \tilde{a}_k) \leqslant$

$\frac{3}{5}\varepsilon$. Therefore, we have

$$\begin{aligned}\tilde{\rho}(\tilde{a}_n, \tilde{a}_k) &\leqslant \tilde{\rho}(\tilde{a}_n, \tilde{a}_{in}) + \tilde{\rho}(\tilde{a}_{in}, \tilde{a}_{ik}) + \tilde{\rho}(\tilde{a}_{ik}, \tilde{a}_k) \\ &\leqslant \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{3}{5}\varepsilon = \varepsilon\end{aligned}$$

for $n, k > N$. Since ε is arbitrary, $\{\tilde{a}_n\}$ is a Cauchy sequence and hence $\{\tilde{a}_n\}$ converges. Therefore, since the sequence $\{\tilde{a}_{in}\}_{i=1}^{\infty}$ has already converged to the fuzzy number $\tilde{a}_n$(say) for each $n \in \mathbb{N}$ by Theorem 2.2, $\{\tilde{a}_n\} \in \mathcal{C}$, this proves the completeness of $\mathcal{C}$. □ □

Let $\mathcal{B}$ denote the set of all bounded sequences in $F(\mathbb{R})$. Let $A_i \in \mathcal{B}$, $A_i = \{\tilde{a}_{in}\}_{n=1}^{\infty}$. Since A_i is bounded for each i, there exists a convergent subsequence $\{\tilde{a}_{in_k}\}_{k=1}^{\infty}$ of $\{\tilde{a}_{in}\}_{n=1}^{\infty}$ including $\tilde{a}_{in}$, and hence $\{\tilde{a}_{in_k}\}_{k=1}^{\infty}$ is a Cauchy sequence. From this result and main theorem, we obtain

Corollary 1. *$(\mathcal{B}, \tilde{\theta})$ is a fuzzy complete metric space with the fuzzy metric $\tilde{\theta}$ defined by*

$$\begin{aligned}\tilde{\theta}(A_i,\, A_j) = \bigcup_{\lambda\in[0,1]} \lambda\Big[&\sup_n |(a_{in})_1^- - (a_{jn})_1^-|, \\ &\sup_n \sup_{\lambda\le\eta\le1} |(a_{in})_\eta^- - (a_{jn})_\eta^-| \ \vee |(a_{in})_\eta^+ - (a_{jn})_\eta^+|\Big]\end{aligned}$$

where $A_i = \{\tilde{a}_{in}\}_{n=1}^{\infty}$ and $A_j = \{\tilde{a}_{jn}\}_{n=1}^{\infty}$ are bounded sequences in $F(\mathbb{R})$.

Corollary 2. $(\mathcal{C}, \tilde{\theta}) \subset (\mathcal{B}, \tilde{\theta})$.

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