SOME PROPERTIES OF FUZZY QUASI-PROXIMITY SPACES

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Abstract. We will define the fuzzy quasi-proximity space and investigate some properties of fuzzy quasi-proximity spaces. We will prove the existences of initial fuzzy quasi-proximity structures. From this fact, we can define subspaces and products of fuzzy quasi-proximity spaces.

1. Introduction and preliminaries

In [8, 9, 10, 11], S.K. Samanta introduced the concept of gradations of openness and proximity. M.H. Ghanim et al.[5] introduced fuzzy proximity spaces with somewhat different definition of S.K. Samanta [8].

In this paper, we will define the fuzzy quasi-proximity space in view of the definition of M.H. Ghanim et al.[5]. We will investigate some properties of fuzzy quasi-proximity spaces. We will study relationships between fuzzy quasi-proximity spaces and fuzzy topological spaces.

In particular, we will prove the existence of initial fuzzy quasi-proximity structures. From this fact, we can define subspaces and products of fuzzy quasi-proximity spaces.

In this paper, all the notations and the definitions are standard in fuzzy set theory.

Definition 1.1. [11] Let $X$ be a nonempty set. A function $T : I^X \to I$ is called a gradation of openness on $X$ if it satisfies the following conditions:

1. $T(\emptyset) = T(\bar{1}) = 1,$
(c2) \( T(\mu_1 \land \mu_2) \geq T(\mu_1) \land T(\mu_2) \),
(c3) \( T(\bigvee_{i \in \Delta} \mu_i) \geq \bigwedge_{i \in \Delta} T(\mu_i) \).
The pair \((X, T)\) is called a fuzzy topological space.

Let \( T \) be a gradation of openness on \( X \) and \( F : \mathcal{I}^X \rightarrow \mathcal{I} \) be defined by \( F(\mu) = T(\mu^c) \). Then \( F \) is called a gradation of closedness on \( X \).

Let \((X, T)\) be a fuzzy topological space, then for each \( r \in \mathcal{I} \), \( T_r = \{ \mu \in \mathcal{I}^X : T(\mu) \geq r \} \) is a Chang’s fuzzy topology on \( X \).

Let \((X, T)\) and \((Y, T^*)\) be fuzzy topological spaces. A function \( f : (X, T) \rightarrow (Y, T^*) \) is called a gradation preserving map ( gp-map) if \( T^*(\mu) \leq T(f^{-1}(\mu)) \) for all \( \mu \in \mathcal{I}^X \).

A function \( \delta : \mathcal{I}^X \times \mathcal{I}^X \rightarrow \mathcal{I} \) is a fuzzy proximity on \( X \) [8] if it satisfies the following conditions:

\( \text{(SFP1)} \) \( \delta(\tilde{0}, \tilde{1}) = 0 \).
\( \text{(SFP2)} \) \( \delta(\lambda, \mu) = \delta(\mu, \lambda) \).
\( \text{(SFP3)} \) \( \delta(\lambda_1 \lor \lambda_2, \mu) = \delta(\lambda_1, \mu) \lor \delta(\lambda_2, \mu) \).
\( \text{(SFP4)} \) If \( \delta(\lambda, \mu) < 1 - r \), then \( \delta(\text{cl}(\lambda, r), \mu) < 1 - r \), where

\[
\text{cl}(\lambda, r) = \tilde{1} - \bigvee \{ \rho \leq \lambda^c | \delta(\lambda, \rho) < 1 - r \}.
\]

The pair \((X, \delta)\) is called a Samanta type fuzzy proximity space.

We can easily prove the following lemma.

**Lemma 1.2.** If \( f : X \rightarrow Y \), then we have the following properties for direct and inverse image of fuzzy sets under mappings:

1. \( \mu \geq f(f^{-1}(\mu)) \) with equality if \( f \) is surjective,
2. \( \nu \leq f^{-1}(f(\nu)) \) with equality if \( f \) is injective,
3. \( f^{-1}(\mu^c) = f^{-1}(\mu)^c \),
4. \( f^{-1}(\bigvee_{i \in I} \mu_i) = \bigvee_{i \in I} f^{-1}(\mu_i) \),
5. \( f^{-1}(\bigwedge_{i \in I} \mu_i) = \bigwedge_{i \in I} f^{-1}(\mu_i) \),
6. \( f(\bigvee_{i \in I} \nu_i) = \bigvee_{i \in I} f(\nu_i) \),
7. \( f(\bigwedge_{i \in I} \mu_i) \leq \bigwedge_{i \in I} f(\mu_i) \) with equality if \( f \) is injective.

**2. Fuzzy quasi-proximity and fuzzy topological spaces**

From the definition of M.H. Ghanim [5], we can define a fuzzy quasi-proximity.
DEFINITION 2.1. A function $\delta : I^X \times I^X \to I$ is said to be a fuzzy quasi-proximity on $X$ which satisfies the following conditions:

(FQP1) $\delta(\tilde{0}, \tilde{1}) = 0$ and $\delta(\tilde{1}, \tilde{0}) = 0$.

(FQP2) $\delta(\lambda \vee \rho, \mu) = \delta(\lambda, \mu) \vee \delta(\rho, \mu)$ and $\delta(\lambda, \mu \vee \nu) = \delta(\lambda, \mu) \vee \delta(\lambda, \nu)$.

