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MAPPINGS ON FUZZY PROXIMITY AND FUZZY UNIFORM SPACES

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Abstract. We define the fuzzy uniformly continuous map and investigate some properties of fuzzy uniformly continuous maps. We will prove the existences of initial fuzzy uniform structures induced by some functions. From this fact, we construct the product of two fuzzy uniform spaces.

1. Introduction and preliminaries

In [8,9,10], S.K. Samanta introduced the fuzziness in the concept of openness of a fuzzy subset as a generalization of Chang's fuzzy topology. A.A. Ramadan [5,6] expand it in view of a lattice.

On the other hand, S.K. Samanta [7] introduced new definitions for fuzzy proximity and fuzzy uniformity. It was shown that this fuzzy proximity is more general than that of Artico and Moresco [1].

In this paper, we define the fuzzy proximity map and the fuzzy uniformly continuous map in view of definitions of Samanta [7]. We will investigate the relationship among fuzzy topological spaces, fuzzy proximity spaces and fuzzy uniform spaces. We will prove the existences of initial fuzzy uniform structures induced by some functions. From this fact, we construct the product of two fuzzy uniform spaces.

Throughout this paper, for mappings on fuzzy uniform spaces, we refer to A.S. Mashhour, R. Badard and A.A. Ramadan [5].

In this paper, X will denote a set; $I = [0,1]$; $I_0 = (0,1]$; I^X= the set of all fuzzy subsets of X denoted by Greek letters, such as λ, μ, ν, etc. For any $\alpha \in I$, $\tilde{\alpha}$ denotes the constant fuzzy subset whose value is α. All the other notations are standard in fuzzy set theory.

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Definition 1.1. [10] A function $\tau : I^X \to I$ is called a *gradation of openness* on X if it satisfies the following conditions:

(c1) $\tau(\tilde{0}) = \tau(\tilde{1}) = 1$,
(c2) $\tau(\mu_1) \wedge (\mu_2) \geq \tau(\mu_1) \wedge \tau(\mu_2)$,
(c3) $\tau(\bigvee_{i\in\triangle} \mu_i) \geq \bigwedge_{i\in\triangle} \tau(\mu_i)$.

The pair (X, τ) is called a *fuzzy topological space.*

Let τ be a gradation of openness on X and $\mathcal{F} : I^X \to I$ be defined by $\mathcal{F}(\mu) = \tau(\mu^c)$. Then $\mathcal{F}$ is called a *gradation of closedness* on X.

Let (X, τ) be a fuzzy topological space, then for each $r \in I$, $\tau_r = \{\mu \in I^X \mid \tau(\mu) \geq r\}$ is a Chang's fuzzy topology on X.

If $f : X \to Y$ is a function, then for any $\mu \in I^X$, the *direct image* of μ is defined as

$$f(\mu)(y) = \begin{cases} sup_{x\in f^{-1}(\{y\})}\mu(x) & \text{if } f^{-1}(\{y\}) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

and for $\nu \in I^Y$, the *preimage* of ν is defined as $f^{-1}(\nu) = \nu \circ f$.

Let (X, τ) and (Y, τ^*) be fuzzy topological spaces. A function $f : (X, \tau) \to (Y, \tau^*)$ is called a *gradation preserving map* (gp-map) if $\tau^*(\mu) \leq \tau(f^{-1}(\mu))$ for all $\mu \in I^X$.

Lemma 1.2. [10] *Let (X, τ) and (Y, τ') be fuzzy topological spaces. Then $f : (X, \tau) \to (Y, \tau')$ is a gp-map iff $f : (X, \tau_r) \to (Y, \tau'_r)$ is continuous with respect to Chang's fuzzy topology for all $r \in I_0$; i.e. iff $f(cl(\lambda, r)) \leq cl(f(\lambda), r)$, for all $r \in I_0$, for all $\lambda \in I^Y$.*

We can easily prove the following lemma.

Lemma 1.3. *If $f : X \to Y$, then we have the following properties for direct and inverse image of fuzzy sets under mappings:*

(1) $\mu \geq f(f^{-1}(\mu))$ *with equality if f is surjective,*
(2) $\nu \leq f^{-1}(f(\nu))$ *with equality if f is injective,*
(3) $f^{-1}(\mu^c) = f^{-1}(\mu)^c$,
(4) $f^{-1}(\bigvee_{i\in I} \mu_i) = \bigvee_{i\in I} f^{-1}(\mu_i)$,
(5) $f^{-1}(\bigwedge_{i\in I} \mu_i) = \bigwedge_{i\in I} f^{-1}(\mu_i)$,
(6) $f(\bigvee_{i\in I} \nu_i) = \bigvee_{i\in I} f(\nu_i)$,
(7) $f(\bigwedge_{i\in I} \mu_i) \leq \bigwedge_{i\in I} f(\mu_i)$ *with equality if f is injective.*

2. Fuzzy proximity spaces and Fuzzy topological spaces

DEFINITION 2.1. [7] A function $\delta : I^X \times I^X \to I$ is said to be a *fuzzy proximity* on X which satisfies the following conditions:

(FP1) $\delta(\tilde{0}, \tilde{1}) = 0$,
(FP2) $\delta(\lambda, \mu) = \delta(\mu, \lambda)$,
(FP3) $\delta(\lambda_1 \vee \lambda_2, \mu) = \delta(\lambda_1, \mu) \vee \delta(\lambda_2, \mu)$,
(FP4) $\delta(\lambda, \mu) < 1 - r$ implies $\delta(cl(\lambda, r), \mu) < 1 - r$, where

$$cl(\lambda, r) = \tilde{1} - \bigvee\{\eta \leq \lambda^c; \delta(\lambda, \eta) < 1 - r\}, \quad 0 \leq r < 1.$$

The pair (X, δ) is called a *fuzzy proximity space*.

