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SOME PROPERTIES OF FUZZY QUASI-PROXIMITY SPACES

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Abstract. We will define the fuzzy quasi-proximity space and investigate some properties of fuzzy quasi-proximity spaces. We will prove the existences of initial fuzzy quasi-proximity structures. From this fact, we can define subspaces and products of fuzzy quasi-proximity spaces.

1. Introduction and preliminaries

In [8, 9,10,11], S.K. Samanta introduced the concept of gradations of openness and proximity. M.H. Ghanim et al.[5] introduced fuzzy proximity spaces with somewhat different definition of S.K. Samanta [8].

In this paper, we will define the fuzzy quasi-proximity space in view of the definition of M.H. Ghanim et al.[5]. We will investigate some properties of fuzzy quasi-proximity spaces. We will study relationships between fuzzy quasi-proximity spaces and fuzzy topological spaces.

In particular, we will prove the existence of initial fuzzy quasi-proximity structures. From this fact, we can define subspaces and products of fuzzy quasi-proximity spaces.

In this paper, all the notations and the definitions are standard in fuzzy set theory.

Definition 1.1. [11] Let X be a nonempty set. A function $\mathcal{T} : I^X \to I$ is called a *gradation of openness* on X if it satisfies the following conditions:

(c1) $\mathcal{T}(\tilde{0}) = \mathcal{T}(\tilde{1}) = 1$,

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(c2) $\mathcal{T}(\mu_1 \wedge \mu_2) \geq \mathcal{T}(\mu_1) \wedge \mathcal{T}(\mu_2)$,
(c3) $\mathcal{T}(\bigvee_{i\in\triangle} \mu_i) \geq \bigwedge_{i\in\triangle} \mathcal{T}(\mu_i)$.

The pair $(X, \mathcal{T})$ is called a *fuzzy topological space.*

Let $\mathcal{T}$ be a gradation of openness on X and $\mathcal{F} : I^X \to I$ be defined by $\mathcal{F}(\mu) = \mathcal{T}(\mu^c)$. Then $\mathcal{F}$ is called a *gradation of closedness* on X.

Let $(X, \mathcal{T})$ be a fuzzy topological space, then for each $r \in I$, $\mathcal{T}_r = \{\mu \in I^X \mid \mathcal{T}(\mu) \geq r\}$ is a Chang's fuzzy topology on X.

Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}^*)$ be fuzzy topological spaces. A function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}^*)$ is called a *gradation preserving map* (gp-map) if $\mathcal{T}^*(\mu) \leq \mathcal{T}(f^{-1}(\mu))$ for all $\mu \in I^X$.

A function $\delta : I^X \times I^X \to I$ is a *fuzzy proximity* on X [8] if it satisfies the following conditions:

(SFP1) $\delta(\tilde{0}, \tilde{1}) = 0$.
(SFP2) $\delta(\lambda, \mu) = \delta(\mu, \lambda)$.
(SFP3) $\delta(\lambda_1 \vee \lambda_2, \mu) = \delta(\lambda_1, \mu) \vee \delta(\lambda_2, \mu)$.
(SFP4) If $\delta(\lambda, \mu) < 1 - r$, then $\delta(cl(\lambda, r), \mu) < 1 - r$, where

$$cl(\lambda, r) = \tilde{1} - \bigvee \{\rho \leq \lambda^c \mid \delta(\lambda, \rho) < 1 - r\}.$$

The pair (X, δ) is called a *Samanta type fuzzy proximity space.*

We can easily prove the following lemma.

LEMMA 1.2. *If $f : X \to Y$, then we have the following properties for direct and inverse image of fuzzy sets under mappings:*

(1) $\mu \geq f(f^{-1}(\mu))$ *with equality if f is surjective,*
(2) $\nu \leq f^{-1}(f(\nu))$ *with equality if f is injective,*
(3) $f^{-1}(\mu^c) = f^{-1}(\mu)^c$,
(4) $f^{-1}(\bigvee_{i\in I} \mu_i) = \bigvee_{i\in I} f^{-1}(\mu_i)$,
(5) $f^{-1}(\bigwedge_{i\in I} \mu_i) = \bigwedge_{i\in I} f^{-1}(\mu_i)$,
(6) $f(\bigvee_{i\in I} \nu_i) = \bigvee_{i\in I} f(\nu_i)$,
(7) $f(\bigwedge_{i\in I} \mu_i) \leq \bigwedge_{i\in I} f(\mu_i)$ *with equality if f is injective.*

2. Fuzzy quasi-proximity and fuzzy topological spaces

From the definition of M.H. Ghanim [5], we can define a fuzzy quasi-proximity.