(FQP3) If $\delta(\lambda, \mu) < r$, then there exists $\rho \in I^X$ such that $\delta(\lambda, \rho) < r$ and $\delta(\tilde{1} - \rho, \mu) < r$.

(FQP4) If $\delta(\lambda, \mu) \neq 1$, then $\lambda \leq \tilde{1} - \mu$.

The pair $(X, \delta)$ is called a fuzzy quasi-proximity space.

A fuzzy quasi-proximity space $(X, \delta)$ is called a fuzzy proximity space if (FP) $\delta(\lambda, \mu) = \delta(\mu, \lambda)$ for any $\lambda, \mu \in I^X$.

Let $(X, \delta_1), (X, \delta_2)$ be given. We say $\delta_2$ is finer than $\delta_1$ ( $\delta_1$ is coarser than $\delta_2$), denoted by $\delta_1 \prec \delta_2$, iff for any $\lambda, \mu \in I^X$, $\delta_2(\lambda, \mu) \leq \delta_1(\lambda, \mu)$.

REMARK 1. (1) If $(X, \delta)$ is a fuzzy quasi-proximity space and $\lambda \leq \mu$, then, by (FQP2), we have $\delta(\lambda, \nu) \leq \delta(\mu, \nu)$ and $\delta(\lambda, \rho) \leq \delta(\mu, \rho)$.

(2) Let $(X, \delta)$ be a fuzzy quasi-proximity space. For each $\lambda, \mu \in I^X$, we define $\delta^{-1}(\lambda, \mu) = \delta(\mu, \lambda)$. Then the structure $\delta^{-1}$ is a fuzzy quasi-proximity on $X$.

(3) Every fuzzy proximity space in our sense is a Samanta type fuzzy proximity space. If $\delta(\lambda, \mu) < 1 - r$, by (FQP3), then there exists $\rho \in I^X$ such that $\delta(\lambda, \rho) < 1 - r$ and $\delta(\tilde{1} - \rho, \mu) < 1 - r$. Since $\delta(\lambda, \rho) < 1 - r$, by (FQP4) and the definition of $\text{cl}(\lambda, r)$, we have $\rho \leq \tilde{1} - \lambda$ and $\text{cl}(\lambda, r) \leq \tilde{1} - \rho$. So, by (FQP2), $\delta(\text{cl}(\lambda, r), \mu) < 1 - r$. Thus (SFP4) holds.

THEOREM 2.2. [5] Let $(X, \delta)$ be a fuzzy quasi-proximity space, then for each $r \in (0, 1]$, the family $\delta_r = \{ (\lambda, \mu) \in I^X \times I^X \mid \delta(\lambda, \mu) \geq r \}$ is a (classical) fuzzy quasi-proximity space on $X$.

THEOREM 2.3. [5] Let $\delta$ be a fuzzy quasi-proximity on $X$. For each $r \in (0, 1], \lambda \in I^X$, we define

$$i_\delta(\lambda, r) = \bigvee \{ \rho \in I^X \mid \delta(\rho, \tilde{1} - \lambda) < r \}.$$

The family $\{ i_\delta(\lambda, r) \mid r \in (0, 1] \}$ satisfies the following properties:

(i) $i_\delta(\tilde{1}, r) = \tilde{1}$.

(ii) $i_\delta(\lambda, r) \leq \lambda$ and $i_\delta(\lambda_1, r) \leq i_\delta(\lambda_2, r)$, if $\lambda_1 \leq \lambda_2$. 
(iii) \(i_\delta(i_\delta(\lambda, r), r) = i_\delta(\lambda, r)\).
(iv) \(i_\delta(\lambda \wedge \mu, r) = i_\delta(\lambda, r) \wedge i_\delta(\mu, r)\).
(v) \(i_\delta(\lambda, r) \leq i_\delta(\lambda, r')\), if \(r \leq r'\), where \(r, r' \in (0, 1]\).

**Theorem 2.4.** Let \(\delta\) be a fuzzy quasi-proximity on \(X\). For each \(r \in (0, 1], \lambda \in I^X\), we define

\[
c_\delta(\lambda, r) = \bigwedge \{\rho^c \in I^X \mid \delta(\rho, \lambda) < r\}.
\]

The family \(\{c_\delta(\lambda, r) \mid r \in (0, 1]\}\) satisfies the following properties:

(i) \(c_\delta(\tilde{0}, r) = \tilde{0}, c_\delta(\tilde{1}, r) = \tilde{1}\).
(ii) \(c_\delta(\lambda, r) \geq \lambda\).
(iii) \(c_\delta(\lambda_1, r) \leq c_\delta(\lambda_2, r), \text{ if } \lambda_1 \leq \lambda_2\).
(iv) \(c_\delta(\lambda \vee \mu, r) = c_\delta(\lambda, r) \vee c_\delta(\mu, r)\).
(v) \(c_\delta(c_\delta(\lambda, r), r) = c_\delta(\lambda, r)\).
(vi) \(c_\delta(\lambda, r) \geq c_\delta(\lambda, r'), \text{ if } r \leq r', \text{ where } r, r' \in (0, 1]\).