Let (X, δ_1) and (X, δ_2) be given. We say that δ_2 is *finer* than δ_1 (δ_1 is *coarser* than δ_2) iff for any $\lambda, \mu \in I^X$, $\delta_2(\lambda, \mu) \leq \delta_1(\lambda, \mu)$.

REMARK 1. If (X, δ) is a fuzzy proximity space and $\lambda \leq \mu$, then, by (FP3), $\delta(\lambda, \nu) \leq \delta(\mu, \nu)$.

DEFINITION 2.2. A function $f : (X, \delta) \to (Y, \delta^*)$ is a *fuzzy proximity map* if it satisfies $\delta(\mu, \nu) \leq \delta^*(f(\mu), f(\nu))$, for every $\mu, \nu \in I^X$.

THEOREM 2.3. [7] *Let δ be a fuzzy proximity on X. Then $cl(\lambda, r) : I^X \times I \to I^X$ satifies the followings:*

(cl1) $cl(\tilde{0}, r) = \tilde{0}, cl(\tilde{1}, r) = \tilde{1}$,
(cl2) $\lambda \leq cl(\lambda, r)$,
(cl3) $cl(\lambda, r) \leq cl(\lambda, r')$ *if* $r \leq r'$,
(cl4) $cl(\lambda \vee \mu, r) = cl(\lambda, r) \vee cl(\mu, r)$,
(cl5) $cl(cl(\lambda, r), r) = cl(\lambda, r)$.

THEOREM 2.4. [8] *Let δ be a fuzzy proximity on X, and let $\mathcal{F}_\delta : I^X \to I$ be a mapping defined by*

$$\mathcal{F}_\delta(\lambda) = \bigvee\{r \in I \mid cl(\lambda, r) = \lambda\}, \ \lambda \in I^X.$$

Then $\mathcal{F}_\delta$ is a gradation of closedness on X.

We define $\tau_\delta(\mu) = \mathcal{F}_\delta(\mu^c)$, then (X, τ_δ) is a fuzzy topological space. We will investigate the relationship between fuzzy topological spaces and fuzzy proximity spaces.

Theorem 2.5. *If $f:(X,\delta)\to(Y,\delta^*)$ is a fuzzy proximity map, then:*

(1) *$f:(X,\tau_\delta)\to(Y,\tau_{\delta^*})$ is a gp-map,*
(2) *$cl(f^{-1}(\lambda),r)\le f^{-1}(cl(\lambda,r))$, for all $r\in I_0$, for all $\lambda\in I^Y$.*

Proof. (1) By Lemma 1.2 and Theorem 2.4, we will show that

$$f(cl(\lambda,r))\le cl(f(\lambda),r),\quad \text{for all } r\in I_0, \text{ for all } \lambda\in I^X.$$

Suppose that, for some $y_0\in Y, c\in I_0$, such that

$$f(cl(\lambda,r))(y_0)>c>cl(f(\lambda),r)(y_0).$$

By the definition of $cl(f(\lambda),r)$, there exists $\eta_0\le f(\lambda)^c$ such that $\delta^*(f(\lambda),\eta_0)<1-r$ and $1-\eta_0(y_0)\le c$.

On the other hand, since $f^{-1}(f(\lambda)^c)=\tilde{1}-f^{-1}\circ f(\lambda)$, we have $f^{-1}(\eta_0)\le f^{-1}(f(\lambda)^c)\le\tilde{1}-\lambda$. Since f is a fuzzy proximity map and $f\circ f^{-1}(\eta_0)\le\eta_0$, we have $\delta(\lambda,f^{-1}(\eta_0))\le\delta^*(f(\lambda),\eta_0)<1-r$. Therefore $cl(\lambda,r)\le\tilde{1}-f^{-1}(\eta_0)$. So, the image of it is

$$f(cl(\lambda,r))\le f(\tilde{1}-f^{-1}(\eta_0))=f(f^{-1}(\tilde{1}-\eta_0))\le\tilde{1}-\eta_0.$$

It is a contradiction.

(2) Suppose that there exist $x_0\in X,\ c\in I_0$ such that

$$cl(f^{-1}(\lambda),r))(x_0)>c>f^{-1}(cl(\lambda,r))(x_0).$$

By the definition of $cl(\lambda,r)$, there exists $\eta_0\le\lambda^c$ such that

$$\delta^*(\lambda,\eta_0)<1-r,\ \ 1-f^{-1}(\eta_0)(x_0)\le c.$$

Here, we have

$$\delta(f^{-1}(\lambda),f^{-1}(\eta_0))\le\delta^*(\lambda,\eta_0)<1-r,$$

because f is a fuzzy proximity map and $f\circ f^{-1}(\lambda)\le\lambda$.

On the other hand, since $\eta_0\le\lambda^c$, we have $f^{-1}(\eta_0)\le\tilde{1}-f^{-1}(\lambda)$. So, It follows that $cl(f^{-1}(\lambda),r)(x_0)\le 1-f^{-1}(\eta_0)(x_0)\le c$. It is a contradiction. □ □

3. Fuzzy uniform spaces

Let Ω_X denote the family of all functions $\alpha : I^X \to I^X$ with the following properties:

(1) $\alpha(\tilde{0}) = \tilde{0}$, $\mu \leq \alpha(\mu)$, for every $\mu \in I^X$,
(2) $\alpha(\bigvee \mu_i) = \bigvee \alpha(\mu_i)$, for $\mu_i \in I^X$.

