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DILATION THEOREM OF OPERATORS WHICH HAVE COMMON NONCYCLIC VECTORS

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Abstract. In this paper, we construct new classes from the idea of [6, Theorem 2.1] and show that the property of operators belonging to the classes is inherited by certain dilations. And we also prove that the existence of common noncyclic vectors for certain families is equivalent to the existence of infinite dimensional common semi-invariant subspace of operators.

1. Introduction

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. A *dual algebra* is a subalgebra of $\mathcal{L}(\mathcal{H})$ that contains the identity operator $1_{\mathcal{H}}$ and is closed in the ultraweak operator topology on $\mathcal{L}(\mathcal{H})$. For $T \in \mathcal{L}(\mathcal{H})$, let $\mathcal{A}_T$ denote the smallest subalgebra of $\mathcal{L}(\mathcal{H})$ that contains T and $1_{\mathcal{H}}$ and is closed in the ultraweak operator topology. Moreover, let $Q_{\mathcal{A}_T}$ denote the quotient space $\mathcal{C}_1(\mathcal{H})/\perp_{\mathcal{A}_T}$, where $\mathcal{C}_1(\mathcal{H})$ is the trace class ideal in $\mathcal{L}(\mathcal{H})$ under the trace norm, and $\perp_{\mathcal{A}_T}$ denotes the preannihilator of $\mathcal{A}_T$ in $\mathcal{C}_1(\mathcal{H})$. For a brief notation, we shall denote $Q_{\mathcal{A}_T}$ by Q_T. One knows that $\mathcal{A}_T$ is the dual space of Q_T and that the duality is given by

$$\langle A, [L]\rangle = tr(AL), \quad A \in \mathcal{A}_T, [L] \in Q_T. \tag{1}$$

The Banach space Q_T is called a predual of $\mathcal{A}_T$. For x and y in $\mathcal{H}$, we can write $x \otimes y$ for the rank one operator in $\mathcal{C}_1(\mathcal{H})$ defined by

$$(x \otimes y)(u) = (u, y)x \quad \text{for all } u \in \mathcal{H}. \tag{2}$$

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The theory of dual algebras is applied to the study of invariant subspaces, dilation theory, and reflexivity. The class $\mathbb{A}_{m,n}$ were defined by H. Bercovici, C. Foias and C. Pearcy in [2]. Also these classes are closely related to the study of the theory of dual algebras. Especially, B. Chevreau and C. Pearcy [6] proved the existence of *common noncyclic vectors* (to be defined in section 2) for certain families $\{T_n\}_{n=1}^{\infty}$ of (in general) noncommuting operators and also proved that the existence of a common noncyclic vector for a family of operators does not imply that the family has a common nontrivial invariant subspace. In this paper, we construct new classes from the idea of Theorem 2.1 in [6] and show that the property of operators belonging to the classes is inherited by certain dilations. And we also prove that the existence of common noncyclic vectors for certain families is equivalent to the existence of infinite dimensional common *semi-invariant subspace* (to be defined in section 2) of operators.

2. Notation and Preliminaries

The notation and terminology employed herein agree with those in [3], [5], [9].

We shall denote by $\mathbb{D}$ the open unit disc in the complex plane $\mathbb{C}$, and we write $\mathbb{T}$ for the boundary of $\mathbb{D}$. The space $L^p = L^p(\mathbb{T}), 1 \leq p \leq \infty$, is the usual Lebesgue function space relative to normalized Lebesgue measure m on $\mathbb{T}$. The space $H^p = H^p(\mathbb{T}), 1 \leq p \leq \infty$, is the usual Hardy space. It is well-known that the space H^∞ is the dual space of L^1/H_0^1, where

$$(3) \quad H_0^1 = \{f \in L^1 : \int_0^{2\pi} f(e^{it})e^{int}dt = 0, \qquad for \quad n = 0, 1, 2, \cdots\}$$

and the duality is given by the pairing

$$(4) \qquad < f, [g] >= \int_T fg\, dm \quad \text{for } f \in H^\infty, [g] \in L^1/H_0^1.$$

Recall that any contraction T can be written as a direct sum $T = T_1 \oplus T_2$, where T_1 is a completely nonunitary contraction and T_2 is a unitary operator. If T_2 is absolutely continuous or acts on the space

$(0), T$ will be called an *absolutely continuous contraction.* The following Foias–Sz.Nagy functional calculus of Theorem 4.1 in [3] provides a good relationship between the function space H^∞ and a dual algebra $\mathcal{A}_T$.