**Proof.** (i),(iii) and (vi) are easily proved from the definition of \(c_\delta\).
(ii). Suppose that there exists \(\lambda \in I^X\) such that for some \(x_0 \in X\),
\(c_\delta(\lambda, r)(x_0) < \lambda(x_0)\). By the definition of \(c_\delta(\lambda, r)\), there exists \(\rho \in I^X\) such that \((\tilde{1} - \rho)(x_0) < \lambda(x_0)\) and \(\delta(\rho, \lambda) < r\).

On the other hand, since \((\tilde{1} - \rho)(x_0) < \lambda(x_0)\), by (FQP4), we have \(\delta(\rho, \lambda) = 1\). It is a contradiction.
(iv). By (iii), we have \(c_\delta(\lambda \vee \mu, r) \geq c_\delta(\lambda, r) \vee c_\delta(\mu, r)\).

We will show that \(c_\delta(\lambda \vee \mu, r) \leq c_\delta(\lambda, r) \vee c_\delta(\mu, r)\).

Suppose that there exist \(\lambda, \mu \in I^X\) such that for some \(x_0 \in X\),
\(c_\delta(\lambda \vee \mu, r)(x_0) > c_\delta(\lambda, r)(x_0) \vee c_\delta(\mu, r)(x_0)\).

There exists \(\rho_1, \rho_2 \in I^X\) such that \(\delta(\rho_1, \lambda) < r, \delta(\rho_2, \mu) < r\) and
\(c_\delta(\lambda \vee \mu, r)(x_0) > (\tilde{1} - \rho_1)(x_0) \vee (\tilde{1} - \rho_2)(x_0)\).

On the other hand, by Remark 1 and (FQP2), since
\(\delta(\rho_1 \wedge \rho_2, \lambda \vee \mu) \leq \delta(\rho_1, \lambda) \vee \delta(\rho_2, \mu) < r\)
we have $c_\delta(\lambda \lor \mu, r) \leq (\tilde{1} - \rho_1) \lor (\tilde{1} - \rho_2)$. It is a contradiction.

(v). By (ii), it suffices to show that $c_\delta(c_\delta(\lambda, r), r) \leq c_\delta(\lambda, r)$.

Suppose that there exists $\lambda \in I^X$ such that for some $x_0 \in X$, $c_\delta(c_\delta(\lambda, r), r)(x_0) > c_\delta(\lambda, r)(x_0)$. There exists $\rho \in I^X$ such that

$$c_\delta(c_\delta(\lambda, r), r)(x_0) > (\tilde{1} - \rho)(x_0), \quad \delta(\rho, \lambda) < r.$$ 

Since $\delta(\rho, \lambda) < r$, there exists $\mu \in I^X$ such that $\delta(\rho, \mu) < r$ and $\delta(\tilde{1} - \mu, \lambda) < r$. Hence $c_\delta(\lambda, r) \leq \mu$. It follow that $\delta(\rho, c_\delta(\lambda, r)) < r$. Therefore $c_\delta(c_\delta(\lambda, r), r) \leq \tilde{1} - \rho$. It is a contradiction. $\Box \Box$

**Theorem 2.5.** [5] Let $(X, \delta)$ be a fuzzy quasi-proximity space. Define the function $T_\delta : I^X \to I$ on $X$ by

$$T_\delta(\lambda) = \bigvee \{ r \in (0, 1] \mid i_\delta(\lambda, r) = \lambda \}.$$ 

Then $T_\delta$ is a gradation of openness on $X$.

**Remark 2.** Let $T_\delta$ be a gradation of openness on $X$. By definitions of $c_\delta$ and $i_\delta$, a function $F_\delta(\mu) = T_\delta(\mu^c) = \bigvee \{ r \in (0, 1] \mid c_\delta(\mu, r) = \mu \}$ is a gradation of closedness on $X$.

Let $(X, \delta_1)$ and $(X, \delta_2)$ be fuzzy quasi-proximity spaces. Unfortunately, a structure $\delta_1 \land \delta_2$ defined by $\delta_1 \land \delta_2(\lambda, \mu) = \delta_1(\lambda, \mu) \land \delta_2(\lambda, \mu)$ is not a fuzzy quasi-proximity on $X$.

We will construct the coarsest fuzzy quasi-proximity on $X$ finer than $\delta_1$ and $\delta_2$.

**Theorem 2.6.** Let $(X, \delta_1)$ and $(X, \delta_2)$ be fuzzy quasi-proximity spaces. We define, for all $\lambda, \mu \in I^X$,

$$\delta_1 \sqcap \delta_2(\lambda, \mu) = \inf \{ \bigvee_{j,k} (\delta_1(\lambda_j, \mu_k) \land \delta_2(\lambda_j, \mu_k)) \}$$

where for every finite families $(\lambda_j), (\mu_k)$ such that $\lambda = \bigvee \lambda_j$ and $\mu = \bigvee \mu_k$. Then the structure $\delta_1 \sqcap \delta_2$ is the coarsest fuzzy quasi-proximity on $X$ finer than $\delta_1$ and $\delta_2$. 


**Proof.** First, we will show that $\delta_1 \sqcap \delta_2$ is a fuzzy quasi-proximity on $X$.

(FQP1). Since, for each $\lambda \in I^X$, $\delta_i(\lambda, \tilde{0}) = 0$, it is easily proved.