For $\alpha \in \Omega_X$, the function $\alpha^{-1} \in \Omega_X$ is defined by

$$\alpha^{-1}(\mu) = \bigwedge\{\rho \mid \alpha(\tilde{1} - \rho) \leq \tilde{1} - \mu\}$$

and $(\alpha \wedge \beta)(\mu) = \bigwedge\{\alpha(\mu_1) \vee \beta(\mu_2) \mid \mu_1 \vee \mu_2 = \mu\}$ and $\alpha \circ \beta(\mu) = \alpha(\beta(\mu)))$.

Definition 3.1. [7] A function $\mathcal{U} : \Omega_X \to I$ is said to be a *fuzzy uniformity* on X if it satisfies the following conditions:

(FU1) $\mathcal{U}(\alpha_1 \wedge \alpha_2) \geq \mathcal{U}(\alpha_1) \wedge \mathcal{U}(\alpha_2)$,
(FU2) for $\alpha \in \Omega_X$, there exists $\alpha_1 \in \Omega_X$ with $\alpha_1 \circ \alpha_1 \leq \alpha$ such that $\mathcal{U}(\alpha_1) \geq \mathcal{U}(\alpha)$,
(FU3) for $\alpha \in \Omega_X$, there exists $\alpha_1 \in \Omega_X$ such that $\alpha_1 \leq \alpha^{-1}$ and $\mathcal{U}(\alpha_1) \geq \mathcal{U}(\alpha)$,
(FU4) if $\alpha_1 \geq \alpha$, then $\mathcal{U}(\alpha_1) \geq \mathcal{U}(\alpha)$,
(FU5) there exists $\alpha \in \Omega_X$ such that $\mathcal{U}(\alpha) = 1$.

If $\mathcal{U}$ is a fuzzy uniformity on X, then $(X,\mathcal{U})$ is said to be a *fuzzy uniform space.*

Remark 2. By (FU1) and (FU4), we have $\mathcal{U}(\alpha_1 \wedge \alpha_2) = \mathcal{U}(\alpha_1) \wedge \mathcal{U}(\alpha_2)$.

Let $(X,\mathcal{U}_1)$ and $(X,\mathcal{U}_2)$ be fuzzy uniform spaces. We say $\mathcal{U}_1$ is *finer* than $\mathcal{U}_2$ (or $\mathcal{U}_2$ is *coarser* than $\mathcal{U}_1$), denoted by $\mathcal{U}_2 \preceq \mathcal{U}_1$, iff for any $\alpha \in \Omega_X, \mathcal{U}_2(\alpha) \leq \mathcal{U}_1(\alpha)$.

Lemma 3.2. *Let $f : X \to Y$ be a function. We define, for every $\beta \in \Omega_Y$, $f^{*-1}(\beta) = f^{-1} \circ \beta \circ f$; i.e. for all $\mu \in I^X$, $f^{*-1}(\beta)(\mu) = f^{-1}(\beta(f(\mu)))$. Then:*

(1) $f^{*-1}(\beta) \in \Omega_X$,
(2) $(f^{*-1}(\beta))^{-1} \leq f^{*-1}(\beta^{-1})$.

Proof. (1) For $\beta \in \Omega_Y$, we have $f^{*-1}(\beta)(\tilde{0}) = \tilde{0}$. Since $\mu \leq f^{-1}(f(\mu)) \leq f^{-1}(\beta(f(\mu)))$, we have $\mu \leq f^{*-1}(\beta)(\mu)$. By Lemma 1.3, it is easy that $f^{*-1}(\beta)(\bigvee u_i) = \bigvee f^{*-1}(\beta)(u_i)$. Then we have $f^{*-1}(\beta) \in \Omega_X$.

(2) Suppose that there exist $\mu \in I^X$, $x_0 \in X$ such that

$$(f^{*-1}(\beta))^{-1}(\mu)(x_0) > f^{*-1}(\beta^{-1})(\mu)(x_0).$$

Since $f^{*-1}(\beta^{-1})(\mu) = f^{-1}(\beta^{-1}(f(\mu)))$, by the definition of $\beta^{-1}(f(\mu))$, there exists $\rho_0 \in I^X$ such that

$$(f^{*-1}(\beta))^{-1}(\mu)(x_0) > f^{-1}(\rho_0)(x_0)$$

and $\beta(\tilde{1} - \rho_0) \leq \tilde{1} - f(\mu)$.

On the other hand, by Lemma 1.3, we have $\beta(f(\tilde{1} - f^{-1}(\rho_0))) = \beta(f(f^{-1}(\tilde{1} - \rho_0))) \leq \beta(\tilde{1} - \rho_0)$ and $f^{-1}(\beta(\tilde{1} - \rho_0)) \leq f^{-1}(\tilde{1} - f(\mu)) \leq \tilde{1} - \mu$. It follows that $f^{-1}(\beta(f(\tilde{1} - f^{-1}(\rho_0)))) \leq \tilde{1} - \mu$. Hence we have $(f^{*-1}(\beta))^{-1}(\mu) \leq f^{-1}(\rho_0)$. It is a contradiction. □ □

By using Lemma 3.2, we define a fuzzy uniformly continuous map.

Definition 3.3. Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be fuzzy uniform spaces. A function $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is *fuzzy uniformly continuous* if it satisfies $\mathcal{V}(\beta) \leq \mathcal{U}(f^{*-1}(\beta))$, for every $\beta \in \Omega_Y$, where $f^{*-1}(\beta) = f^{-1} \circ \beta \circ f$.