Definition 2.1. A function $\delta : I^X \times I^X \to I$ is said to be a *fuzzy quasi-proximity* on X which satisfies the following conditions:

(FQP1) $\delta(\tilde{0}, \tilde{1}) = 0$ and $\delta(\tilde{1}, \tilde{0}) = 0$.

(FQP2) $\delta(\lambda \vee \rho, \mu) = \delta(\lambda, \mu) \vee \delta(\rho, \mu)$ and $\delta(\lambda, \mu \vee \nu) = \delta(\lambda, \mu) \vee \delta(\lambda, \nu)$.

(FQP3) If $\delta(\lambda, \mu) < r$, then there exists $\rho \in I^X$ such that $\delta(\lambda, \rho) < r$ and $\delta(\tilde{1} - \rho, \mu) < r$.

(FQP4) If $\delta(\lambda, \mu) \neq 1$, then $\lambda \leq \tilde{1} - \mu$.

The pair (X, δ) is called a *fuzzy quasi-proximity space.*

A fuzzy quasi-proximity space (X, δ) is called a *fuzzy proximity space* if (FP) $\delta(\lambda, \mu) = \delta(\mu, \lambda)$ for any $\lambda, \mu \in I^X$.

Let $(X, \delta_1), (X, \delta_2)$ be given. We say δ_2 is *finer* than δ_1 (δ_1 is *coarser* than δ_2), denoted by $\delta_1 \prec \delta_2$, iff for any $\lambda, \mu \in I^X$, $\delta_2(\lambda, \mu) \leq \delta_1(\lambda, \mu)$.

Remark 1. (1) If (X, δ) is a fuzzy quasi-proximity space and $\lambda \leq \mu$, then, by (FQP2), we have $\delta(\lambda, \nu) \leq \delta(\mu, \nu)$ and $\delta(\rho, \lambda) \leq \delta(\rho, \mu)$.

(2) Let (X, δ) be a fuzzy quasi-proximity space. For each $\lambda, \mu \in I^X$, we define $\delta^{-1}(\lambda, \mu) = \delta(\mu, \lambda)$. Then the structure δ^{-1} is a fuzzy quasi-proximity on X.

(3) Every fuzzy proximity space in our sense is a Samanta type fuzzy proximity space. If $\delta(\lambda, \mu) < 1 - r$, by (FQP3), then there exists $\rho \in I^X$ such that $\delta(\lambda, \rho) < 1 - r$ and $\delta(\tilde{1} - \rho, \mu) < 1 - r$. Since $\delta(\lambda, \rho) < 1 - r$, by (FQP4) and the definition of $cl(\lambda, r)$, we have $\rho \leq \tilde{1} - \lambda$ and $cl(\lambda, r) \leq \tilde{1} - \rho$. So, by (FQP2), $\delta(cl(\lambda, r), \mu) < 1 - r$. Thus (SFP4) holds.

Theorem 2.2. [5] *Let (X, δ) be a fuzzy quasi-proximity space, then for each $r \in (0, 1]$, the family $\delta_r = \{(\lambda, \mu) \in I^X \times I^X \mid \delta(\lambda, \mu) \geq r\}$ is a (classical) fuzzy quasi-proximity space on X.*

Theorem 2.3. [5] *Let δ be a fuzzy quasi-proximity on X. For each $r \in (0, 1], \lambda \in I^X$, we define*

$$i_\delta(\lambda, r) = \bigvee \{\rho \in I^X \mid \delta(\rho, \tilde{1} - \lambda) < r\}.$$

The family $\{i_\delta(\lambda, r) \mid r \in (0, 1]\}$ satisfies the followings properties:

(i) $i_\delta(\tilde{1}, r) = \tilde{1}$.

(ii) $i_\delta(\lambda, r) \leq \lambda$ *and* $i_\delta(\lambda_1, r) \leq i_\delta(\lambda_2, r)$, *if* $\lambda_1 \leq \lambda_2$.

