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A DUAL LIMIT THEOREMS IN A MEAN-FIELD MODEL ^{+}II : -SPECIFIC FREE ENERGY

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1. Introduction

Let $\{X_i^{(n)} : i = 1, 2, \cdots, n\}$ $(n = 1, 2, \cdots)$ triangular array of dependent and identically distributed random variables with the joint distribution given by

$$d\mu_n(x_1, \cdots, x_n) = c_n^{-1} \exp\Big\{\Big(\sum_{i=1}^{n} x_i\Big)^2 / 2n\Big\} \prod_{i=1}^{n} x_i dP(x_i), \tag{1.1}$$

where c_n is a normalizing constant and P is a Borel probability measure on R^1. The model (1.1) is usually called the Curie-Weiss Model in the area of statistical mechanics. In many case, an explicit evaluation of c_n is very difficult and physicists usually try to the specific free energy defined as follows:

$$-\lim_{n\to\infty} \frac{1}{n} \log c_n, \tag{1.2}$$

It was shown by Ellis and Newman(1978) that

$$-\lim_{n\to\infty} \frac{1}{n} \log c_n = \inf_t G(t), \tag{1.3}$$

where $G(t) = t^2/2 - \psi_P(t)$ and $\psi_P(t)$ is the cumulant generating function(c. g. f.) of the probability measure P. The limit (1.3) known

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as the specific free energy of model (1.1) in statistical mechanics. Ellis and Rosen(1982) considered the limit (1.2) for circle model on the space $C[0,1]$ of continuous real-valued functions on $[0,1]$. Park and Jeon(1985) dealt with the limit (1.2) for more general integrals on a separable Banach space by using the result of Bolthausen(1984), large deviation theorem and a limit theorem of Varadhen(1966). On the other hand, Eisele and Ellis(1983) proved for the Curie-Weiss model that

$$(1.4) \qquad -\lim_{n\to\infty}\frac{1}{n}\log c_n = \inf_t\left\{\frac{t^2}{2}-\psi_P(t)\right\} = \inf_t\left\{\gamma_P(t)-\frac{t^2}{2}\right\},$$

where P is the symmetric Bernoulli measure and $\gamma_P(t)$ is the large deviation rate of the probability measure P. In this paper, we extended the result of Eisele and Ellis(1983) to the generalized Curie-Weiss model and its dual model. Chaganty and Sethuraman(1987) considered the limit theorems for further extended models. Lee, Kim and Jeon(1993) obtained the limit theorems and the basic relationships between two models with the unique global minimum. Lee, Kim and Jeon(1995) considered the case of multiple global minima case.

We introduce these two models briefly in section 2. In section 3, we first state the result of Park and Jeon(1985) which will be used to prove the main result and then state and prove the main result.

2. Preliminaries

Let a probability measure Q with the moment generating function(m. g. f.) $\phi_Q(t)<\infty$, for all t, be given. And let the corresponding distribution F_Q be such that $F_Q(x)=0$ for $x<a$, $0<F_Q(x)<1$ for $a<x<b$, $F_Q(x)=1$ for $x>b$, and let $D_Q=(a,b)$, where $-\infty\le a<b\le\infty$.

Define L_Q be the class of probability measures P such that

$$(2.1) \qquad \phi_P(t)=\int_{|t|<k}\exp(tx)dP(x)<\infty, \qquad k>0$$

and

$$(2.2) \qquad \int_R \phi_Q(x)dP(x)<\infty.$$

Let $\{X_j^{(n)} : j = 1, 2, \cdots, n\}(n = 1, 2, \cdots)$ be a triangular array of dependent and identically distributed random variables with the joint distribution given by

$$d\mu_n^G(x_1, \cdots, x_n) = z_n^{-1} \exp\Big\{n\psi_Q\Big(\sum_{i=1}^n x_i/n\Big)\Big\} \prod_{j=1}^n dP(x_j), \tag{2.3}$$

where $P \in L_Q, \psi_Q(t) = \log \phi_Q(t)$ and z_n is a normalizing constant,

$$z_n = \int_{R^n} \exp\Big\{n\psi_Q\Big(\sum_{i=1}^n x_i/n\Big)\Big\} \prod_{j=1}^n dP(x_j). \tag{2.4}$$

The model in (2.3) defines the mean field model $^+$ II which is a direct generalization of the classical mean field model or Curie - Weiss model in which the joint distribution is postulated as

$$d\mu_n^{CW}(x_1, \cdots, x_n) = z_n^{-1} \exp\Big\{\Big(\sum_{i=1}^n x_i\Big)^2 \Big/ 2n\Big\} \prod_{j=1}^n dP(x_i). \tag{2.5}$$

Given a probability measure Q, define a function γ_Q as

$$\gamma_Q(t) = \sup_{s \in R} \Big\{ts - \psi_Q(s)\Big\}, \qquad t \in R, \tag{2.6}$$

where $\psi_Q(s) = \log \int_R \exp(sx) dQ(x)$.