Remark 3. The identity function $i : (X, \mathcal{U}_1) \to (X, \mathcal{U}_2)$ is fuzzy uniformly continuous iff $\mathcal{U}_1$ is finer than $\mathcal{U}_2$.

Theorem 3.4. *Let $(X, \mathcal{U}), (Y, \mathcal{V}), (Z, \mathcal{W})$ be fuzzy uniform spaces. If $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ and $g : (Y, \mathcal{V}) \to (Z, \mathcal{W})$ are fuzzy uniformly continuous, then $g \circ f : (X, \mathcal{U}) \to (Z, \mathcal{W})$ is fuzzy uniformly continuous.*

Proof. It follows from, for every $\gamma \in \Omega_Z$,

$$\mathcal{W}(\gamma) \leq \mathcal{V}(g^{*-1}(\gamma)) \leq \mathcal{U}(f^{*-1}(g^{*-1}(\gamma))) = \mathcal{U}((g \circ f)^{*-1}(\gamma)).$$

□ □

In the following theorem, we obtain a fuzzy proximity space from a fuzzy uniform space.

THEOREM 3.5. [7] *Let* $(X,\mathcal{U})$ *be a fuzzy uniform space. Define*

$$\delta_{\mathcal{U}}(\lambda,\mu)=\bigwedge\{1-\mathcal{U}(\alpha)\mid \alpha(\mu)\le 1-\lambda\}.$$

Then $(X,\delta_{\mathcal{U}})$ *is a fuzzy proximity space.*

We will investigate the relationship between fuzzy uniform spaces and fuzzy proximity spaces.

THEOREM 3.6. *Let* $(X,\mathcal{U})$ *and* $(Y,\mathcal{V})$ *be fuzzy uniform spaces. If* $f:(X,\mathcal{U})\to(Y,\mathcal{V})$ *is fuzzy uniformly continuous, then* $f:(X,\delta_{\mathcal{U}})\to(Y,\delta_{\mathcal{V}})$ *is a fuzzy proximity map.*

Proof. Suppose that f is not a fuzzy proximity map. There exist $\lambda,\mu\in I^X$ such that $\delta_{\mathcal{U}}(\lambda,\mu)>\delta_{\mathcal{V}}(f(\lambda),f(\mu))$. By the definition of $\delta_{\mathcal{V}}$, there exists $\beta_0\in\Omega_Y$ such that

$$\delta_{\mathcal{U}}(\lambda,\mu)>1-\mathcal{V}(\beta_0)>\delta_{\mathcal{V}}(f(\lambda),f(\mu)),\quad \beta_0(f(\mu))\le\tilde{1}-f(\lambda).$$

Therefore, since $f^{-1}(\tilde{1}-f(\lambda))=\tilde{1}-f^{-1}(f(\lambda))$ and $f^{-1}(f(\lambda))\ge\lambda$,

$$f^{*-1}(\beta_0)(\mu)=f^{-1}\circ\beta_0\circ f(\mu)\le f^{-1}(\tilde{1}-f(\lambda))\le\tilde{1}-\lambda.$$

By the definition of $\delta_{\mathcal{U}}$, we have

$$1-\mathcal{U}(f^{*-1}(\beta_0))\ge\delta_{\mathcal{U}}(\lambda,\mu)>1-\mathcal{V}(\beta_0).$$

But f is fuzzy uniformly continuous, it is a contradiction. □ □

LEMMA 3.7. [7] *Let* $(X,\mathcal{U})$ *be a fuzzy uniform space. We define* $cl_{\mathcal{U}}(\lambda,r)=\bigwedge\{\alpha(\lambda)\mid\mathcal{U}(\alpha)>r\}$. *Then* $cl_{\mathcal{U}}(\lambda,r)=cl_{\delta_{\mathcal{U}}}(\lambda,r)$.

By Theorem 2.5 and Lemma 3.7, it is clear that if $f:(X,\mathcal{U})\to(Y,\mathcal{V})$ is fuzzy uniformly continuous, then $f:(X,\tau_{\mathcal{U}})\to(Y,\tau_{\mathcal{V}})$ is a gp-map.

LEMMA 3.8. *A family* Ω_X *have the following properties:*

(1) $(\alpha^{-1})^{-1}=\alpha$, *for every* $\alpha\in\Omega_X$,
(2) $\alpha_1\le\alpha_2$ *iff* $(\alpha_1)^{-1}\le(\alpha_2)^{-1}$,
(3) $\mathcal{U}(\alpha)=\mathcal{U}(\alpha^{-1})$ *if* $(X,\mathcal{U})$ *is a fuzzy uniform space.*

Proof. (1) Since $\alpha^{-1}(\tilde{1}-\alpha(\mu)) = \bigwedge\{\eta \mid \alpha(\tilde{1}-\eta) \le \alpha(\mu)\} \le \tilde{1}-\mu$, we have $(\alpha^{-1})^{-1}(\mu) \le \alpha(\mu)$. On the other hand, we must show that $(\alpha^{-1})^{-1}(\mu) \ge \alpha(\mu)$, for all $\mu \in I^X$. Suppose that there exist $\mu \in I^X, x \in X$ such that $(\alpha^{-1})^{-1}(\mu)(x) < \alpha(\mu)(x)$. By the definition of $(\alpha^{-1})^{-1}$, there exists $\rho \in I^X$ such that

$$(\alpha^{-1})^{-1}(\mu)(x) < \rho(x) < \alpha(\mu)(x),\ \ \alpha^{-1}(\tilde{1}-\rho)(x) \le 1-\mu(x).$$

Put $\eta_0 = \alpha^{-1}(\tilde{1}-\rho) = \bigwedge\{\eta \mid \alpha(\tilde{1}-\eta) \le \rho\}$. Then $\eta_0(x) \le 1-\mu(x)$ such that $\alpha(\tilde{1}-\eta_0)(x) \le \rho(x) < \alpha(\mu)(x)$. But, since $\mu(x) \le 1-\eta_0(x)$ and α is an increasing function, we have $\alpha(\mu)(x) \le \alpha(\tilde{1}-\eta_0)(x)$. It is a contradiction.