THEOREM 2.1. [3, Theorem 4.1] *Let T be an absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$. Then there is an algebra homomorphism $\Phi_T : H^\infty \to \mathcal{A}_T$ defined by $\Phi_T(f) = f(T)$ such that*

(a) $\Phi_T(1) = 1_{\mathcal{H}}, \Phi_T(\xi) = T,$
(b) $\| \Phi_T(f) \| \leq \| f \|_\infty, f \in H^\infty,$
(c) *Φ_T is continuous if both H^∞ and $\mathcal{A}_T$ are given their weak* topologies,*
(d) *the range of Φ_T is weak* dense in $\mathcal{A}_T$,*
(e) *there exists a bounded, linear, one–to–one map $\phi_T : Q_T \to L^1/H_0^1$ such that ${\phi_T}^* = \Phi_T$, and*
(f) *if Φ_T is an isometry, then Φ_T is a weak* homeomorphism of H^∞ onto $\mathcal{A}_T$ and ϕ_T is an isometry of Q_T onto L^1/H_0^1.*

DEFINITION 2.2. [8] Let $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ be a dual algebra and let m and n be any cardinal numbers such that $1 \leq m, n \leq \aleph_0$. A dual algebra $\mathcal{A}$ will be said to have property $(\mathbb{A}_{m,n})$ if $m \times n$ system of simultaneous equations of the form

$$[x_i \otimes y_j] = [L_{ij}], 0 \leq i < m, 0 \leq j < n \tag{5}$$

where $\{[L_{ij}]\}_{\substack{0 \leq i < m \\ 0 \leq j < n}}$ is an arbitrary $m \times n$ array from $Q_{\mathcal{A}}$, has a solution consisting of a pair of sequences $\{x_i\}_{0 \leq i < m}, \{y_j\}_{0 \leq j < n}$ of vectors from $\mathcal{H}$.

For brief notation, we shall denote $(\mathbb{A}_{n,n})$ by $(\mathbb{A}_n)$. We denote by $\mathbb{A} = \mathbb{A}(\mathcal{H})$ the class of all absolutely continuous contractions T in $\mathcal{L}(\mathcal{H})$ for which the Foias-Sz.Nagy functional calculus $\Phi_T : \mathcal{H}^\infty \to \mathcal{A}_T$ is an isometry. Furthermore, if m and n are cardinal numbers such that $1 \leq m, n \leq \aleph_0$, we denote by $\mathbb{A}_{m,n} = \mathbb{A}_{m,n}(\mathcal{H})$ the set of all T in $\mathbb{A}(\mathcal{H})$ such that the singly generated dual algebra $\mathcal{A}_T$ has property $(\mathbb{A}_{m,n})$.

DEFINITION 2.3. [6] If $\{T_\alpha\}_{\alpha \in I}$ is a family of operators in $\mathcal{L}(\mathcal{H})$ and x is a nonzero vector in $\mathcal{H}$ such that for each $\alpha \in I$,

$$\mathcal{M}_\alpha = \bigvee_{n=0}^{\infty} T_\alpha^n x \neq \mathcal{H}, \tag{6}$$

then x is said to be a *common noncyclic vector* for the family $\{T_\alpha\}_{\alpha\in I}$.

