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MAPPING TORUS AND THE ASYMPTOTIC EXPANSION OF $\log T(M,\varphi)(t)$

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Abstract. In this paper we define a torsion function $\log T(M,\varphi)(t)$ for $t \gg 0$ and show that it has an asymptotic expansion $\frac{1}{2}\chi(M)t$ as $t \to \infty$.

1. Introduction

Let (M, g) be a closed oriented Riemannian manifold of dimension n. Given an orientation-preserving diffeomorphism $\varphi : M \to M$, we define a mapping torus M_φ by $M_\varphi = M \times I \Big/ (x,1) \sim (\varphi(x), 0)$, where $I = [0,1]$. Then M_φ is a fiber bundle over S^1 and each fiber bundle over S^1 can be obtained in this way.

Let $\pi : M_\varphi \to S^1$ be the natural projection and denote by $d\theta$ the 1-form on S^1 with $\int_{S^1} d\theta = 1$. Choose a Riemannian metric g_1 on M_φ. We define for $t > 0$

$$d_q(t) : \Omega^q(M_\varphi) \to \Omega^{q+1}(M_\varphi)$$
$$d_q(t) = d_q + t\pi^* d\theta \wedge,$$

where $\Omega^q(M_\varphi)$ is the set of smooth q-forms on M_φ and d_q is the exterior differential operator. Since $d_q(t) d_{q-1}(t) = 0$, we can define the cohomology associated to $d_q(t)$ by

$$H^q(M_\varphi, d_q(t), \mathbb{R}) = ker d_q(t) \Big/ Im d_{q-1}(t).$$

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We also define the Laplacian $\Delta_q(t)$ associated to $d_q(t)$ by $\Delta_q(t) = d_q(t)^* d_q(t) + d_{q-1}(t) d_{q-1}(t)^*$, where $d_q(t)^*$ is the adjoint of $d_q(t)$ with respect to the given metric g_1 on M_φ. Then $\Delta_q(t) = \Delta_q + tA + t^2 \|\pi^* d\theta\|^2$, where A is a zero order operator and Δ_q is the usual Laplacian acting on $\Omega^q(M_\varphi)$. It is a known fact (cf. [CFKS]) that $\Delta_q(t)$ does not have a zero eigenvalue for sufficiently large $t > 0$ and hence $\Delta_q(t)$ is a positive definite elliptic differential operator for t large enough. By Hodge theorem

$$H^q(M_\varphi, d_q(t), \mathbb{R}) = ker \Delta_q(t) = 0.$$

We define the torsion function $T_0(M, \varphi, g_1)(t)$ for $t \gg 0$ by

$$T_0(M, \varphi, g_1)(t) = \frac{1}{2} \sum_{q=0}^{n+1} (-1)^{q+1} \cdot q \cdot logDet(\Delta_q(t)).$$

Since $H^q(M_\varphi, d_q(t), \mathbb{R}) = 0$ for $t \gg 0$, $T_0(M, \varphi, g_1)(t)$ does not depend on the choice of a Riemannian metric g_1 on M_φ (cf. [RS]). Hence we can write $T_0(M, \varphi)(t)$ rather than $T_0(M, \varphi, g_1)(t)$. In this paper we are going to prove the following theorem.

THEOREM 1. *Define* $T(M, \varphi)(t) = \frac{1}{2}(T_0(M, \varphi)(t) + T_0(M, \varphi^{-1})(t))$. *Then the followings hold.*

(1) *If dimM is odd, then* $T_0(M, \varphi)(t) = -T_0(M, \varphi^{-1})(t)$ *so that* $T(M, \varphi)(t) \equiv 0$.
(2) *If dimM is even,* $T(M, \varphi)(t) = T_0(M, \varphi)(t) = T_0(M, \varphi^{-1})(t)$.
(3) $T(M, \varphi)(t) \sim \frac{1}{2}\chi(M) t$ *as* $t \to \infty$.
(4) *If* $\varphi = Id$, $T(M, \varphi)(t) = \frac{1}{2}\chi(M)(t + 2log(1 - e^{-t}))$.