(2) By the definition of α^{-1} and (1), it is trivial.

(3) For $\alpha \in \Omega_X$, there exists $\alpha_1 \in \Omega_X$ such that $\alpha_1 \le \alpha^{-1}$ and $\mathcal{U}(\alpha_1) \ge \mathcal{U}(\alpha)$. By (FU4), it follows that $\mathcal{U}(\alpha^{-1}) \ge \mathcal{U}(\alpha)$. By (1), we have $\mathcal{U}((\alpha^{-1})^{-1}) \ge \mathcal{U}(\alpha^{-1})$. □ □

Definition 3.9. Let $(X_i, \mathcal{U}_i)_{i\in\triangle}$ be fuzzy uniform spaces. Let X be a set and, for each $i \in \triangle$, $f_i : X \to X_i$ a function. The *initial structure* $\mathcal{U}$ is the coarsest fuzzy uniformity on X with respect to which for each $i \in \triangle$, f_i is fuzzy uniformly continuous.

Theorem 3.10. *Let X be a set and $(Y, \mathcal{V})$ a fuzzy uniform space. Let $f : X \to Y$ be a function and $\Gamma_\alpha = \{\beta \in \Omega_Y \mid {f^*}^{-1}(\beta) \le \alpha\}$ given. Define, for every $\alpha \in \Omega_X$,*

$$\mathcal{U}(\alpha) = \begin{cases} sup_{\beta\in\Gamma_\alpha}\mathcal{V}(\beta) & \text{if } \Gamma_\alpha \ne \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathcal{U}$ is the initial fuzzy uniformity on X with respect to which f is fuzzy uniformly continuous.

Proof. By Lemma 3.2, for $\beta \in \Omega_Y$, we have ${f^*}^{-1}(\beta) \in \Omega_X$.

(FU1) We will show that $\mathcal{U}(\alpha_1 \wedge \alpha_2) \ge \mathcal{U}(\alpha_1) \wedge \mathcal{U}(\alpha_2)$.

First, if $\mathcal{U}(\alpha_1) = 0$ or $\mathcal{U}(\alpha_2) = 0$, then it is trivial.

Second, suppose $\mathcal{U}(\alpha_1 \wedge \alpha_2) < \mathcal{U}(\alpha_1) \wedge \mathcal{U}(\alpha_2)$. By the definition of $\mathcal{U}$, for $\alpha_1, \alpha_2 \in \Omega_X$ there exist $\beta_1, \beta_2 \in \Omega_Y$ with ${f^*}^{-1}(\beta_1) \le \alpha_1$, ${f^*}^{-1}(\beta_2) \le \alpha_2$ such that

$$\mathcal{U}(\alpha_1 \wedge \alpha_2) < \mathcal{V}(\beta_1) \wedge \mathcal{V}(\beta_2) < \mathcal{U}(\alpha_1) \wedge \mathcal{U}(\alpha_2).$$

On the other hand, since $f^{*-1}(\beta_1 \wedge \beta_2) \leq \alpha_1 \wedge \alpha_2$, we have

$$\mathcal{U}(\alpha_1 \wedge \alpha_2) \geq \mathcal{V}(\beta_1 \wedge \beta_2) \geq \mathcal{V}(\beta_1) \wedge \mathcal{V}(\beta_2).$$

It is a contradiction.

(FU2) For all $\alpha \in \Omega_X$, if $\mathcal{U}(\alpha) = 0$, then there exists the identity function $i \in \Omega_X$ with $i \circ i \leq \alpha$ such that $\mathcal{U}(i) \geq \mathcal{U}(\alpha) = 0$.

If $\mathcal{U}(\alpha) \neq 0$, for $\epsilon > 0$, by the definition of $\mathcal{U}$, there exists $\beta \in \Omega_Y$ such that $\mathcal{V}(\beta) \geq \mathcal{U}(\alpha) - \epsilon$ and $f^{*-1}(\beta) \leq \alpha$. By (FU2), for $\beta \in \Omega_Y$, there exists $\beta_1 \in \Omega_Y$ such that $\beta_1 \circ \beta_1 \leq \beta$ and $\mathcal{V}(\beta_1) \geq \mathcal{V}(\beta)$. By Lemma 1.3, we have $f^{*-1}(\beta_1) \circ f^{*-1}(\beta_1) \leq f^{*-1}(\beta_1 \circ \beta_1)$. Hence, for all $\alpha \in \Omega_X$, there exists $f^{*-1}(\beta_1) \in \Omega_X$ such that

$$f^{*-1}(\beta_1) \circ f^{*-1}(\beta_1) \leq \alpha, \quad \mathcal{U}(f^{*-1}(\beta_1)) \geq \mathcal{V}(\beta_1) \geq \mathcal{V}(\beta) \geq \mathcal{U}(\alpha) - \epsilon.$$

(FU3) For all $\alpha \in \Omega_X$, if $\mathcal{U}(\alpha) = 0$, then there exists the identity function $i \in \Omega_X$ with $i \leq \alpha^{-1}$ such that $\mathcal{U}(i) \geq \mathcal{U}(\alpha) = 0$.