REMARK 2.4. [3] Let $\mathcal{K}$ be arbitrary complex Hilbert space such that dim $\mathcal{K} \le \aleph_0$. If $T_n \in \mathcal{L}(\mathcal{K})$ and $\mathcal{M} \subset \mathcal{K}$ is a *semi-invariant* subspace for T_n (i.e., there exist $\mathcal{N}_1$ and $\mathcal{N}_2$ in $\mathrm{Lat}(T_n)$ such that $\mathcal{M} = \mathcal{N}_1 \ominus \mathcal{N}_2$ and $\mathcal{N}_2 \subset \mathcal{N}_1$), we write $(T_n)_{\mathcal{M}}$ for the compression of T_n to $\mathcal{M}$. In other words,

$$(T_n)_{\mathcal{M}} = P_{\mathcal{M}} T_n|_{\mathcal{M}}, \tag{7}$$

where $P_{\mathcal{M}}$ is the orthogonal projection whose range is $\mathcal{M}$.

We shall employ the notation $C_{\cdot 0} = C_{\cdot 0}(\mathcal{H})$ for the class of all (completely nonunitary) contractions T in $\mathcal{L}(\mathcal{H})$ such that the sequences $\{T^*\}^n$ converges to zero in the strong operator topology and is denoted by, as usual, $C_{0\cdot} = (C_{\cdot 0})^*$.

LEMMA 2.5. [6] *Suppose $\{T_k\}_{k=1}^\infty$ is any sequence of operators contained in the class $\mathbb{A}_{\aleph_0} \cap C_{\cdot 0}, \{[L_k]_{T_k}\}_{k=1}^\infty$ is an arbitrary sequence (where $[L_k]_{T_k} \in Q_{T_k}$), and $\{\epsilon_k\}_{k=1}^\infty$ is any sequence of positive numbers. Then there exists a dense set $\mathcal{D} \subset \mathcal{H}$ such that for every x in $\mathcal{D}$, there exists a sequence $\{y_k^x\}_{k=1}^\infty \subset \mathcal{H}$ satisfying*

$$[x \otimes y_k^x]_{T_k} = [L_k]_{T_k}, \quad k \in \mathbb{N}, \tag{8}$$

and

$$\|y_k^x\| > \epsilon_k, \quad k \in \mathbb{N}. \tag{9}$$

3. Main Results

From the idea of Lemma 2.5, we construct a new classes as following.

DEFINITION 3.1. Let m, n and l be any cardinal numbers such that $1 \le m, n, l \le \aleph_0$. We denote by $\mathbb{A}_{m,n}^l(\mathcal{H})$ the class of all sets $\{T_k\}_{k=1}^l$ such that T_k belongs to $\mathbb{A}(\mathcal{H})$ for all $k = 1, \cdots, l$ and that every $m\times n\times l$ system of simultaneous equations of the form

$$[x_i \otimes y_j^{(k)}]_{T_k} = [L_{ij}^{(k)}]_{T_k}, \tag{10}$$

where $\{[L_{ij}^{(k)}]_{T_k}\}_{\substack{0\le i<m\\0\le j<n}}$ is an arbitrary $m\times n$ array from Q_{T_k} for each $k=1,\cdots,l$, has a solution consisting of a pair of sequences $\{x_i\}_{0\le i<m},\{y_j^{(k)}\}_{\substack{0\le j<n\\1\le k\le l}}$ of vectors from $\mathcal{H}$.

LEMMA 3.2. *Suppose T_1 and T_2 are absolutely continuous contraction in $\mathcal{L}(\mathcal{K})$, m and n are nonzero cardinal number less than or equal to $\aleph_0$, and $\{[L_{ij}^{(k)}]_{T_k}\}_{\substack{0\le i<m\\0\le j<n}}$ is a doubly indexed sequence from Q_{T_k} for each $k, k=1,2$. Then sequence $\{x_i\}_{0\le i<m}$ and $\{y_j^{(k)}\}_{\substack{0\le j<n\\1\le k\le 2}}$ in $\mathcal{K}$ solve the system*

$$[x_i\otimes y_j^{(k)}]_{T_k}=[L_{ij}^{(k)}]_{T_k},\quad 0\le i<m,0\le j<n,1\le k\le 2, \tag{11}$$

if and only if

$$\begin{aligned}&C_{-w}(\phi_{T_k}([L_{ij}^{(k)}]_{T_k}))=(T^wx_i,y_j^{(k)}),\\&0\le i<m,0\le j<n,1\le k\le 2,w=0,1,2,\cdots,\end{aligned} \tag{12}$$

where ϕ_{T_k} is as in theorem 2.1(e).

