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ON FUZZY UNIFORM CONVERGENCE

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Abstract. In this note, we study on fuzzy uniform convergences of sequences of fuzzy numbers, and sequences of fuzzy functions.

1. Introduction

Zhang [1] provided "The Cauchy criterion for sequences of fuzzy numbers" under a restricted condition. In this note, we prove the criterion of fuzzy uniform convergences for fuzzy numbers and fuzzy functions.

2. Preliminaries

All fuzzy sets, considered in this paper, are functions defined in the set $\mathbf{R}$ of real numbers to $[0,1]$.

Definition 2.1. [1] A fuzzy set A is called a *fuzzy number* if the following conditions are satisfied:

(1) there exists $x \in \mathbf{R}$ such that $A(x) = 1$;
(2) for any $\lambda \in (0,1]$, the set $\{x|A(x) \geq \lambda\}$ is a closed interval, denoted by $[A_\lambda^-, A_\lambda^+]$.

Note that every fuzzy point a_1 $(a \in \mathbf{R})$ defined by

$$a_1(x) = \begin{cases} 1 & \text{for } x = a \\ 0 & \text{for } x \neq a \end{cases}$$

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is a fuzzy number.

Let $\mathcal{F}(\mathbf{R})$ be the set of all fuzzy numbers. Remark that for any $A \in \mathcal{F}(\mathbf{R})$,

$$A = \sup_{\lambda\in[0,1]} \lambda\chi_{[A_\lambda^-, A_\lambda^+]},$$

where each $\chi_{[A_\lambda^-, A_\lambda^+]}$ denotes the characteristic function. For notational convenience, we shall denote $\chi_{[A_\lambda^-, A_\lambda^+]}$ by $[A_\lambda^-, A_\lambda^+]$.

Definition 2.2. [1] For any $a \in \mathbf{R}$, any $\{A_k | k = 1, \cdots, n\} \subset \mathcal{F}(\mathbf{R})$ and any $A, B, C \in \mathcal{F}(\mathbf{R})$,

(1) $C = A \pm B$ if for every $\lambda \in (0, 1]$, $C_\lambda^- = A_\lambda^- \pm B_\lambda^-$ and $C_\lambda^+ = A_\lambda^+ \pm B_\lambda^+$. We denote

$$A_1 + \cdots + A_n = \sum_{k=1}^{n} A_k;$$

(2) $C = A * B$ if for every $\lambda \in (0, 1]$, $C_\lambda^- = A_\lambda^- B_\lambda^-$ and $C_\lambda^+ = A_\lambda^+ B_\lambda^+$. We denote

$$\underbrace{A * \cdots * A}_{\text{n-copies}} = A^n;$$

(3) $A \le B$ if for every $\lambda \in (0, 1]$, $A_\lambda^- \le B_\lambda^-$ and $A_\lambda^+ \le B_\lambda^+$;
(4) $A < B$ if $A \le B$ and there exists $\lambda_0 \in (0, 1]$ such that $A_{\lambda_0}^- < B_{\lambda_0}^-$ or $A_{\lambda_0}^+ < B_{\lambda_0}^+$;
(5) $A = B$ if $A \le B$ and $B \le A$.
(6)

$$aA = \begin{cases} \sup_{\lambda\in[0,1]} \lambda[aA_\lambda^-, aA_\lambda^+] & \text{for } a \ge 0 \\ \sup_{\lambda\in[0,1]} \lambda[aA_\lambda^+, aA_\lambda^-] & \text{for } a < 0. \end{cases}$$

Lemma 2.3. *For any* $A \in \mathcal{F}(\mathbf{R})$ *and any* $a, b \in \mathbf{R}$,

(1) $A + 0_1 = A$;
(2) $A * 1_1 = A$;
(3) $a_1 + b_1 = (a + b)_1$;
(4) $a_1 \le b_1$, *if* $a \le b$ *in* $\mathbf{R}$.

DEFINITION 2.4. [1] A fuzzy number A in $\mathcal{F}(\mathbf{R})$ is said to *belong to fuzzy infinity*, denoted by $A \in \infty$, if for any positive real number M, there exists $\lambda_0 \in (0,1]$ such that $A^-_{\lambda_0} \leq -M$ or $A^+_{\lambda_0} \geq M$.

DEFINITION 2.5. [1] A function $d : \mathcal{F}(\mathbf{R}) \times \mathcal{F}(\mathbf{R}) \to \mathcal{F}(\mathbf{R})$ is called a *fuzzy distance* on $\mathcal{F}(\mathbf{R})$ if

(1) $d(A,B) \geq 0_1$, $d(A,B) = 0_1$ if and only if $A = B$;
(2) $d(A,B) = d(B,A)$;
(3) $d(A,B) \leq d(A,C) + d(C,B)$.