This function is called the large deviation rate of Q in statistics. For the given probability measures Q and $P \in L_Q$, define

$$G_{QP}(t) = \gamma_Q(t) - \psi_P(t) \qquad \text{for all } t \in D_Q, \tag{2.7}$$

where $\psi_P(t) = \log \int_R \exp(tx) dP(x)$.

Definition 2.1. A real number $m (\in D_Q)$ is said to be a global minimum for G_{QP} if

$$G_{QP}(m) \leq G_{QP}(u) \qquad \text{for all } u \in D_Q.$$

When $D_Q = (-\infty, \infty)$, we have the following Lemma.

LEMMA 2.2. *Given a probability measure Q, let $P \in L_Q$. Then G_{QP} is analytic and $G_{QP}(s) \longrightarrow \infty$ as $|s| \to \infty$. Thus, G_{QP} has a finite number of minima. See Lee, Kim and Jeon(1993).*

DEFINITION 2.3. For given probability measures Q and $P \in L_Q$, a local minimum m for G_{QP} is said to be of type k if, as $u \to 0$,

$$G_{QP}(m+u) - G_{QP}(m) = \frac{c_{2k}u^{2k}}{(2k)!} + o(u^{2k}), \tag{2.8}$$

where $c_{2k} = G_{QP}^{2k} > 0$.

We define the dual of the generalized Curie - Weiss model. The dual model is defined as the joint distribution

$$d\mu_n^{GD}(x_1, \cdots, x_n) = d_n^{-1} \exp\Big\{n\psi_P\Big(\sum_{i=1}^{n} x_i \Big/ n\Big)\Big\} \prod_{j=1}^{n} dQ(x_j) \tag{2.9}$$

Note that dual of the original model is obtained simply by exchange the role of Q and P in (2.3). The dual model is well defined. See Lee, Kim and Jeon(1993).

For the dual model, let F_P, the distribution function of P, be such that $F_P(x) = 0$ for $x < c, 0 < F_P(x) < 1$, for $c < x < d, F_P(x) = 1$ for $x < d$ and let $D_P = (c, d)$ where $-\infty \le c \le d \le \infty$. The function corresponding to G_{QP} for the original model is defined by

$$G_{PQ}(t) = \gamma_P(t) - \psi_Q(t) \quad \text{for all} \quad t \in D_P, \tag{2.10}$$

where γ_P is the large deviation rate of P and ψ_Q is the cumulant generating function(c.g.f.) of Q. Note that $\psi_Q'(t)$ is strictly increasing on (c, d) [see Daniels(1954)]. See Lee, Kim and Jeon(1993) for more details for these models.

3. Main result

We first state the result of Park and Jeon(1985). Let $\{\nu_n\}$ be a sequence of probability measures on a measurable space $(V, \mathcal{B})$, where V is a separable Banach space with a norm $\|\cdot\|$ and $\mathcal{B}$ is a Borel σ -field on V. Suppose that ν_n converges weakly to some probability measure ν on $(V, \mathcal{B})$ and further suppose that $\sup_{n\geq 1}\int_V \exp(a\cdot\|x\|)d\nu_n(x) < \infty$: for all $a > 0$.

Let $\{X_j^{(n)} : j = 1, 2, \cdots, n\}$, $n = 1, 2, \cdots$, be a sequence of independent and identically distributed V-valued random variables with common probability measure ν_n and let ν_n^n be the probability measure of $(X_1^{(n)} + \cdots + X_n^{(n)})/n$.

Theorem 3.1. *(Park and Jeon) Assume that following;*

(A1) *There exist positive constant α, β, γ (not depending on n) such that for all sufficiently large n and $t > \gamma$, $\nu_n^n(y; \|y\|^2 > t) \leq \alpha\cdot\exp(-n\beta t)$.*

(A2) *$\{F_n\}$ is a sequence of real-valued functions on V such that, for all sufficiently large $y \in V$, $-F_n(y) \leq a + b\|y\|^2$ for some constants a and $b(0 < b < \beta)$.*

(A3) *For any $y \in V$ with $I_\nu(y) < \infty$ and for any sequence $\{y_n\}$ such that $y_n \to y$ as $n \to \infty$, $F_n(y_n)$ converges to $F(y)$ for some real-valued function F on V. Then*

$$-\lim_{n\to\infty}\frac{1}{n}\int_V \exp\Big\{-nF_n(y)\Big\}d\nu_n^n(y) = \sup_{y\in V}\Big\{F(y) + I_\nu(y)\Big\}. \tag{3.1}$$

The entropy functional of ν is defined by

$$I_\nu(y) = \sup_{\theta\in V^+}\Big\{\theta(y) - \log\int_V \exp\{\theta(x)\}d\nu(x)\Big\} \tag{3.2}$$

where V^+ is the dual of V.

Now. we state and prove the main result.