Proof. Since the polynomials $p(T_k)$ are weak*-dense in $\mathcal{A}_{T_k}$, $k=1,2$, vectors x and $y^{(k)}$ in $\mathcal{K}$ solve $[x\otimes y^{(k)}]_{T_k}=[L^{(k)}]_{T_k}$ if and only if

$$\langle T_k^w,[L^{(k)}]_{T_k}\rangle=\langle T_k^w,[x\otimes y^{(k)}]_{T_k}\rangle,\quad k=1,2,w=0,1,2,\cdots. \tag{13}$$

But the right-hand side of (13) equals $(T_k^wx,y^{(k)})$, and for the left-hand side we have

$$\begin{aligned}\langle T_k^w,[L^{(k)}]_{T_k}\rangle&=\langle\Phi_{T_k}(e^{iwt}),[L^{(k)}]_{T_k}\rangle\\&=\langle e^{iwt},\phi_{T_k}([L^{(k)}]_{T_k})\rangle\\&=C_{-w}(\phi_{T_k}([L^{(k)}]_{T_k})),\quad k=1,2.\end{aligned}$$

So the proof is complete. □ □

Now we are ready to prove main results. The idea of the following theorem comes from [3,Proposition 4.11].

THEOREM 3.3. *If T_1 and T_2 are absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$, and m and n are cardinal numbers such that $1 \leq m, n \leq \aleph_0$, then the set $\{T_1, T_2\} \in \mathbb{A}^2_{m,n}$ if and only if the set made by compressing $T_k{}'s$, $k = 1, 2$, to some infinite dimensional common semi-invariant subspace $\mathcal{P}$ belongs to $\mathbb{A}^2_{m,n}(\mathcal{P})$.*

Proof. If a set $\{T_1, T_2\} \in \mathbb{A}^2_{m,n}$, we may take $\mathcal{P} = \mathcal{H} \ominus (0)$. Thus it suffices to show that if $\mathcal{P} = \mathcal{M} \ominus \mathcal{N}$, where $\mathcal{M}, \mathcal{N} \in \mathrm{Lat}(T_k)$, $k = 1, 2$, with $\mathcal{N} \subset \mathcal{M}$, and if the set $\{(T_1)_{\mathcal{P}}, (T_2)_{\mathcal{P}}\} \in \mathbb{A}^2_{m,n}(\mathcal{P})$, then $\{T_1, T_2\} \in \mathbb{A}^2_{m,n}(\mathcal{H})$. Since for any function f in $\mathbb{H}^\infty$ we have

$$\|f\|_\infty \geq \|f(T_k)\| \geq \|P_{\mathcal{P}} f(T_k)|_{\mathcal{P}}\| = \|f(\tilde{T}_k)\| = \|f\|_\infty, k = 1, 2,$$

where $\tilde{T}_k = (T_k)_{\mathcal{P}}$, it follows easily that $T_k \in \mathbb{A}$, $k = 1, 2$. Now suppose that (11) is a given system of equations, where the $[L^{(k)}_{ij}]_{T_k}$ are arbitrary elements of $Q_{T_k}, k = 1, 2$. For each pair i, j choose $[\tilde{L}^{(k)}_{ij}]_{\tilde{T}_k}$ in $Q_{\tilde{T}_k}$ such that

$$\phi_{T_k}([L^{(k)}_{ij}]_{T_k}) = \phi_{\tilde{T}_k}([\tilde{L}^{(k)}_{ij}]_{T_k}), k = 1, 2. \tag{14}$$