2. The case of $M \times S^1$

If $\varphi = Id$, then $M_\varphi = M \times S^1$ and we can choose the product metric $g \oplus d\theta^2$ on $M \times S^1$. Consider $\pi : M \times S^1 \to S^1$ and $d(t) = d + \frac{t}{2\pi} d\theta$, where $d\theta$ is the canonical 1-form on S^1, *i.e.* $\int_{S^1} d\theta = 2\pi$.

Then one can show that $\Delta_q(t) = \Delta_q^{M \times S^1} + \frac{t^2}{4\pi^2} Id$, where $\Delta_q^{M \times S^1}$ is the usual Laplacian acting on $\Omega^q(M \times S^1)$. Set $\lambda = \frac{t^2}{4\pi^2}$ and note that

$$\Omega^q(M \times S^1) = C^\infty(M \times S^1) \Omega^q(M) \otimes$$

$$\Omega^0(S^1) \oplus C^\infty(M \times S^1)\Omega^{q-1}(M) \otimes \Omega^1(S^1).$$

Then

$$\Delta_q(t) = \Delta_q^{M\times S^1} + \lambda Id =$$

$$\begin{pmatrix} \Delta_q^M \otimes Id_{S^1} + Id_M \otimes \Delta_0^{S^1} + \lambda Id & 0 \\ 0 & \Delta_{q-1}^M \otimes Id_{S^1} + Id_M \otimes \Delta_1^{S^1} + \lambda Id \end{pmatrix}.$$

$$tre^{-t\Delta_q(t)}$$

$$= tr\left(e^{-t(\Delta_q^M \otimes Id_{S^1} + Id_M \otimes \Delta_0^{S^1} + \lambda Id)} + e^{-t(\Delta_{q-1}^M \otimes Id_{S^1} + Id_M \otimes \Delta_1^{S^1} + \lambda Id)}\right)$$

$$= e^{-\lambda t}\left(tre^{-t\Delta_q^M} \otimes e^{-t\Delta_0^{S^1}} + tre^{-t\Delta_{q-1}^M} \otimes e^{-t\Delta_1^{S^1}}\right)$$

$$= e^{-\lambda t}\left(tre^{-t\Delta_q^M} \cdot tre^{-t\Delta_0^{S^1}} + tre^{-t\Delta_{q-1}^M} \cdot tre^{-t\Delta_1^{S^1}}\right)$$

$$= e^{-\lambda t} tre^{-t\Delta_0^{S^1}}\left(tre^{-t\Delta_q^M} + tre^{-t\Delta_{q-1}^M}\right),$$

since $\Delta_0^{S^1}$ and $\Delta_1^{S^1}$ are isospectral.

Now

$$\frac{1}{2}\sum_{q=0}^{n+1}(-1)^{q+1} \cdot q \cdot trexp\left(-t(\Delta_q^{M\times S^1} + \lambda Id)\right)$$

$$= \frac{1}{2}e^{-\lambda t} tre^{-t\Delta_0^{S^1}}\sum_{q=0}^{n+1}(-1)^{q+1} \cdot q \cdot (tre^{-t\Delta_q^M} + tre^{-t\Delta_{q-1}^M})$$

$$= \frac{1}{2}e^{-\lambda t} tre^{-t\Delta_0^{S^1}}\sum_{q=0}^{n}(-1)^q tre^{-t\Delta_q^M}.$$

Since $\sum_{q=0}^n (-1)^q tre^{-t\Delta_q^M}$ is equal to $\chi(M)$, the Euler characteristic of M (cf. [Gi]), we get

$$\frac{1}{2}\sum_{q=0}^{n+1}(-1)^{q+1} \cdot q \cdot trexp\left(-t(\Delta_q^{M\times S^1} + \lambda Id)\right) =$$

$$\frac{1}{2}\chi(M) trexp\left(-t(\Delta_0^{S^1} + \lambda Id)\right).$$

Define $Z_q(s) = \sum_\mu \mu^{-s}$, where μ runs over the eigenvalues of $\Delta_q^{M\times S^1} + \lambda Id$. Then $Z_q(s)$ is holomorphic for $Res > \frac{n+1}{2}$ and it has a meromorphic continuation to the whole complex plane with a regular value at 0 (cf. [Se]). Then