Theorem 3.2. *Let $\{X_i^{(n)}; i = 1, 2, \cdots, n\}(n = 1, 2, \cdots)$ be a triangular array of dependent and identically distributed random variables with the joint distribution given by (2.3) and its dual (2.9). Assume that G_{QP} has the unique global minimum of type k at m. Then*

$$\begin{aligned} -\lim_{n\to\infty}\log z_n &= \inf_{i\in D_Q} G_{QP}(t) = G_{QP}(m), \\ -\lim_{n\to\infty}\log d_n &= \inf_{i\in D_P} G_{PQ}(t) = G_{PQ}(m^D) \end{aligned} \tag{3.3}$$

and also $G_{QP}(m) = G_{PQ}(m^D)$, where $m^D = \psi'_P(m)$.

Proof. Let $Y_1, Y_2, \cdots$ be a sequence of independent and identically distributed random variables with common probability measure Q and let Q_n be the probability measure of $(Y_1 + Y_2 + \cdots Y_n)/n$. Since we have

$$\begin{aligned} z_n &= \int_{R^n} \exp\Big\{n\psi_Q\Big(\sum_{i=1}^n x_i/n\Big)\Big\} \prod_{i=1}^n dP(x_i) \\ &= \int_{R^n}\int_{R^1} \exp\Big\{\Big(\sum_{i=1}^n x_i\Big)y\Big\} dQ_n(x) \prod_{i=1}^n dP(x_i) \\ &= \int_{R^1} \exp\{n\psi_P(y)\} dQ_n(y), \end{aligned}$$

Q_n satisfies (A1) in Theorem 3.1 by the result of Prokhorov(1968). Since $xy \le (w/2)y^2 + (x^2/w)$ for any $w > 0$ and for any $x, y (\in R^1)$, we get $\psi_P(y) \le y^2 w/2 + \log \int_{R^1} \exp(x^2/2w) dP(x)$.

Thus $F_n(y) = -\psi_P(y)$ satisfies (A2). Hence, by Theorem 3.1, we have

$$- \lim_{n\to\infty} \log z_n = \inf_{t \in D_Q} G_{QP}(t) = G_{QP}(m). \tag{3.4}$$

Similarly, for the dual model, we have

$$- \lim_{n\to\infty} \log d_n = \inf_{t \in D_P} G_{PQ}(t) = G_{PQ}(m^D), \tag{3.5}$$

where $m^D = \psi'_P(m)$.

And also from Theorem 3.1 of Lee, Kim and Jeon(1993), we obtain $G_{QP}(m) = G_{PQ}(m^D)$. □ □

4. Example

Let Q be a standard normal distribution and P a symmetric Bernoulli measure. Then we have

$$d\mu_n^G(x_1, \cdots, x_n) = z_n^{-1} \exp\{(x_1 + \cdots + x_n)^2/2n\} \prod_{j=1}^n dP(x_j) \tag{4.1}$$

and

$$(4.2) \quad d\mu_n^D(x_1, \cdots, x_n) = d_n^{-1}[\cosh((x_1 + \cdots + x_n)/n)]^n \prod_{j=1}^{n} dQ(x_j).$$

The model (4.1) is the familiar Curie - Weiss model. Let us now check some the condition. In case of (2.5) we obtain $\gamma_Q(t) = t^2/2$ and $\sigma^2(t) = 1$. Hence uniformity condition holds trivially. Also condition (4.2) holds by Remark 4.1(See Lee, Kim and Jeon(1993)). Using a simple calculation, we obtain

$$\begin{aligned}
\Psi_P(t) &= \log\cosh(t), \quad \gamma_Q(t) = t^2/2, \\
G_{QP}(t) &= \gamma_Q(t) - \Psi_P(t) = t^2/2 - \log\cosh(t) = t^4/12 + t^6 + \cdots , \\
\gamma_P(t) &= \frac{1}{2}\Big\{t \ln \frac{1+t}{1-t} - \ln(1-t^2)\Big\}, \quad |t| < 1, \\
\Psi_Q(t) &= t^2/2 \quad \textit{and} \\
G_{PQ}(t) &= \gamma_P(t) - \Psi_Q(t) = \frac{1}{2}\Big\{t \ln \frac{1+t}{1-t} + \ln(-t^2) - t^2\Big\}, \quad |t| < 1.
\end{aligned}$$

It is seen from the above expression of $G_{QP}(t)$ the condition (3.7) of Theorem 3.2 in Lee, Kim and Jeon(1993) is satisfied. Thus $G_{QP}(t)$ is the type $k = 2$. Hence by Theorem 3.2 in Lee, Kim and Jeon(1993) G_{QP} is also type $k = 2$ and $c_2 = c_4 = 2$.

Thus we obtain

$$(4.3) \qquad \inf_{t \in D_Q} G_{QP}(t) = G_{QP}(0) = G_{PQ}(0) = \inf_{t \in D_P} G_{PQ}(t).$$

Hence zero is a unique global minimum of $G_{QP}(t)$ and $G_{PQ}(t)$. And also $m_1 = m_1^D = 1$. Now from Theorem 4.1 in Lee, Kim and Jeon(1993) we have

$$S_n/n^{3/4} \quad \xrightarrow{d} \quad \exp(-z^4/12),$$

where $S_n = X_1 + X_2 + \cdots + X_n$.

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