(FQP2). For any $\lambda, \mu, \nu \in I^X$, we will show that

$$\delta_1 \sqcap \delta_2(\lambda \lor \nu) \leq \delta_1 \sqcap \delta_2(\lambda, \mu) \lor \delta_1 \sqcap \delta_2(\lambda, \nu).$$

Suppose that there exist $\lambda, \mu, \nu \in I^X$

$$c = \delta_1 \sqcap \delta_2(\lambda, \mu \lor \nu) > \delta_1 \sqcap \delta_2(\lambda, \mu) \lor \delta_1 \sqcap \delta_2(\lambda, \nu).$$

There are finite families $(\lambda_j), (\lambda'_m), (\mu_k)$ and $(\nu_l)$ such that $\lambda = \bigvee \lambda_j = \bigvee \lambda'_m, \mu = \bigvee \mu_k$ and $\nu = \bigvee \nu_l$ with

$$c > \bigvee_{j,k} (\delta_1(\lambda_j, \mu_k) \land \delta_2(\lambda_j, \mu_k)), \quad c > \bigvee_{m,l} (\delta_1(\lambda'_m, \nu_l) \land \delta_2(\lambda'_m, \nu_l)).$$

It follows that $\lambda = \bigvee_{j,m} (\lambda_j \land \lambda'_m)$ and $\mu \lor \nu = (\bigvee \mu_k) \lor (\bigvee \nu_l)$. Since

$$\delta_1(\lambda_j, \mu_k) \land \delta_2(\lambda_j, \mu_k) \geq \delta_1(\lambda_j \land \lambda'_m, \mu_k) \land \delta_2(\lambda_j \land \lambda'_m, \mu_k),$$

$$\delta_1(\lambda'_m, \nu_l) \land \delta_2(\lambda'_m, \nu_l) \geq \delta_1(\lambda'_m \land \lambda'_m, \nu_l) \land \delta_2(\lambda'_m \land \lambda'_m, \nu_l),$$

we have

$$c > \left( \bigvee_{j,k} (\delta_1(\lambda_j, \mu_k) \land \delta_2(\lambda_j, \mu_k)) \right) \lor \left( \bigvee_{m,l} (\delta_1(\lambda'_m, \nu_l) \land \delta_2(\lambda'_m, \nu_l)) \right) \geq \delta_1 \sqcap \delta_2(\lambda, \mu \lor \nu) = c.$$

It is a contradiction.

Similarly, we have $\delta_1 \sqcap \delta_2(\lambda \lor \rho, \mu) \leq \delta_1 \sqcap \delta_2(\lambda, \mu) \lor \delta_1 \sqcap \delta_2(\rho, \mu)$.

(FQP3). If for any $\lambda, \mu \in I^X$, $\delta_1 \sqcap \delta_2(\lambda, \mu) < r$, we will show that there exists $\rho \in I^X$ such that $\delta_1 \sqcap \delta_2(\lambda, \rho) < r$ and $\delta_1 \sqcap \delta_2(\bar{1} - \rho, \mu) < r$. 
It follows that
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Similarly, we have
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Some properties of fuzzy quasi-proximity spaces

Putting
\[ \rho_j = \bigvee_{k=1}^q \rho_{jk}, \quad \rho = \bigwedge_{j=1}^p \rho_j. \]

Then, by the definition of \( \delta_1 \cap \delta_2 \), we have \( \delta_1 \cap \delta_2(\lambda, \rho_j) < r \). Using (FQP2) and Remark 1, we have \( \delta_1 \cap \delta_2(\lambda, \rho) < r \).

In a similar way, we have \( \delta_1 \cap \delta_2(1 - \rho_{jk}, \mu_k) \leq \delta_1(1 - \rho_{jk}, \mu_k) < r \). Thus \( \delta_1 \cap \delta_2(1 - \rho_j, \mu_k) < r \). By (FQP2), we have \( \delta_1 \cap \delta_2(1 - \rho, \mu_k) < r \) and \( \delta_1 \cap \delta_2(1 - \rho, \mu) < r \).

(FQP4). We will show that if \( \lambda \not\leq 1 - \mu \), then \( \delta_1 \cap \delta_2(\lambda, \mu) = 1 \).

If \( \lambda \not\leq 1 - \mu \), then, for every finite families \( (\lambda_j), (\mu_k) \) such that \( \lambda = \bigvee \lambda_j \) and \( \mu = \bigvee \mu_k \), there exist \( j_0, k_0, x_0 \) such that \( \lambda_{j_0}(x_0) + \mu_{k_0}(x_0) > 1 \). Since \( \delta_1, \delta_2 \) are fuzzy quasi-proximities on \( X \), we have \( \delta_1(\lambda_{j_0}, \mu_{k_0}) = 1 \) and \( \delta_2(\lambda_{j_0}, \mu_{k_0}) = 1 \). Hence we have \( \delta_1 \cap \delta_2(\lambda, \mu) = 1 \).

Second, it is proved that \( \delta_1 \cap \delta_2 \succ \delta_1 \) from the following:
\[
\delta_1 \cap \delta_2(\lambda, \mu) = \inf \{ \bigvee_{j,k}(\delta_1(\lambda_j, \mu_k) \land \delta_2(\lambda_j, \mu_k)) \}
\leq \inf \{ \bigvee_{j,k}(\delta_1(\lambda_j, \mu_k)) \}
= \delta_1(\lambda, \mu) \quad \text{(by FQP2)}.\]

Similarly, we have \( \delta_1 \cap \delta_2 \succ \delta_2 \).