(iii) $i_\delta(i_\delta(\lambda, r), r) = i_\delta(\lambda, r)$.
(iv) $i_\delta(\lambda \wedge \mu, r) = i_\delta(\lambda, r) \wedge i_\delta(\mu, r)$.
(v) $i_\delta(\lambda, r) \leq i_\delta(\lambda, r')$, *if* $r \leq r'$, *where* $r, r' \in (0, 1]$.

THEOREM 2.4. *Let* δ *be a fuzzy quasi-proximity on* X. *For each* $r \in (0, 1], \lambda \in I^X$, *we define*

$$c_\delta(\lambda, r) = \bigwedge\{\rho^c \in I^X \mid \delta(\rho, \lambda) < r\}.$$

The family $\{c_\delta(\lambda, r) \mid r \in (0, 1]\}$ *satisfies the following properties:*
(i) $c_\delta(\tilde{0}, r) = \tilde{0}, c_\delta(\tilde{1}, r) = \tilde{1}$.
(ii) $c_\delta(\lambda, r) \geq \lambda$.
(iii) $c_\delta(\lambda_1, r) \leq c_\delta(\lambda_2, r)$, *if* $\lambda_1 \leq \lambda_2$.
(iv) $c_\delta(\lambda \vee \mu, r) = c_\delta(\lambda, r) \vee c_\delta(\mu, r)$.
(v) $c_\delta(c_\delta(\lambda, r), r) = c_\delta(\lambda, r)$.
(vi) $c_\delta(\lambda, r) \geq c_\delta(\lambda, r')$, *if* $r \leq r'$, *where* $r, r' \in (0, 1]$.

Proof. (i),(iii) and (vi) are easily proved from the definition of c_δ.
(ii). Suppose that there exists $\lambda \in I^X$ such that for some $x_0 \in X$, $c_\delta(\lambda, r)(x_0) < \lambda(x_0)$. By the definition of $c_\delta(\lambda, r)$, there exists $\rho \in I^X$ such that $(\tilde{1} - \rho)(x_0) < \lambda(x_0)$ and $\delta(\rho, \lambda) < r$.

On the other hand, since $(\tilde{1} - \rho)(x_0) < \lambda(x_0)$, by (FQP4), we have $\delta(\rho, \lambda) = 1$. It is a contradiction.

(iv). By (iii), we have $c_\delta(\lambda \vee \mu, r) \geq c_\delta(\lambda, r) \vee c_\delta(\mu, r)$.

We will show that $c_\delta(\lambda \vee \mu, r) \leq c_\delta(\lambda, r) \vee c_\delta(\mu, r)$.

Suppose that there exist $\lambda, \mu \in I^X$ such that for some $x_0 \in X$,

$$c_\delta(\lambda \vee \mu, r)(x_0) > c_\delta(\lambda, r)(x_0) \vee c_\delta(\mu, r)(x_0).$$

There exists $\rho_1, \rho_2 \in I^X$ such that $\delta(\rho_1, \lambda) < r, \;\; \delta(\rho_2, \mu) < r$ and

$$c_\delta(\lambda \vee \mu, r)(x_0) > (\tilde{1} - \rho_1)(x_0) \vee (\tilde{1} - \rho_2)(x_0).$$

On the other hand, by Remark 1 and (FQP2), since

$$\delta(\rho_1 \wedge \rho_2, \lambda \vee \mu) \leq \delta(\rho_1, \lambda) \vee \delta(\rho_2, \mu) < r,$$

we have $c_\delta(\lambda \vee \mu, r) \leq (\tilde{1} - \rho_1) \vee (\tilde{1} - \rho_2)$. It is a contradiction.

(v). By (ii), it suffices to show that $c_\delta(c_\delta(\lambda, r), r) \leq c_\delta(\lambda, r)$.

Suppose that there exists $\lambda \in I^X$ such that for some $x_0 \in X$, $c_\delta(c_\delta(\lambda, r), r)(x_0) > c_\delta(\lambda, r)(x_0)$. There exists $\rho \in I^X$ such that

$$c_\delta(c_\delta(\lambda, r), r)(x_0) > (\tilde{1} - \rho)(x_0), \ \ \delta(\rho, \lambda) < r.$$

Since $\delta(\rho, \lambda) < r$, there exists $\mu \in I^X$ such that $\delta(\rho, \mu) < r$ and $\delta(\tilde{1} - \mu, \lambda) < r$. Hence $c_\delta(\lambda, r) \leq \mu$. It follow that $\delta(\rho, c_\delta(\lambda, r)) < r$. Therefore $c_\delta(c_\delta(\lambda, r), r) \leq \tilde{1} - \rho$. It is a contradiction. □ □

THEOREM 2.5. [5] *Let (X, δ) be a fuzzy quasi-proximity space. Define the function $\mathcal{T}_\delta : I^X \to I$ on X by*

$$\mathcal{T}_\delta(\lambda) = \bigvee \{r \in (0, 1] \mid i_\delta(\lambda, r) = \lambda\}.$$

Then $\mathcal{T}_\delta$ is a gradation of openness on X.