$$T(M\times S^1) = \frac{1}{2}\sum_{q=0}^{n+1}(-1)^{q+1}\cdot q\cdot logDet(\Delta_q^{M\times S^1} + \lambda Id)$$

$$= -\frac{1}{2}\sum_{q=0}^{n+1}(-1)^{q+1}\cdot q\cdot Z_q'(0).$$

By Melline transformation

$$\frac{1}{2}\sum_{q=0}^{n+1}(-1)^{q+1}\cdot q\cdot Z_q(s) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\frac{1}{2}\chi(M)trexp(-t(\Delta_0^{S^1} + \lambda Id))dt$$

$$= \frac{1}{2}\chi(M)\{\lambda^{-s} + 2\sum_{n=1}^{\infty}(\lambda+n^2)^{-s}\}.$$

$$T(M\times S^1) = -\frac{1}{2}\chi(M)\{-log\lambda + 2\frac{d}{dt}\mid_{s=0}\sum_{n=1}^{\infty}(\lambda+n^2)^{-s}\}.$$

From [Vo] we get

$$\frac{d}{dt}\mid_{s=0}\sum_{n=1}^{\infty}(\lambda+n^2)^{-s} = -log\left(e^{-2\zeta'(0)}\frac{sin(\pi\sqrt{\lambda}i)}{\pi\sqrt{\lambda}i}\right),$$

where $\zeta(s)$ is the Riemann zeta function. Since $\zeta'(0) = -log\sqrt{2\pi}$,

$$\frac{d}{dt}\mid_{s=0}\sum_{n=1}^{\infty}(\lambda+n^2)^{-s} = -\pi\sqrt{\lambda} - log(1-e^{-2\pi\sqrt{\lambda}}) + \frac{1}{2}log\lambda.$$

Therefore

$$T(M\times S^1) = \frac{1}{2}\chi(M)\{2\pi\sqrt{\lambda} + 2log(1-e^{-2\pi\sqrt{\lambda}})\}.$$

Setting $\lambda = \frac{t^2}{4\pi^2}$, we get

$$T(M\times S^1)(t) = \frac{1}{2}\chi(M)(t + 2log(1-e^{-t})).$$

3. The case of a general mapping torus

Let $\varphi : M \to M$ be an orientation preserving diffeomorphism of M and $d\theta$ be a 1-form on S^1 with $\int_{S^1} d\theta = 1$. Consider the fiber bundle $M \to M_\varphi \xrightarrow{\pi} S^1$ with $d(t) = d + t\pi^* d\theta$.

Let $\{U_k\}$ be an atlas of M_φ and $\{\rho_k\}$ be a partition of unity subordinate to $\{U_k\}$. Suppose that $\sigma(\mu - \Delta_q(t))^{-1} \sim \sum_{j=0}^{\infty} r_{-2-j}(\mu, t, x, \xi)$ on each U_k, where r_{-2-j} is the homogeneous component of the asymptotic symbol of $(\mu - \Delta_q(t))^{-1}$ on U_k. Set

$$J_j^q(s,x) = \frac{1}{2\pi i} \int_{\mathbb{R}^{n+1}} d\xi \int_\gamma \mu^{-s} r_{-2-j}(\mu, 1, x, \xi) d\mu,$$

where γ is a contour enclosing all the eigenvalues of $\Delta_q(t)$, *i.e.* for sufficiently small $\epsilon > 0$,

$$\gamma = \{ue^{i\pi} | \infty > u \geq \epsilon\} \cup \{\epsilon e^{i\psi} | \pi \geq \psi \geq -\pi\} \cup \{ue^{-i\pi} | \epsilon \leq u < \infty\}.$$

Set

$$\pi_j = \frac{1}{(2\pi)^{n+1}} \frac{d}{ds}|_{s=0} \sum_k \int_{M_\varphi} J_j^q(s,x)\rho_k(x) dvol(x),$$

and

$$q_j = \frac{1}{(2\pi)^{n+1}} \sum_k \int_{M_\varphi} J_j^q(0,x)\rho_k(x) dvol(x).$$

Then from the appendix of [BFK] we get the following theorem.