LEMMA 2.6. [1] *The function* $\rho : \mathcal{F}(\mathbf{R}) \times \mathcal{F}(\mathbf{R}) \to \mathcal{F}(\mathbf{R})$, *defined by*

$$\rho(A,B) = \sup_{\lambda \in [0,1]} \lambda[|A^-_1 - B^-_1|, \sup_{\lambda \leq \mu \leq 1} \max\{|A^-_\mu - B^-_\mu|, |A^+_\mu - B^+_\mu|\}]$$

is a fuzzy distance on $\mathcal{F}(\mathbf{R})$.

Throughout this paper, $\mathcal{F}(\mathbf{R})$ is the set of all fuzzy numbers with a fuzzy distance ρ defined in the above lemma.

The equality $\rho(A \pm C, B \pm C) = \rho(A,B)$ for $A, B, C \in \mathcal{F}(\mathbf{R})$ ([1]) is needed in the proof of Theorem 3.7.

DEFINITION 2.7. [1] Let $\{A_n\} \subset \mathcal{F}(\mathbf{R})$ and $A \in \mathcal{F}(\mathbf{R})$. $\{A_n\}$ is said to *converge* to A, denoted by $\lim_{n\to\infty} A_n = A$, if for any $\epsilon > 0$, there exists a positive integer N such that $\rho(A_n, A) < \epsilon_1$ for every $n \geq N$.

LEMMA 2.8. [1] *Let* $\{A_n\} \subset \mathcal{F}(\mathbf{R})$. *Then* $\{A_n\}$ *converges to a fuzzy number* A *if and only if* $\{(A_n)^-_\lambda\}$ *and* $\{(A_n)^+_\lambda\}$ *converges uniformly to* A^-_λ *and* A^+_λ, *respectively, for every* $\lambda \in (0,1]$ *in the usual distance of real numbers.*

DEFINITION 2.9. [1] Let $\mathcal{A} \subset \mathcal{F}(\mathbf{R})$. A mapping $f : \mathcal{A} \to \mathcal{F}(\mathbf{R})$ is called a *fuzzy function* on $\mathcal{A}$. For any fuzzy functions $f, g : \mathcal{A} \to \mathcal{F}(\mathbf{R})$, we define for any $A \in \mathcal{A}$, $(f \pm g)(A) = f(A) \pm g(A)$ and $(f * g)(A) = f(A) * g(A)$.

3. Fuzzy Uniform Convergence

DEFINITION 3.1. [1] Let $\{A_n\} \subset \mathcal{F}(\mathbf{R})$. Then $\{A_n\}$ is called a *fuzzy Cauchy sequence* if for any $\epsilon > 0$, there exists a positive integer N such that $\rho(A_n, A_m) < \epsilon_1$ for all $m, n \geq N$.

LEMMA 3.2. *Let $\{A_n\} \subset \mathcal{F}(\mathbf{R})$. Then $\{A_n\}$ is fuzzy Cauchy if and only if $\{(A_n)_\lambda^-\}$ and $\{(A_n)_\lambda^+\}$ are uniformly Cauchy sequence of real numbers for every $\lambda \in (0, 1]$ in the usual distance of real numbers.*

Proof. ($\Rightarrow$) Assume that $\{A_n\}$ is a fuzzy Cauchy sequence. Then for any $\epsilon > 0$, there exists a positive integer N such that for all $n, m \geq N$, $\rho(A_n, A_m) < (\epsilon/2)_1$. This means that for all $n, m \geq N$ and all $\lambda \in (0, 1]$,

$$\max\{|(A_n)_\lambda^- - (A_m)_\lambda^-|, |(A_n)_\lambda^+ - (A_m)_\lambda^+|\} \leq \epsilon/2 < \epsilon,$$

and hence the desired result follows.

($\Leftarrow$) Assume the given condition is satisfied. Let $\epsilon > 0$ be given. Then there exists a positive real number N such that

$$|(A_n)_\lambda^- - (A_m)_\lambda^-| < \epsilon/2 \text{ and } |(A_n)_\lambda^+ - (A_m)_\lambda^+| < \epsilon/2$$

for all $n, m \geq N$ and all $\lambda \in (0, 1]$. Thus

$$sup_{\lambda \leq \mu \leq 1} \max\{|(A_n)_\lambda^- - (A_m)_\lambda^-|, |(A_n)_\lambda^+ - (A_m)_\lambda^+|\} < \epsilon$$

for all $n, m \geq N$ and all $\lambda \in (0, 1]$. Consequently, $\rho(A_n, A_m) < \epsilon_1$ for all $n, m \leq N$. $\square$ $\square$