Finally, if \( \delta_1 \prec \delta \) and \( \delta_2 \prec \delta \), then we have
\[
\delta_1 \cap \delta_2(\lambda, \mu) = \inf \{ \bigvee_{j,k}(\delta_1(\lambda_j, \mu_k) \land \delta_2(\lambda_j, \mu_k)) \}
\geq \inf \{ \bigvee_{j,k}\delta(\lambda_j, \mu_k) \} \quad \text{(since \( \delta_i \prec \delta \))}
= \delta(\lambda, \mu) \quad \text{(by FQP2)}.\]

It follows that \( \delta_1 \cap \delta_2 \prec \delta \).
Let \((X, \delta)\) be a fuzzy quasi-proximity space. For each \(\lambda, \mu \in I^X\), we define \(\delta^*(\lambda, \mu) = \delta \cap \delta^{-1}(\lambda, \mu)\). By the above theorem, we can easily prove that \((X, \delta^*)\) is a fuzzy proximity space.

**Theorem 2.7.** Let \(\lambda, \mu \in I^X\) be given in a fuzzy quasi-proximity space on \((X, \delta)\). Then for each \(r \in (0, 1]\), the followings are equivalent:

1. \(\delta(\lambda, \mu) \geq r\),
2. \(\delta(c_{\delta^*}(\lambda, r), c_{\delta^*}(\mu, r)) \geq r\),
3. \(\delta(c_{\delta^{-1}}(\lambda, r), c_{\delta}(\mu, r)) \geq r\).

**Proof.** Since \(\lambda \leq c_{\delta^*}(\lambda, r) \leq c_{\delta^{-1}}(\lambda, r)\) and \(\mu \leq c_{\delta^*}(\mu, r) \leq c_{\delta}(\mu, r)\), we have (1) \(\Rightarrow\) (2) \(\Rightarrow\) (3) from the following:

\[
\delta(\lambda, \mu) \leq \delta(c_{\delta^*}(\lambda, r), c_{\delta^*}(\mu, r)) \leq \delta(c_{\delta^{-1}}(\lambda, r), c_{\delta}(\mu, r)).
\]

We will show that (3) \(\Rightarrow\) (1). Suppose that there exists \(\lambda, \mu \in I^X\) such that \(\delta(\lambda, \mu) < r \leq \delta(c_{\delta^{-1}}(\lambda, r), c_{\delta}(\mu, r))\).

Since \(\delta(\lambda, \mu) < r\), by (FQP3), there exists \(\rho \in I^X\) such that

\[
\delta(\lambda, \rho) < r, \quad \delta(\tilde{1} - \rho, \mu) < r.
\]

Since \(\delta(\tilde{1} - \rho, \mu) < r\), we have \(c_{\delta}(\mu, r) \leq \rho\). Since \(\delta(\lambda, \rho) \leq \delta(\lambda, \mu) < r\) and \(c_{\delta}(\mu, r) \leq \rho\), we have \(\delta(\lambda, c_{\delta}(\mu, r)) < r\). Again, since \(\delta(\lambda, c_{\delta}(\mu, r)) < r\), there exists \(\nu \in I^X\) such that

\[
\delta(\lambda, \nu) < r, \quad \delta(\tilde{1} - \nu, c_{\delta}(\mu, r)) < r.
\]

Now, since \(\delta^{-1}(\nu, \lambda) = \delta(\lambda, \nu) < r\), we have \(c_{\delta^{-1}}(\lambda, r) \leq \tilde{1} - \nu\). Hence \(\delta(c_{\delta^{-1}}(\lambda, r), c_{\delta}(\mu, r)) < r\). It is a contradiction. \(\square\)

**Definition 2.8.** Let \((X, \delta_1)\) and \((Y, \delta_2)\) be fuzzy quasi-proximity spaces. A function \(f : (X, \delta_1) \rightarrow (Y, \delta_2)\) is a fuzzy quasi-proximity map if it satisfies \(\delta_1(\mu, \nu) \leq \delta_2(f(\mu), f(\nu))\), for every \(\mu, \nu \in I^X\).

Using the above definition, we can easily prove the following lemma.
Lemma 2.9. If \( f : (X, \delta_1) \to (Y, \delta_2) \) is a fuzzy quasi-proximity map, then:

(a) \( f : (X, \delta_1^{-1}) \to (Y, \delta_2^{-1}) \) is a fuzzy quasi-proximity map,
(b) \( f : (X, \delta_1^*) \to (Y, \delta_2^*) \) is a fuzzy quasi-proximity map.

We will investigate relationships between fuzzy topological spaces and fuzzy quasi-proximity spaces.

Theorem 2.10. If a function \( f : (X, \delta_1) \to (Y, \delta_2) \) is a fuzzy quasi-proximity map, then:

(a) \( f : (X, T_{\delta_1}) \to (Y, T_{\delta_2}) \) is a gp-map,
(b) \( f : (X, T_{\delta_1^{-1}}) \to (Y, T_{\delta_2^{-1}}) \) is a gp-map,
(c) \( f : (X, T_{\delta_1^*}) \to (Y, T_{\delta_2^*}) \) is a gp-map.