If $\mathcal{U}(\alpha) \neq 0$, for $\epsilon > 0$, by the definition of $\mathcal{U}$, there exists $\beta \in \Omega_Y$ such that $\mathcal{V}(\beta) \geq \mathcal{U}(\alpha) - \epsilon$ and $f^{*-1}(\beta) \leq \alpha$. By (FU3), for $\beta \in \Omega_Y$, there exists $\beta_1 \in \Omega_Y$ such that $\beta_1 \leq \beta^{-1}$ and $\mathcal{V}(\beta_1) \geq \mathcal{V}(\beta)$. By Lemma 3.8, we have $\beta_1^{-1} \leq (\beta^{-1})^{-1} = \beta$ and $f^{*-1}(\beta_1^{-1}) \leq f^{*-1}(\beta) \leq \alpha$. By Lemma 3.2, since $f^{*-1}(\beta_1)^{-1} \leq f^{*-1}(\beta_1^{-1})$, there exists $f^{*-1}(\beta_1) \in \Omega_X$ such that

$$f^{*-1}(\beta_1) \leq f^{*-1}(\beta_1^{-1})^{-1} \leq \alpha^{-1}, \quad \mathcal{U}(f^{*-1}(\beta_1)) \geq \mathcal{V}(\beta) \geq \mathcal{U}(\alpha) - \epsilon.$$

(FU4) It is trivial.

(FU5) There exists $\beta \in \Omega_Y$ such that $\mathcal{V}(\beta) = 1$. Then $f^{*-1}(\beta) \in \Omega_X$ such that $\mathcal{U}(f^{*-1}(\beta)) = 1$.

By the definition of $\mathcal{U}$, we have $\mathcal{U}(f^{*-1}(\beta)) \geq \mathcal{V}(\beta)$. Hence $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is fuzzy uniformly continuous.

If $f : (X, \mathcal{U}') \to (Y, \mathcal{V})$ is fuzzy uniformly continuous, we will show that for any $\alpha \in \Omega_X$, $\mathcal{U}(\alpha) \leq \mathcal{U}'(\alpha)$. Suppose that there exists $\alpha \in \Omega_X$ such that $\mathcal{U}(\alpha) > \mathcal{U}'(\alpha)$. There exists $\beta \in \Omega_Y$ such that $\mathcal{U}(\alpha) > \mathcal{V}(\beta) > \mathcal{U}'(\alpha)$ and $f^{*-1}(\beta) \leq \alpha$. But , since $f^{*-1}(\beta) \leq \alpha$, by (FU4), we have $\mathcal{U}'(f^{*-1}(\beta)) \leq \mathcal{U}'(\alpha) < \mathcal{V}(\beta)$. It is a contradiction. $\square$ $\square$

Let $(X, \mathcal{U}_1)$ and $(X, \mathcal{U}_2)$ be fuzzy uniform spaces. A structure $\mathcal{U}_1 \vee \mathcal{U}_2$ is not a fuzzy uniform structure on X, because it does not satify (FU1).

We will construct the coarsest fuzzy uniformity on X finer than $\mathcal{U}_1$ and $\mathcal{U}_2$.

THEOREM 3.11. *Let* $(X,\mathcal{U}_1)$ *and* $(X,\mathcal{U}_2)$ *be fuzzy uniform spaces. We define, for all* $\gamma \in \Omega_X$,

$$\mathcal{U}_1 \odot \mathcal{U}_2(\gamma) = sup\{\mathcal{U}_1(\alpha) \wedge \mathcal{U}_2(\beta) \mid \alpha \wedge \beta \leq \gamma\}.$$

Then the structure $\mathcal{U}_1 \odot \mathcal{U}_2$ *is the coarsest fuzzy uniformity on* X *finer than* $\mathcal{U}_1$ *and* $\mathcal{U}_2$.

Proof. (FU1) For any γ_1, γ_2, we will show that

$$\mathcal{U}_1 \odot \mathcal{U}_2(\gamma_1 \wedge \gamma_2) \geq \mathcal{U}_1 \odot \mathcal{U}_2(\gamma_1) \wedge \mathcal{U}_1 \odot \mathcal{U}_2(\gamma_2).$$

Put $c_1 = \mathcal{U}_1 \odot \mathcal{U}_2(\gamma_1)$ and $c_2 = \mathcal{U}_1 \odot \mathcal{U}_2(\gamma_2)$. If $c_1 = 0$ or $c_2 = 0$, it is trivial.

Otherwise, for ϵ such that $c_1 \wedge c_2 > \epsilon > 0$, there exist $\alpha_1, \beta_1, \alpha_2, \beta_2$ such that $\mathcal{U}_1(\alpha_1) \wedge \mathcal{U}_2(\beta_1) \geq c_1 - \epsilon,\ \ \alpha_1 \wedge \beta_1 \leq \gamma_1$ and $\mathcal{U}_1(\alpha_2) \wedge \mathcal{U}_2(\beta_2) \geq c_2 - \epsilon,\ \ \alpha_2 \wedge \beta_2 \leq \gamma_2$. Since $(\alpha_1 \wedge \alpha_2) \wedge (\beta_1 \wedge \beta_2) \leq \gamma_1 \wedge \gamma_2$, by Remark 2,

$$\mathcal{U}_1 \odot \mathcal{U}_2(\gamma_1 \wedge \gamma_2) \geq \mathcal{U}_1(\alpha_1 \wedge \alpha_2) \wedge \mathcal{U}_2(\beta_1 \wedge \beta_2) \geq (c_1 - \epsilon) \wedge (c_2 - \epsilon).$$

(FU2) For all $\gamma \in \Omega_X$, put $c = \mathcal{U}_1 \odot \mathcal{U}_2(\gamma)$. If $c = 0$, then there exists the identity function $i \in \Omega_X$ with $i \circ i \leq \gamma$ such that $\mathcal{U}_1 \odot \mathcal{U}_2(i) \geq 0$.