THEOREM 3.3. *Every fuzzy Cauchy sequence converges.*

Proof. Let $\{A_n\}$ be a fuzzy Cauchy sequence. By Lemma 3.2, $\{(A_n)_\lambda^-\}$ and $\{(A_n)_\lambda^+\}$ are uniformly Cauchy for every $\lambda \in (0, 1]$, in the usual distance of real numbers. By the well known theorem for uniform Cauchy sequence of real valued functions, there exist real valued functions $f(\lambda)$ and $g(\lambda)$ defined on $(0, 1]$ such that $\{(A_n)_\lambda^-\}$ and $\{(A_n)_\lambda^+\}$ converges uniformly to $f(\lambda)$ and $g(\lambda)$, respectively. Let $f(\lambda) = A_\lambda^-$, $g(\lambda) = A_\lambda^+$ and $A = \sup_{\lambda \in [0,1]} \lambda[A_\lambda^-, A_\lambda^+]$. Then A is a fuzzy number. By Lemma 2.8, $\lim_{n \to \infty} A_n = A$. $\square$ $\square$

Definition 3.4. Let $\mathcal{A} \subset \mathcal{F}(\mathbf{R})$ and let $\{f_n\}$ be a sequence of fuzzy functions from $\mathcal{A}$ to $\mathcal{F}(\mathbf{R})$.

(1) We say that $\{f_n\}$ *converges pointwise* to a fuzzy function $f : \mathcal{A} \to \mathcal{F}(\mathbf{R})$ if for any $A \in \mathcal{A}$, $\{f_n(A)\}$ converges to $f(A)$.
(2) We say that $\{f_n\}$ *converges uniformly* to a fuzzy function $f : \mathcal{A} \to \mathcal{F}(\mathbf{R})$ if for any $\epsilon > 0$ and any $A \in \mathcal{A}$, there exists a positive integer $N(\epsilon)$ such that $\rho(f_n(A), f(A)) < \epsilon_1$ for all $n \geq N(\epsilon)$ and all $A \in \mathcal{A}$.
(3) The sequence $\{f_n\}$ is said to be *uniformly fuzzy Cauchy* if for any $\epsilon > 0$, there exists a positive integer $N(\epsilon)$ such that $\rho(f_m(A), f_n(A)) < \epsilon_1$ for all $m, n \geq N(\epsilon)$ and all $A \in \mathcal{A}$.
(4) An infinite series $\sum_{k=1}^{\infty} f_k$ of fuzzy functions is said to *converge uniformly* to a fuzzy function $f : \mathcal{A} \to \mathcal{F}(\mathbf{R})$ if the sequence of partial sums $\{S_n\} = \{\sum_{k=1}^{n} f_k\}$ of the series converges uniformly to f

If we use Theorem 3.3, the proof of the following theorem follows that of corresponding classical theorem in real analysis.

Theorem 3.5. *Let $\mathcal{A} \subset \mathcal{F}(\mathbf{R})$ and let $\{f_n\}$ be a sequence of fuzzy functions from $\mathcal{A}$ to $\mathcal{F}(\mathbf{R})$. Then the sequence $\{f_n\}$ is uniformly Cauchy if and only if there exists a fuzzy function $f : \mathcal{A} \to \mathcal{F}(\mathbf{R})$ such that $\{f_n\}$ converges uniformly to f.*

Lemma 3.6. *For any $A \in \mathcal{F}(\mathbf{R})$, the following are equivalent:*

(1) *$A \notin \infty$;*
(2) *there exists a positive real number M such that*

$$\max\{|A_\lambda^-|, |A_\lambda^+|\} < M \text{ for all } \lambda \in (0, 1];$$

(3) *there exists a positive real number M such that $\rho(A, 0_1) < M_1$.*

Proof. (1) $\Rightarrow$ (2). Assume $A \notin \infty$. Then there exists a positive real number M such that for every $\lambda \in (0, 1]$, $-M < A_\lambda^- \leq A_\lambda^+ < M$. Thus, $\max\{|A_\lambda^-|, |A_\lambda^+|\} < M$ for all $\lambda \in (0, 1]$.

(2) $\Rightarrow$ (3). If such an M exists, then

$$\begin{aligned}\rho(A, 0_1) &= \sup_{\lambda\in[0,1]} \lambda[|A_1^-|, \sup_{\lambda\le\mu\le 1} \max\{|A_\mu^-|, |A_\mu^+|\}] \\ &< \sup_{\lambda\in[0,1]} \lambda[M, M] \\ &= M_1.\end{aligned}$$

(3) $\Rightarrow$ (1). Assume to the contrary that $A \in \infty$. Then for every positive integer M, there exists $\lambda_0 \in (0, 1]$ such that $M \le A_{\lambda_0}^+$ or $A_{\lambda_0}^- \le -M$. Therefore, $\sup_{\lambda_0\le\mu\le 1} \max\{|A_\mu^-|, |A_\mu^+|\} \ge M$, contrary to the hypothesis. □ □