Proof. (a). Suppose that \( f \) is not a gp-map. Then there exists \( \lambda \in I^Y \) such that \( T_{\delta_2}(\lambda) > T_{\delta_1}(f^{-1}(\lambda)) \). Hence there exists \( r \in I \) such that \( T_{\delta_4}(\lambda) > r > T_{\delta_4}(f^{-1}(\lambda)) \). Since \( T_{\delta_4}(\lambda) > r \), for some \( c > r \), then

\[
\lambda = i_{\delta_2}(\lambda, c) = \bigvee \{ \rho \mid \delta_2(\rho, \tilde{1} - \lambda) < c \}.
\]

Since \( f \) is a fuzzy quasi-proximity map, by Lemma 1.2, we have

\[
f^{-1}(\lambda) = \bigvee \{ f^{-1}(\rho) \mid \delta_2(\rho, \tilde{1} - \lambda) < c \}
\leq \bigvee \{ f^{-1}(\rho) \mid \delta_1(f^{-1}(\rho), \tilde{1} - f^{-1}(\lambda)) < c \}
\leq i_{\delta_1}(f^{-1}(\lambda), c).
\]

So, by Theorem 2.3 (ii), we have \( i_{\delta_1}(f^{-1}(\lambda), c) = f^{-1}(\lambda) \). It follows that \( T_{\delta_4}(f^{-1}(\lambda)) \geq c > r \). It is a contradiction.

(b) and (c) are easy from Lemma 2.9 and (a).

3. Initial fuzzy quasi-proximity structures

Now we will prove the existence of initial fuzzy quasi-proximity structures.
Definition 3.1. Let \((X_i, \delta_i)_{i \in \Delta}\) be a family of fuzzy quasi-proximity spaces. Let \(X\) be a set and, for each \(i \in \Delta\), \(f_i : X \to X_i\) a function. The initial structure \(\delta\) is the coarsest fuzzy quasi-proximity on \(X\) with respect to which for each \(i \in \Delta\), \(f_i\) is a fuzzy quasi-proximity map.

Theorem 3.2. (Existence of initial structures) Let \((X_i, \delta_i)_{i \in \Delta}\) be a family of fuzzy quasi-proximity spaces. Let \(X\) be a set and, for each \(i \in \Delta\), \(f_i : X \to X_i\) a mapping. Define the function \(\delta : I^X \times I^X \to I\) on \(X\) by
\[
\delta(\lambda, \mu) = \inf \left\{ \bigvee_{j,k} \inf_{i \in \Delta} \delta_i(f_i(\lambda_j), f_i(\mu_k)) \right\},
\]
where for every finite families \((\lambda_j), (\mu_k)\) such that \(\lambda = \bigvee_{j=1}^n \lambda_j\) and \(\mu = \bigvee_{k=1}^m \mu_k\). Then:

1. \(\delta\) is the coarsest fuzzy quasi-proximity on \(X\) with respect to which for each \(i \in \Delta\), \(f_i\) is a fuzzy quasi-proximity map.
2. A map \(f : (Y, \delta') \to (X, \delta)\) is a fuzzy quasi-proximity map iff each \(f_i \circ f : (Y, \delta') \to (X_i, \delta_i)\) is a fuzzy quasi-proximity map.

Proof. (1). First, we will show that \(\delta\) is a fuzzy quasi-proximity on \(X\).

(FQP1). Since \(\delta_i(f_i(\lambda), \tilde{0}) = 0\) for all \(\lambda \in I^X\), it is clear.

(FQP2). For any \(\lambda, \mu, \nu \in I^X\), we will show that
\[
\delta(\lambda, \mu \vee \nu) \leq \delta(\lambda, \mu) \vee \delta(\lambda, \nu).
\]
Suppose that there exist \(\lambda, \mu, \nu \in I^X\) such that
\[
c = \delta(\lambda, \mu \vee \nu) > \delta(\lambda, \mu) \vee \delta(\lambda, \nu).
\]

There are finite families \((\lambda_j), (\lambda'_m), (\mu_k)\) and \((\nu_l)\) such that \(\lambda = \bigvee \lambda_j = \bigvee \lambda'_m\), \(\mu = \bigvee \mu_k\) and \(\nu = \bigvee \nu_l\) with
\[
c > \bigvee_{j,k} \delta_i(f_i(\lambda_j), f_i(\mu_k)), \quad c > \bigvee_{m,l} \delta_i(f_i(\lambda'_m), f_i(\nu_l)).
\]
It follows that \(\lambda = \bigvee_{j,m} (\lambda_j \land \lambda'_m)\) and \(\mu \vee \nu = (\bigvee \mu_k) \lor (\bigvee \nu_l)\).
Since \( \inf_{i \in \Delta} \delta_i(f_i(\lambda_j), f_i(\mu_k)) \geq \inf_{i \in \Delta} \delta_i(f_i(\lambda_j \land \lambda'_m), f_i(\mu_k)) \) and \( \inf_{i \in \Delta} \delta_i(f_i(\lambda'_m), f_i(\nu_l)) \geq \inf_{i \in \Delta} \delta_i(f_i(\lambda_j \land \lambda'_m), f_i(\nu_l)) \),

\[
c > \left( \bigvee_{j,k} \inf_{i \in \Delta} \delta_i(f_i(\lambda_j), f_i(\mu_k)) \right) \lor \left( \bigvee_{m,l} \inf_{i \in \Delta} \delta_i(f_i(\lambda'_m), f_i(\nu_l)) \right) \\
\geq \left( \bigvee_{j,k} \inf_{i \in \Delta} \delta_i(f_i(\lambda_j \land \lambda'_m), f_i(\mu_k)) \right) \\
\lor \left( \bigvee_{m,l} \inf_{i \in \Delta} \delta_i(f_i(\lambda_j \land \lambda'_m), f_i(\nu_l)) \right) \\
\geq \delta(\lambda, \mu \lor \nu) = c.
\]

It is a contradiction.