If $c \neq 0$, for $\epsilon > 0$, there exist α, β such that $\mathcal{U}_1(\alpha) \wedge \mathcal{U}_2(\beta) \geq c - \epsilon,\ \ \alpha \wedge \beta \leq \gamma$. By (FU2), there exist α_1, β_1 such that $\alpha_1 \circ \alpha_1 \leq \alpha,\ \ \mathcal{U}_1(\alpha_1) \geq \mathcal{U}_1(\alpha)$ and $\beta_1 \circ \beta_1 \leq \beta,\ \ \mathcal{U}_2(\beta_1) \geq \mathcal{U}_2(\beta)$. Then there exists $\alpha_1 \wedge \beta_1 \in \Omega_X$ with $(\alpha_1 \wedge \beta_1) \circ (\alpha_1 \wedge \beta_1) \leq \alpha \wedge \beta \leq \gamma$ such that

$$\mathcal{U}_1 \odot \mathcal{U}_2(\alpha_1 \wedge \beta_1) \geq \mathcal{U}_1(\alpha_1) \wedge \mathcal{U}_2(\beta_1) \geq \mathcal{U}_1(\alpha) \wedge \mathcal{U}_2(\beta) \geq c - \epsilon.$$

(FU3) For all $\gamma \in \Omega_X$, put $c = \mathcal{U}_1 \odot \mathcal{U}_2(\gamma)$. If $c = 0$, then there exists the identity function $i \in \Omega_X$ with $i \leq \gamma^{-1}$ such that $\mathcal{U}_1 \odot \mathcal{U}_2(i) \geq 0$.

If $c \neq 0$, for $\epsilon > 0$, there exist $\alpha, \beta \in \Omega_X$ such that $\mathcal{U}_1(\alpha) \wedge \mathcal{U}_2(\beta) \geq c - \epsilon, \alpha \wedge \beta \leq \gamma$. By (FU3), for $\alpha, \beta \in \Omega_X$, there exist α_1, β_1 such that $\alpha_1 \leq \alpha^{-1}, \mathcal{U}_1(\alpha_1) \geq \mathcal{U}_1(\alpha)$ and $\beta_1 \leq \beta^{-1}, \mathcal{U}_2(\beta_1) \geq \mathcal{U}_2(\beta)$. It follows that, by Lemma 3.2,

$$(\alpha_1 \wedge \beta_1)^{-1} \leq (\alpha_1)^{-1} \wedge (\beta_1)^{-1} \leq \alpha \wedge \beta.$$

Hence there exists $\alpha_1 \wedge \beta_1 \in \Omega_X$ such that $\alpha_1 \wedge \beta_1 \leq (\alpha \wedge \beta)^{-1} \leq \gamma^{-1}$,

$$\mathcal{U}_1 \odot \mathcal{U}_2(\alpha_1 \wedge \beta_1) \geq \mathcal{U}_1(\alpha_1) \wedge \mathcal{U}_2(\beta_1) \geq \mathcal{U}_1(\alpha) \wedge \mathcal{U}_2(\beta) \geq c - \epsilon.$$

(FU4) It is trivial.

(FU5) There exist α, β such that $\mathcal{U}_1(\alpha) = 1, \quad \mathcal{U}_2(\beta) = 1$. Hence $\mathcal{U}_1 \odot \mathcal{U}_2(\alpha \wedge \beta) = 1$.

We will show that

$$\mathcal{U}_1 \odot \mathcal{U}_2(\gamma) \geq \mathcal{U}_1(\gamma).$$

Put $\iota(\tilde{0}) = \tilde{0}$ and $\iota(\mu) = \tilde{1}$, for all $\mu \in I^X$, then $\iota \in \Omega_X$ and $\iota \wedge \alpha = \alpha$ for all $\alpha \in \Omega_X$. By (FU4) and (FU5), we have $\mathcal{U}_2(\iota) = 1$.

$$\begin{aligned}
\mathcal{U}_1 \odot \mathcal{U}_2(\gamma) &= sup\{\mathcal{U}_1(\alpha) \wedge \mathcal{U}_2(\beta) \mid \alpha \wedge \beta \leq \gamma\} \\
&\geq sup\{\mathcal{U}_1(\alpha) \wedge \mathcal{U}_2(\iota) \mid \alpha \wedge \iota \leq \gamma\} \\
&= sup\{\mathcal{U}_1(\alpha) \mid \alpha \leq \gamma\} = \mathcal{U}_1(\gamma).
\end{aligned}$$

Similarly, we have $\mathcal{U}_1 \odot \mathcal{U}_2(\gamma) \geq \mathcal{U}_2(\gamma)$.