THEOREM 3.7. [Weierstrass M-test] *Let $\mathcal{A} \subset \mathcal{F}(\mathbf{R})$ and let $\{f_n\}$ be a sequence of fuzzy functions from $\mathcal{A}$ to $\mathcal{F}(\mathbf{R})$. If for each n, there exists a positive real number M_n such that*

$$\max\{|(f_n(A))_\lambda^-|, |(f_n(A))_\lambda^+|\} \le M_n$$

for all $A \in \mathcal{A}$ and all $\lambda \in (0, 1]$, and if the series $\sum_{n=1}^{\infty} M_n$ converges, then there exists a fuzzy function $f : \mathcal{A} \to \mathcal{F}(\mathbf{R})$ such that $\sum_{n=1}^{\infty} f_n$ converges uniformly to f.

Proof. Let $\epsilon > 0$ be given. Since $\sum_{k=1}^{\infty} M_k$ converges, there exists a positive integer N such that for all $m \ge n \ge N$,

$$|\sum_{k=1}^{m} M_k - \sum_{k=1}^{n} M_k| = \sum_{k=n+1}^{m} M_k < \epsilon.$$

Note that

$$\max\{|\sum_{k=n+1}^{m} (f_k(A))_\lambda^-|, |\sum_{k=n+1}^{m} (f_k(A))_\lambda^+|\} \le \sum_{k=n+1}^{m} M_k.$$

By Lemma 3.6,

$$\rho(S_m(A), S_n(A)) = \rho(\sum_{k=n+1}^{m} f_k(A), 0_1) \le (\sum_{k=n+1}^{m} M_k)_1 < \epsilon_1$$

for all $m \geq n \geq N$ and all $A \in \mathcal{A}$. This shows that the sequence of partial sums $\{S_n\}$ of $\sum_{k=1}^{\infty} f_k$ is uniformly fuzzy Cauchy. By Theorem 3.5, there exists a function $f : \mathcal{A} \to \mathcal{F}(\mathbf{R})$ such that $\{S_n\}$ converges uniformly to f. By Definition 3.4, the series $\sum_{k=1}^{\infty} f_k$ converges uniformly to f. $\square$ $\square$

DEFINITION 3.8. [1] Let f be a fuzzy function defined on a subset $\mathcal{A}$ of $\mathcal{F}(\mathbf{R})$ and let $A \in \mathcal{A}$. If for any $\epsilon > 0$, there exists $\delta > 0$ such that $\rho(f(A), f(X)) < \epsilon_1$ whenever $X \in \mathcal{A}$ and $\rho(A, X) < \delta_1$, then f is called *fuzzy continuous* at A.

The proof of the following theorem is completely analogous to that of real uniform limit theorem, and hence omitted.

THEOREM 3.9. [Uniform limit theorem] *Let $\mathcal{A} \subset \mathcal{F}(\mathbf{R})$ and let $\{f_n\}$ be a sequence of fuzzy continuous functions from $\mathcal{A}$ to $\mathcal{F}(\mathbf{R})$. If $\{f_n\}$ converges uniformly to a fuzzy function $f : \mathcal{A} \to \mathcal{F}(\mathbf{R})$, then f is fuzzy continuous.*

REMARK 3.10. Obviously, pointwise convergence implies uniform convergence. But the converse is not necessarily true as is seen from the following example.

DEFINITION 3.11. [1] A subset $\mathcal{A}$ of $\mathcal{F}(\mathbf{R})$ is called a *M-closed interval* if there exist $A, B \in \mathcal{A}$ with $A \leq B$ such that for any $C, D \in \mathcal{A}$, $A \leq C \leq D \leq B$ and $(C + D)/2 \in \mathcal{A}$.

EXAMPLE 3.12. Let $\mathcal{A} = \{a_1 | 0 \leq a \leq 1\}$. Then $\mathcal{A}$ is M-closed interval. For each n, define $f_n : \mathcal{A} \to \mathcal{F}(\mathbf{R})$ by $f_n(a_1) = (a_1)^n$. It is easy to show that every f_n is fuzzy continuous on $\mathcal{A}$. Let $f : \mathcal{A} \to \mathcal{F}(\mathbf{R})$ be a fuzzy function defined by

$$f(a_1) = \begin{cases} 1_1 & \text{for } a = 1 \\ 0_1 & \text{for } a \neq 1. \end{cases}$$

Clearly, f is not fuzzy continuous at 1_1. Since $(f_n(a_1))_\lambda^- = (f_n(a_1))_\lambda^+ = a^n$, we have by Lemma 2.8, $\lim_{n\to\infty} f_n(a_1) = f(a_1)$ for every $a_1 \in \mathcal{A}$. Thus, $\{f_n\}$ converges pointwise to f. On the other hand, Theorem 3.9 implies that $\{f_n\}$ does not converge uniformly to f.

References

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