REMARK 2. Let $\mathcal{T}_\delta$ be a gradation of openness on X. By definitions of c_δ and i_δ, a function $\mathcal{F}_\delta(\mu) = \mathcal{T}_\delta(\mu^c) = \bigvee\{r \in (0, 1] \mid c_\delta(\mu, r) = \mu\}$ is a gradation of closedness on X.

Let (X, δ_1) and (X, δ_2) be fuzzy quasi-proximity spaces. Unfortunately, a structure $\delta_1 \wedge \delta_2$ defined by $\delta_1 \wedge \delta_2(\lambda, \mu) = \delta_1(\lambda, \mu) \wedge \delta_2(\lambda, \mu)$ is not a fuzzy quasi-proximity on X.

We will construct the coarsest fuzzy quasi-proximity on X finer than δ_1 and δ_2.

THEOREM 2.6. *Let (X, δ_1) and (X, δ_2) be fuzzy quasi-proximity spaces. We define, for all $\lambda, \mu \in I^X$,*

$$\delta_1 \sqcap \delta_2(\lambda, \mu) = inf\{\bigvee_{j,k} (\delta_1(\lambda_j, \mu_k) \wedge \delta_2(\lambda_j, \mu_k))\}$$

where for every finite families $(\lambda_j), (\mu_k)$ such that $\lambda = \bigvee \lambda_j$ and $\mu = \bigvee \mu_k$. Then the structure $\delta_1 \sqcap \delta_2$ is the coarsest fuzzy quasi-proximity on X finer than δ_1 and δ_2.

Proof. First, we will show that $\delta_1 \sqcap \delta_2$ is a fuzzy quasi-proximity on X.

(FQP1). Since, for each $\lambda \in I^X$, $\delta_i(\lambda, \tilde{0}) = 0$, it is easily proved.

(FQP2). For any $\lambda, \mu, \nu \in I^X$, we will show that

$$\delta_1 \sqcap \delta_2(\lambda, \mu \vee \nu) \leq \delta_1 \sqcap \delta_2(\lambda, \mu) \vee \delta_1 \sqcap \delta_2(\lambda, \nu).$$

Suppose that there exist $\lambda, \mu, \nu \in I^X$

$$c = \delta_1 \sqcap \delta_2(\lambda, \mu \vee \nu) > \delta_1 \sqcap \delta_2(\lambda, \mu) \vee \delta_1 \sqcap \delta_2(\lambda, \nu).$$

There are finite families $(\lambda_j), (\lambda'_m), (\mu_k)$ and (ν_l) such that $\lambda = \bigvee \lambda_j = \bigvee \lambda'_m$, $\mu = \bigvee \mu_k$ and $\nu = \bigvee \nu_l$ with

$$c > \bigvee_{j,k} (\delta_1(\lambda_j, \mu_k) \wedge \delta_2(\lambda_j, \mu_k)), \qquad c > \bigvee_{m,l} (\delta_1(\lambda'_m, \nu_l) \wedge \delta_2(\lambda'_m, \nu_l)).$$

It follows that $\lambda = \bigvee_{j,m} (\lambda_j \wedge \lambda'_m)$ and $\mu \vee \nu = (\bigvee \mu_k) \vee (\bigvee \nu_l)$. Since

$$\delta_1(\lambda_j, \mu_k) \wedge \delta_2(\lambda_j, \mu_k) \geq \delta_1(\lambda_j \wedge \lambda'_m, \mu_k) \wedge \delta_2(\lambda_j \wedge \lambda'_m, \mu_k),$$
$$\delta_1(\lambda'_m, \nu_l) \wedge \delta_2(\lambda'_m, \nu_l) \geq \delta_1(\lambda_j \wedge \lambda'_m, \nu_l) \wedge \delta_2(\lambda_j \wedge \lambda'_m, \nu_l),$$