Theorem 2.

$$logDet(\Delta_q(t)) \sim \sum_{j=0}^{\infty} \pi_j t^{n+1-j} + \sum_{j=0}^{n+1} q_j t^{n+1-j} logt$$

as $t \to +\infty$.

Let us consider $M \times S^1$ with the product metric $g \oplus d\theta^2$, where g is a Riemannian metric on M and $d\theta^2$ is the normalized canonical metric on S^1 with $\int_{S^1} d\theta = 1$. Let $\{U_k\}$ be an atlas of M and $\{\rho_k\}$ be a partition of unity subordinate to $\{U_k\}$. Then $\Delta_q(t) = \Delta_q^{M\times S^1} + t^2 Id$ and from Theorem 2 and the statement (4) of Theorem 1 we get

$$\frac{1}{2}\sum_{q=0}^{n+1}(-1)^{q+1}\cdot q\cdot logDet(\Delta_q(t)) \sim \sum_{j=0}^{\infty} c_j t^{n+1-j} + \sum_{j=0}^{n+1} d_j t^{n+1-j} logt$$

$$= \frac{1}{2}\chi(M)t$$

as $t \to \infty$ for some constants c_j's and d_j's. Hence each $d_j = 0$ and

$$c_j = \frac{1}{2}\sum_{q=0}^{n+1}(-1)^{q+1}\cdot q\cdot \frac{1}{(2\pi)^{n+1}}\frac{d}{ds}\mid_{s=0}$$

$$\sum_k \int_{M\times S^1} J_j^q(s,x,\theta)\rho_k(x)dvol(M\times S^1)$$

$$= \frac{1}{2}\sum_{q=0}^{n+1}(-1)^{q+1}\cdot q\cdot \frac{1}{(2\pi)^{n+1}}\frac{d}{ds}\mid_{s=0}\sum_k \int_M J_j^q(s,x)\rho_k(x)dvol(M)$$

$$= \frac{1}{2}\chi(M)\delta_{nj},$$

since J_j^q does not depend on S^1-variable θ.

Now let us denote $S^1 = [0,1]/0 \sim 1$ and let $V_1 = (\frac{1}{5}, \frac{2}{5})$, $V_2 = (\frac{3}{5}, \frac{4}{5})$, $V_3 = [0, \frac{1}{5}+\epsilon) \cup (\frac{4}{5}-\epsilon, 1]$, $V_4 = (\frac{2}{5}-\epsilon, \frac{3}{5}+\epsilon)$ for sufficiently small $\epsilon > 0$. Let $\{\eta_k\}_{1\le k\le 4}$ be a partition of unity subordinate to $\{V_k\}_{1\le k\le 4}$. We denote by g_1, g_2 Riemannian metrics on M. Choose a nondecreasing function $\omega(r)$ on $\mathbb{R}$ such that $\omega(r) = 0$ for $r \le 0$, 1 for $r \ge 1$ and $\omega(r)$ is symmetric to the line $r = \frac{1}{2}$.

Set $\omega_1(r) = \omega(5r-1)$ and $\omega_2(r) = \omega(5r-3)$. We define a new metric $G(r,\theta)$ on $M \times S^1$ as follows.

$$G(r,\theta) = \begin{cases} g_1 \oplus d\theta^2, & \text{for } 0 \le \theta \le \frac{1}{5} \\ ((1-\omega_1(\theta))g_1 + \omega_1(\theta)g_2) \oplus d\theta^2, & \text{for } \frac{1}{5} \le \theta \le \frac{2}{5} \\ g_2 \oplus d\theta^2, & \text{for } \frac{2}{5} \le \theta \le \frac{3}{5} \\ ((1-\omega_2(\theta))g_2 + \omega_2(\theta)g_1) \oplus d\theta^2, & \text{for } \frac{3}{5} \le \theta \le \frac{4}{5} \\ g_1 \oplus d\theta^2, & \text{for } \frac{4}{5} \le \theta \le 1. \end{cases}$$