Since the set $\{\tilde{T}_1, \tilde{T}_2\} \in \mathbb{A}^2_{m,n}(\mathcal{P})$, by Lemma 3.2 there exist sequences $\{\tilde{x}_i\}_{0 \leq i < m}$ and $\{\tilde{y}^{(k)}_j\}_{\substack{0 \leq j < n \\ 1 \leq k \leq 2}}$ in $\mathcal{P}$ that solve the system

$$\begin{aligned} &C_{-w}(\phi_{\tilde{T}_k}([\tilde{L}^{(k)}_{ij}]_{T_k})) = (\tilde{T}^w_k \tilde{x}_i, \tilde{y}^{(k)}_j) \\ &0 \leq i < m, 0 \leq j < n, 1 \leq k \leq 2, w = 0, 1, 2, \cdots . \end{aligned} \tag{15}$$

We write $\mathcal{H} = \mathcal{N} \oplus \mathcal{P} \oplus \mathcal{M}^\perp$, and define the vectors $x_i, y^{(k)}_j$ in $\mathcal{H}$ by

$$\begin{aligned} &x_i = 0 \oplus \tilde{x}_i \oplus 0, y^{(k)}_j = 0 \oplus \tilde{y}^{(k)}_j \oplus 0, \\ &0 \leq i < m, 0 \leq j < n, k = 1, 2. \end{aligned}$$

Then, since $\mathcal{P}$ is common semi-invariant for $T_k, k = 1, 2$, we deduce easily from (14) and (15) that

$$\begin{aligned} &C_{-w}(\phi_{T_k}([L^{(k)}_{ij}]_{T_k})) = (T^w_k x_i, y^{(k)}_j) \\ &0 \leq i < m, 0 \leq j < n, 1 \leq k \leq 2, w = 0, 1, 2, \cdots . \end{aligned}$$

and the result follows from Lemma 3.2. □ □

Finally, let $\mathbb{N}$ be the set of all natural numbers. If $n \in \mathbb{N}$, we denoted by $\tilde{\mathcal{H}}^{(n)}$ the Hilbert space consisting of the direct sum of n copies of $\mathcal{H}$ and by $T^{(n)}$ the n-fold *ampliation* of T acting on $\tilde{\mathcal{H}}^{(n)}$ defined by

$$T^{(n)}(x_1 \oplus \cdots \oplus x_n) = Tx_1 \oplus \cdots \oplus Tx_n. \tag{16}$$

LEMMA 3.4. *Let T_1 and T_2 be absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$. Then the set $\{T_1, T_2\} \in \mathbb{A}^2_{1,\aleph_0}$ iff the set $\{T_1^{(n)}, T_2^{(n)}\} \in \mathbb{A}^2_{1,\aleph_0}$.*

Proof. Clear. □ □

THEOREM 3.5. *Suppose T_1 and T_2 are absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$. Then the set $\{T_1, T_2\} \in \mathbb{A}^2_{1,\aleph_0}(\mathcal{H})$ if and only if there exist a positive integer n and an infinite dimensional common semi-invariant subspace $\mathcal{P}$ for $T_k^{(n)}$, $k = 1, 2$, such that the set made by compressing $T_k^{(n)'}s$, $k = 1, 2$, to $\mathcal{P}$ belongs to $\mathbb{A}^2_{1,\aleph_0}(\mathcal{P})$.*

Proof. If the set $\{T_1, T_2\} \in \mathbb{A}^2_{1,\aleph_0}(\mathcal{H})$, we may take $n = 1$ and $\mathcal{P} = \mathcal{H}$. Conversely, since T_1 and T_2 are absolutely continuous contraction, so are $T_k^{(n)}$, $k = 1, 2$. Hence by Theorem 3.3, we obtain the set $\{T_1^{(n)}, T_2^{(n)}\} \in \mathbb{A}^2_{1,\aleph_0}$, and it follows from Lemma 3.4 that the set $\{T_1, T_2\} \in \mathbb{A}^2_{1,\aleph_0}$. □ □

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