Finally, if $\mathcal{U} \succeq \mathcal{U}_1$ and $\mathcal{U} \succeq \mathcal{U}_2$, we will show that $\mathcal{U} \succeq \mathcal{U}_1 \odot \mathcal{U}_2$.

$$\begin{aligned}
\mathcal{U}_1 \odot \mathcal{U}_2(\gamma) &= sup\{\mathcal{U}_1(\alpha) \wedge \mathcal{U}_2(\beta) \mid \alpha \wedge \beta \leq \gamma\} \\
&\leq sup\{\mathcal{U}(\alpha) \wedge \mathcal{U}(\beta) \mid \alpha \wedge \beta \leq \gamma\} \\
&= sup\{\mathcal{U}(\alpha \wedge \beta) \mid \alpha \wedge \beta \leq \gamma\} \\
&= \mathcal{U}(\gamma).
\end{aligned}$$

□ □

Let $(X, \mathcal{V}_1)$ and $(Y, \mathcal{V}_2)$ be fuzzy uniform spaces and $X \times Y$ a set. If $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are projection maps, by Theorem 3.10, we have the initial structures $\mathcal{U}_1$ and $\mathcal{U}_2$ induced by π_1 and π_2, respectively. We will show that the structure $\mathcal{U}_1 \odot \mathcal{U}_2$ is the initial fuzzy uniformity structure with respect to π_1 and π_2.

THEOREM 3.12. *Let $(X, \mathcal{V}_1)$ and $(Y, \mathcal{V}_2)$ be fuzzy uniform spaces and $X \times Y$ a set. Then:*

(1) *$\mathcal{U}_1 \odot \mathcal{U}_2$ is the initial fuzzy uniformity on $X \times Y$ with respect to which π_1 and π_2 are fuzzy uniformly continuous.*

(2) *A map $f : (Z, \mathcal{W}) \to (X \times Y, \mathcal{U}_1 \odot \mathcal{U}_2)$ is fuzzy uniformly continuous iff $\pi_1 \circ f$ and $\pi_2 \circ f$ are fuzzy uniformly continuous.*

Proof. (1) Since $\mathcal{U}_1 \odot \mathcal{U}_2$ is finer than $\mathcal{U}_1$, by Remark 3, $\pi_1 : (X \times Y, \mathcal{U}_1 \odot \mathcal{U}_2) \to (X, \mathcal{V})$ is fuzzy uniformly continuous. Similarly, so is π_2. By Theorem 3.10 and Theorem 3.11, all the others are easy.

(2) Necessity of the composition condition is clear since the composition of fuzzy uniformly continuous maps is a fuzzy uniformly continuous map.

Conversely, suppose that $f : (Z, \mathcal{W}) \to (X \times Y, \mathcal{U}_1 \odot \mathcal{U}_2)$ is not fuzzy uniformly continuous. There exists $\gamma \in \Omega_{X \times Y}$ such that $\mathcal{W}(f^{*-1}(\gamma)) < \mathcal{U}_1 \odot \mathcal{U}_2(\gamma)$. By the definition of $\mathcal{U}_1 \odot \mathcal{U}_2$, there exist α, β such that

$$\mathcal{W}(f^{*-1}(\gamma)) < \mathcal{U}_1(\alpha) \wedge \mathcal{U}_2(\beta) < \mathcal{U}_1 \odot \mathcal{U}_2(\gamma), \ \alpha \wedge \beta \le \gamma.$$

By the definitions of $\mathcal{U}_1$ and $\mathcal{U}_2$ in Theorem 3.10, for $\alpha, \beta \in \Omega_{X \times Y}$, there exist α_1, β_1 such that

$$\mathcal{W}(f^{*-1}(\gamma)) < \mathcal{V}_1(\alpha_1) \wedge \mathcal{V}_2(\beta_1), \ \ \pi_1^{*-1}(\alpha_1) \le \alpha, \ \ \pi_2^{*-1}(\beta_1) \le \beta.$$

On the other hand, since $\pi_1 \circ f$ and $\pi_2 \circ f$ are fuzzy uniformly continuous, we have $\mathcal{V}_1(\alpha_1) \le \mathcal{W}((\pi_1 \circ f)^{*-1}(\alpha_1)) = \mathcal{W}(f^{*-1} \circ \pi_1^{*-1}(\alpha_1))$ and $\mathcal{V}_2(\beta_1) \le \mathcal{W}((\pi_2 \circ f)^{*-1}(\beta_1)) = \mathcal{W}(f^{*-1} \circ \pi_2^{*-1}(\beta_1))$. It follows that

$$\begin{aligned}
\mathcal{V}_1(\alpha_1) \wedge \mathcal{V}_2(\beta_1) &\le \mathcal{W}(f^{*-1} \circ \pi_1^{*-1}(\alpha_1)) \wedge \mathcal{W}(f^{*-1} \circ \pi_2^{*-1}(\beta_1)) \\
&= \mathcal{W}(f^{*-1}(\pi_1^{*-1}(\alpha_1) \wedge \pi_2^{*-1}(\beta_1))) \\
&\le \mathcal{W}(f^{*-1}(\alpha \wedge \beta)) \\
&\le \mathcal{W}(f^{*-1}(\gamma)).
\end{aligned}$$

It is a contradiction. □ □

By the above theorem, we can define the product of two fuzzy uniform spaces.

Definition 3.13. Let $(X, \mathcal{V}_1)$ and $(Y, \mathcal{V}_2)$ be fuzzy uniform spaces and $X \times Y$ a set. The initial structure $\mathcal{U}_1 \odot \mathcal{U}_2$ with respect to π_1 and π_2 is called the *product fuzzy uniformity structure* on $X \times Y$.

The pair $(X \times Y, \mathcal{U}_1 \odot \mathcal{U}_2)$ is called a *product fuzzy uniform space.*

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