Then

$$c_j = \frac{1}{2}\sum_{q=0}^{n+1}(-1)^{q+1}\cdot q\cdot\sum_{l,k}\frac{d}{ds}\mid_{s=0}\frac{1}{(2\pi)^{n+1}}$$

$$\int_{M\times S^1}\rho_l(x)\eta_k(\theta)J_j^q(s,x,\theta)dvol(M\times S^1)$$

$$=\frac{1}{2}\chi(M)\delta_{nj}.$$

Note that $J_j(s,x,\theta)$ coming from the product metric of the form $g\oplus d\theta^2$ does not depend on the S^1-variable θ.

$$c_j = \frac{1}{2}\sum_{q=0}^{n+1}(-1)^{q+1}\cdot q\cdot\sum_{l}\sum_{k\neq 1,2}\frac{d}{ds}\mid_{s=0}\frac{1}{(2\pi)^{n+1}}$$

$$\int_{M\times S^1}\rho_l(x)\eta_k(\theta)J_j^q(s,x,\theta)dvol(M\times S^1)$$

$$+\frac{1}{2}\sum_{q=0}^{n+1}(-1)^{q+1}\cdot q\cdot\sum_{l}\frac{d}{ds}\mid_{s=0}\frac{1}{(2\pi)^{n+1}}\times$$

$$\Big(\int_{M\times S^1}\rho_l(x)\eta_1(\theta)J_j^q dvol(M\times S^1)+$$

$$\int_{M\times S^1}\rho_l(x)\eta_2(\theta)J_j^q dvol(M\times S^1)\Big)$$

$$=\left(\sum_{\substack{k\\ k\neq 1,2}}\int_{S^1}\eta_k(\theta)d\theta\right)\cdot\frac{1}{2}\cdot\sum_{q=0}^{n+1}(-1)^{q+1}\cdot q\cdot\sum_{l}\frac{d}{ds}\mid_{s=0}\frac{1}{(2\pi)^{n+1}}$$

$$\int_M\rho_l(x)J_j^q dvol(M)+C(g_1,g_2)+C(g_2,g_1)$$

$$=\frac{1}{2}\chi(M)\delta_{nj}\left(\sum_{\substack{k\\ k\neq 1,2}}\int_{S^1}\eta_k(\theta)d\theta\right)+C(g_1,g_2)+C(g_2,g_1).$$

Here

$$C(g_1,g_2)=\frac{1}{2}\sum_{q=0}^{n+1}(-1)^{q+1}\cdot q\cdot\sum_{l}\frac{d}{ds}\mid_{s=0}\frac{1}{(2\pi)^{n+1}}$$
$$\int_{M\times S^1}\rho_l(x)\eta_1(\theta)J_j^q dvol(M\times S^1),$$
$$C(g_2,g_1)=\frac{1}{2}\sum_{q=0}^{n+1}(-1)^{q+1}\cdot q\cdot\sum_{l}\frac{d}{ds}\mid_{s=0}\frac{1}{(2\pi)^{n+1}}$$
$$\int_{M\times S^1}\rho_l(x)\eta_2(\theta)J_j^q dvol(M\times S^1).$$

Hence

$$\frac{1}{2}\chi(M)\delta_{nj}=\frac{1}{2}\chi(M)\delta_{nj}\left(\sum_{\substack{k\\k\neq 1,2}}\int_{S^1}\eta_k(\theta)d\theta\right)+C(g_1,g_2)+C(g_2,g_1).$$

Since $\sum_{k=1}^4\int_{S^1}\eta_k(\theta)d\theta=1$,

$$(1)\quad C(g_1,g_2)+C(g_2,g_1)=\frac{1}{2}\chi(M)\delta_{nj}\left(\int_{S^1}\eta_1(\theta)d\theta+\int_{S^1}\eta_2(\theta)d\theta\right).$$