Similarly, we have \( \delta(\lambda \lor \rho, \mu) \leq \delta(\lambda, \mu) \lor \delta(\rho, \mu) \).

(FQP3). If for any \( \lambda, \mu \in I^X \), \( \delta(\lambda, \mu) < r \), we will show that there exists \( \rho \in I^X \) such that \( \delta(\lambda, \rho) < r \) and \( \delta(1 - \rho, \mu) < r \).

If for any \( \lambda, \mu \in I^X \), \( \delta(\lambda, \mu) < r \), then there are finite families \((\lambda_j), (\mu_k)\) such that \( \lambda = \bigvee_{j=1}^p \lambda_j \), \( \mu = \bigvee_{k=1}^q \mu_k \) with

\[
\delta(\lambda, \mu) \leq \bigvee_{j,k} \inf_{i \in \Delta} (\delta_i(f_i(\lambda_j), f_i(\mu_k))) < r
\]

i.e., for all \( j, k \), \( \inf_{i \in \Delta} (\delta_i(f_i(\lambda_j), f_i(\mu_k))) < r \). It follows that for any \( j, k \), there exists an \( i_{jk} \in \Delta \) such that \( \delta_{i_{jk}}(f_{i_{jk}}(\lambda_j), f_{i_{jk}}(\mu_k)) < r \). Since \( \delta_{i_{jk}} \) is a fuzzy quasi-proximity on \( X_{i_{jk}} \), by (FQP3), there exists \( \rho_{jk} \in I^{X_{i_{jk}}} \) such that \( \delta_{i_{jk}}(f_{i_{jk}}(\lambda_j), \rho_{jk}) < r \) and \( \delta_{i_{jk}}(1 - \rho_{jk}, f_{i_{jk}}(\mu_k)) < r \).

Put

\[
\rho_j = \bigvee_{k=1}^q f_{i_{jk}}^{-1}(\rho_{jk}), \quad \rho = \bigwedge_{j=1}^p \rho_j.
\]

Since \( f_{i_{jk}}(f_{i_{jk}}^{-1}(\rho_{jk})) \leq \rho_{jk} \), then

\[
\delta(\lambda_j, \rho_j) \leq \bigvee_{k=1}^q \delta_{i_{jk}}(f_{i_{jk}}(\lambda_j), f_{i_{jk}}(f_{i_{jk}}^{-1}(\rho_{jk}))) \\
\leq \bigvee_{k=1}^q \delta_{i_{jk}}(f_{i_{jk}}(\lambda_j), \rho_{jk}) < r.
\]
Using (FQP2) and Remark 1, we have \( \delta(\lambda, \rho) < r \).

In a similar way, by the definition of \( \delta \), for all \( k = 1, \ldots, q \),

\[
\delta(f_{ijk}^{-1}(\tilde{1} - \rho_{jk}), \mu_k) \leq \delta_{ijk}(\tilde{1} - \rho_{jk}, \mu_k) < r,
\]

because \( f_{ijk}(f_{ijk}^{-1}(\tilde{1} - \rho_{jk})) \leq \tilde{1} - \rho_{jk} \). By Remark 1, we have \( \delta(\tilde{1} - \rho, \mu_k) < r \) and \( \delta(\tilde{1} - \rho, \mu_k) < r \).

(FQP4). We will show that if \( \lambda \nleq \tilde{1} - \mu \), then \( \delta(\lambda, \mu) = 1 \).

If \( \lambda \nleq \tilde{1} - \mu \), then, for every finite families \( (\lambda_j), (\mu_k) \) such that \( \lambda = \bigvee \lambda_j \) and \( \mu = \bigvee \mu_k \), there exist \( j_0, k_0, x_0 \) such that \( \lambda_{j_0}(x_0) + \mu_{k_0}(x_0) > 1 \). It follows that, for all \( i \in \vartriangle \),

\[
f_i(\lambda_{j_0})(f_i(x_0)) + f_i(\mu_{k_0})(f_i(x_0)) \geq \lambda_{j_0}(x_0) + \mu_{k_0}(x_0) > 1.
\]

Since for each \( i \in \vartriangle \), \( \delta_i \) is a fuzzy quasi-proximity on \( X_i \), we have \( \delta_i(\lambda_{j_0}, \mu_{k_0}) = 1 \). Hence \( \delta(\lambda, \mu) = 1 \).

Second, from the definition of \( \delta \), since

\[
\delta(\lambda, \mu) = \inf\left\{ \bigvee_{j,k} \inf_{i \in \vartriangle} \delta_i(f_i(\lambda_j), f_i(\mu_k)) \right\}
\]

\[
\leq \inf\left\{ \bigvee_{j,k} \delta_i(f_i(\lambda_j), f_i(\mu_k)) \right\}
\]

\[
= \delta_i(f_i(\lambda), f_i(\mu)) \quad \text{(by FQP 2)},
\]

for each \( i \in \vartriangle \), \( f_i : (X, \delta) \rightarrow (X_i, \delta_i) \) is a fuzzy quasi-proximity map.