Now we consider a general mapping torus. Let (M,g) be an oriented Riemannian manifold and $\varphi: M\to M$ be an orientation preserving diffeomorphism. Then φ is an isometry from (M,φ^*g) to (M,g). Note that

$$M_{\varphi^{-1}}=M\times I/(x,1)\sim(\varphi^{-1}(x),0)=M\times I/(x,0)\sim(\varphi(x),1).$$

Define $\Phi: M_\varphi\to M_{\varphi^{-1}}$ by $[x,t]\mapsto[x,1-t]$. We give metrics $G_1(x,\theta)$ and $G_2(x,\theta)$ on M_φ and $M_{\varphi^{-1}}$ respectively as follows.

$$G_1(x,\theta)=\begin{cases}\varphi^*g\oplus d\theta^2, & \text{for } 0\le\theta\le\frac{1}{5}\\ ((1-\omega_1(\theta))\varphi^*g+\omega_1(\theta)g)\oplus d\theta^2, & \text{for } \frac{1}{5}\le\theta\le\frac{2}{5}\\ g\oplus d\theta^2, & \text{for } \frac{2}{5}\le\theta\le 1.\end{cases}$$
$$G_2(x,\theta)=\begin{cases}g\oplus d\theta^2, & \text{for } 0\le\theta\le\frac{3}{5}\\ ((1-\omega_2(\theta))g+\omega_2(\theta)\varphi^*g)\oplus d\theta^2, & \text{for } \frac{3}{5}\le\theta\le\frac{4}{5}\\ \varphi^*g\oplus d\theta^2, & \text{for } \frac{4}{5}\le\theta\le 1.\end{cases}$$

Then Φ is an (orientation-reversing) isometry from (M_φ,G_1) to $(M_{\varphi^{-1}},G_2)$.

LEMMA 3. *$T_0(M,\varphi)(t) = (-1)^n T_0(M,\varphi^{-1})(t)$ for $t \gg 0$, where n is the dimension of M.*

Proof. Denote by $\Delta_q(t)$, $\tilde{\Delta}_q(t)$ the Laplacians on (M_φ, G_1) and $(M_{\varphi^{-1}}, G_2)$ respectively. By Hodge theorem

$$\Omega^q(M_\varphi) = Imd_{q-1}(t) \oplus Imd_q(t)^* = \Omega^q_+(M_\varphi) \oplus \Omega^q_-(M_\varphi),$$

where $\Omega^q_+(M_\varphi) = Imd_{q-1}(t)$ and $\Omega^q_-(M_\varphi) = Imd_q(t)^*$. Let $\Delta^\pm_q(t)$ be the Laplacians acting on $\Omega^q_\pm(M_\varphi)$ respectively. Then from the fact that

$$\begin{aligned} logDet(\Delta_q(t)) &= logDet(\Delta^+_q(t)) + logDet(\Delta^-_q(t)) \\ &= logDet(\Delta^+_q(t)) + logDet(\Delta^+_{q+1}(t)) \\ &= logDet(\Delta^-_{q-1}(t)) + logDet(\Delta^-_q(t)), \end{aligned}$$

we get

$$T_0(M,\varphi)(t) = \frac{1}{2}\sum_{q=0}^{n+1}(-1)^q logDet(\Delta^-_q(t))$$

$$= -\frac{1}{2}\sum_{q=0}^{n+1}(-1)^q logDet(\Delta^+_q(t)). \tag{2}$$

If we denote by $*$ the Hodge operator on M_φ, then one can check that

$$\begin{aligned} \Delta^+_q(t)* &= (d(t) + t\pi^* d\theta)(d(t)^* + t(\pi^* d\theta)^*)* \\ &= *(d(t)^* - t(\pi^* d\theta)^*)(d(t) - t(\pi^* d\theta)) \\ &= *\Delta^-_{n+1-q}(-t). \end{aligned}$$

Hence $\Delta^+_q(t)$ and $\Delta^-_{n+1-q}(-t)$ are isospectral. From $\Phi : M_\varphi \to M_{\varphi^{-1}}$ defined by $[x,t] \mapsto [x, 1-t]$, one can check that $\Delta^\pm_q(-t) \circ \Phi^* = \Phi^* \circ \tilde{\Delta}^\pm_q(t)$ and $\Delta^\pm_q(-t)$ and $\tilde{\Delta}^\pm_q(t)$ are isospectral. Hence $\Delta^+_q(t)$ and $\tilde{\Delta}^-_{n+1-q}(t)$ are also isospectral. From the equation (2),

$$T_0(M,\varphi)(t) = -\frac{1}{2}\sum_{q=0}^{n+1}(-1)^q logDet(\Delta^+_q(t))$$

$$= -\frac{1}{2}\sum_{q=0}^{n+1}(-1)^{q} logDet(\tilde{\Delta}^{-}_{n+1-q}(t))$$

$$= (-1)^{n}\frac{1}{2}\sum_{q=0}^{n+1}(-1)^{q} logDet(\tilde{\Delta}^{-}_{q}(t)) = (-1)^{n} T_{0}(M,\varphi^{-1})(t).$$

Hence the statements (1) and (2) of the Theorem 1 are proved. □ □

From now on we assume that the dimension of M is even. Suppose that

$$T(M,\varphi)(t) = \frac{1}{2}\sum_{q=0}^{n+1}(-1)^{q+1}\cdot q\cdot logDet(\Delta_{q}(t)) \sim$$

$$\sum_{j=0}^{\infty} c_{j} t^{n+1-j} + \sum_{j=0}^{n+1} d_{j} t^{n+1-j} logt$$

as $t \to \infty$. Then

$$c_{j} =$$

$$\frac{1}{2}\sum_{j=0}^{n+1}(-1)^{q+1}\cdot q\cdot \frac{d}{ds}\mid_{s=0}\sum_{l,k}\int_{M_{\varphi}} \rho_{l}(x)\eta_{k}(\theta) J^{q}_{j}(s,x,\theta) dvol(M\times S^{1})$$

$$= \frac{1}{2}\sum_{q=0}^{n+1}(-1)^{q+1}\cdot q\cdot \frac{d}{ds}\mid_{s=0}\sum_{l}$$

$$\left(\sum_{k\neq 1}\int_{V_{k}}\eta_{k}\int_{U_{l}}\rho_{l} J^{q}_{j} dvol(M) dvol(S^{1}) + \right.$$

$$\left.\int_{V_{1}}\eta_{1}\int_{U_{l}}\rho_{l} J^{q}_{j} dvol(M) dvol(S^{1})\right).$$

If $k \neq 1$, on $V_{k}\times S^{1}$ $J^{q}_{j}(s,x,\theta)$ comes from the product metric and so it does not depend on θ. Hence

$$c_{j} = \frac{1}{2}\chi(M)\delta_{nj}\left(\int_{S^{1}}\sum_{k\neq 1}\eta_{k}(\theta) d\theta\right) + C(\varphi^{*}g, g).$$

From the same argument on $M_{\varphi^{-1}}$,

$$c_j = \frac{1}{2}\chi(M)\delta_{nj}\left(\int_{S^1}\sum_{k\neq 2}\eta_k(\theta)d\theta\right) + C(g,\varphi^* g).$$

From (1), we know that

$$C(\varphi^* g, g) + C(g,\varphi^* g) = \frac{1}{2}\chi(M)\delta_{nj}\left(\int_{S^1}\eta_1(\theta)d\theta + \int_{S^1}\eta_2(\theta)d\theta\right).$$

Therefore

$$c_j = \frac{1}{2}\chi(M)\delta_{nj}.$$

We can use the same argument to show that $d_j = 0$.

Remark. This is a weak result of J. Marcsik (cf. [Ma]) but the method is more elementary. In fact, he proved that on a general orientable mapping torus M_φ, $T(M,\varphi)(t) = \frac{1}{2}\chi(M)t + \sum_{n=1}^{\infty}\frac{L(\varphi^n)e^{-nt}}{n}$ for $t \gg 0$, where $L(\varphi^n)$ is the Lefschetz number of φ^n.

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