If \( f_i : (X, \delta') \rightarrow (X_i, \delta_i) \) is a fuzzy quasi-proximity map, then, for every \( i \in \vartriangle \), since

\[
\delta(\lambda, \mu) = \inf\left\{ \bigvee_{j,k} \inf_{i \in \vartriangle} \delta_i(f_i(\lambda_j), f_i(\mu_k)) \right\}
\]

\[
\geq \inf\left\{ \bigvee_{j,k} \delta'(\lambda_j, \mu_k) \right\}
\]

\[
= \delta'(\lambda, \mu) \quad \text{(by FQP 2)},
\]

we have \( \delta'(\lambda, \mu) \leq \delta(\lambda, \mu) \), \( \forall \lambda, \mu \in I^X \).

(2). Necessity of the composition condition is clear since the composition of fuzzy quasi-proximity maps is a fuzzy quasi-proximity map.
Conversely, suppose \( f \) is not a fuzzy quasi-proximity map. Then there exists \( \lambda, \mu \in I^Y \) such that
\[
\delta'(\lambda, \mu) > \delta(f(\lambda), f(\mu)).
\]
Therefore there are finite families \((\lambda'_j), (\mu'_k)\) such that
\[
f(\lambda) = \bigvee_{j=1}^{p} \lambda'_j, \quad f(\mu) = \bigvee_{k=1}^{q} \mu'_k,
\]
and
\[
\delta'(\lambda, \mu) > \bigvee_{j,k} \inf_{i \in \triangle} \delta_i(f_i(\lambda'_j), f_i(\mu'_k)).
\]
It follows that for any \( j, k \), there exists an \( i_{jk} \in \triangle \) such that
\[
\delta_{i_{jk}}(f_{i_{jk}}(\lambda'_j), f_{i_{jk}}(\mu'_k)) < \delta'(\lambda, \mu).
\]
on the other hand, \( f_i \circ f \) is a fuzzy quasi-proximity map. For any \( j, k \), by Lemma 1.2, since \( f_i(f^{-1}(\lambda'_j))) \leq f_i(\lambda'_j) \),
\[
\delta'(f^{-1}(\lambda'_j), f^{-1}(\mu'_k)) \leq \delta_{i_{jk}}(f_{i_{jk}}(\lambda'_j), f_{i_{jk}}(\mu'_k)).
\]
Since \( \lambda \leq f^{-1}(f(\lambda)) = \bigvee_{j=1}^{p} f^{-1}(\lambda'_j) \), we have
\[
\delta'(\lambda, \mu) \leq \bigvee_{j,k} \delta'(f^{-1}(\lambda'_j), f^{-1}(\mu'_k)) \quad \text{(by FQP 2 and Lemma 1.2)}
\]
\[
\leq \bigvee_{j,k} \delta_{i_{jk}}(f_{i_{jk}}(\lambda'_j), f_{i_{jk}}(\mu'_k))
\]
\[
< \delta'(\lambda, \mu).
\]
It is a contradiction. \( \square \) \( \square \)

By the above theorem, we can define subspaces and products in the obvious way.

**Definition 3.3.** Let \((X, \delta)\) be a fuzzy quasi-proximity and \( A \) be a subset of \( X \). The pair \((A, \delta_A)\) is said to be a *subspace* of \((X, \delta)\) if it is endowed with the initial fuzzy quasi-proximity structure with respect to the inclusion map.
Definition 3.4. Let $X$ be the product $\prod_{i \in \Delta} X_i$ of the family $\{(X_i, \delta_i) \mid i \in \Delta\}$ of fuzzy quasi-proximity spaces. An initial fuzzy quasi-proximity structure $\delta = \otimes \delta_i$ on $X$ with respect to all the projections $\pi_i : X \to X_i$ is called the \textit{product fuzzy quasi-proximity structure} of $\{\delta_i \mid i \in \Delta\}$, and $(X, \otimes \delta_i)$ is called the \textit{product fuzzy quasi-proximity space}.

Using Theorem 3.2, we have the following corollary.

**Corollary 3.5.** Let $(X_i, \delta_i)_{i \in \Delta}$ be a family of fuzzy quasi-proximity spaces. Let $X = \prod_{i \in \Delta} X_i$ be a set and, for each $i \in \Delta$, $\pi_i : X \to X_i$ a mapping. The structure $\delta = \otimes \delta_i$ on $X$ is defined by

$$
\delta(\lambda, \mu) = \inf \{ \inf_{i \in \Delta} \delta_i(\pi_i(\lambda_j), \pi_i(\mu_k)) \},
$$

where for every finite families $(\lambda_j), (\mu_k)$ such that $\lambda = \bigvee_{j=1}^n \lambda_j$ and $\mu = \bigvee_{k=1}^m \mu_k$. Then:

1. $\delta$ is the coarsest fuzzy quasi-proximity on $X$ with respect to which for each $i \in \Delta$, $\pi_i$ is a fuzzy quasi-proximity map.
2. A map $f : (Y, \delta') \to (X, \delta)$ is a fuzzy quasi-proximity map iff each $\pi_i \circ f : (Y, \delta') \to (X_i, \delta_i)$ is a fuzzy quasi-proximity map.

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