we have

$$\begin{aligned} c &> (\bigvee_{j,k} (\delta_1(\lambda_j, \mu_k) \wedge \delta_2(\lambda_j, \mu_k))) \vee (\bigvee_{m,l} (\delta_1(\lambda'_m, \nu_l) \wedge \delta_2(\lambda'_m, \nu_l))) \\ &\geq (\bigvee_{j,k} (\delta_1(\lambda_j \wedge \lambda'_m, \mu_k)) \wedge \delta_2(\lambda_j \wedge \lambda'_m, \mu_k))) \\ &\quad \vee (\bigvee_{m,l} (\delta_1(\lambda_j \wedge \lambda'_m, \nu_l) \wedge \delta_2(\lambda_j \wedge \lambda'_m, \nu_l))) \\ &\geq \delta_1 \sqcap \delta_2(\lambda, \mu \vee \nu) = c. \end{aligned}$$

It is a contradiction.

Similarly, we have $\delta_1 \sqcap \delta_2(\lambda \vee \rho, \mu) \leq \delta_1 \sqcap \delta_2(\lambda, \mu) \vee \delta_1 \sqcap \delta_2(\rho, \mu)$.

(FQP3). If for any $\lambda, \mu \in I^X$, $\delta_1 \sqcap \delta_2(\lambda, \mu) < r$, we will show that there exists $\rho \in I^X$ such that $\delta_1 \sqcap \delta_2(\lambda, \rho) < r$ and $\delta_1 \sqcap \delta_2(\tilde{1} - \rho, \mu) < r$.

If for any $\lambda, \mu \in I^X$, $\delta_1 \sqcap \delta_2(\lambda, \mu) < r$, then there are finite families $(\lambda_j), (\mu_k)$ such that $\lambda = \bigvee_{j=1}^{p} \lambda_j$, $\mu = \bigvee_{k=1}^{q} \mu_k$ with for all j, k, $\delta_1(\lambda_j, \mu_k) \wedge \delta_2(\lambda_j, \mu_k) < r$. For any j, k, there exists $i = i(j,k) \in \{1, 2\}$ such that $\delta_i(\lambda_j, \mu_k) < r$. Since δ_i is a fuzzy quasi-proximity on X, there exists $\rho_{jk} \in I^X$ such that $\delta_i(\lambda_j, \rho_{jk}) < r$ and $\delta_i(\tilde{1} - \rho_{jk}, \mu_k) < r$.

Put

$$\rho_j = \bigvee_{k=1}^{q} \rho_{jk}, \quad \rho = \bigwedge_{j=1}^{p} \rho_j.$$

Then, by the definition of $\delta_1 \sqcap \delta_2$, we have $\delta_1 \sqcap \delta_2(\lambda_j, \rho_j) < r$. Using (FQP2) and Remark 1, we have $\delta_1 \sqcap \delta_2(\lambda, \rho) < r$.

In a similar way, we have $\delta_1 \sqcap \delta_2(\tilde{1} - \rho_{jk}, \mu_k) \leq \delta_i(\tilde{1} - \rho_{jk}, \mu_k) < r$. Thus $\delta_1 \sqcap \delta_2(\tilde{1} - \rho_j, \mu_k) < r$. By (FQP2), we have $\delta_1 \sqcap \delta_2(\tilde{1} - \rho, \mu_k) < r$ and $\delta_1 \sqcap \delta_2(\tilde{1} - \rho, \mu) < r$.

(FQP4). We will show that if $\lambda \not\leq \tilde{1} - \mu$, then $\delta_1 \sqcap \delta_2(\lambda, \mu) = 1$.

If $\lambda \not\leq \tilde{1} - \mu$, then, for every finite families $(\lambda_j), (\mu_k)$ such that $\lambda = \bigvee \lambda_j$ and $\mu = \bigvee \mu_k$, there exist j_0, k_0, x_0 such that $\lambda_{j_0}(x_0) + \mu_{k_0}(x_0) > 1$. Since δ_1, δ_2 are fuzzy quasi-proximities on X, we have $\delta_1(\lambda_{j_0}, \mu_{k_0}) = 1$ and $\delta_2(\lambda_{j_0}, \mu_{k_0}) = 1$. Hence we have $\delta_1 \sqcap \delta_2(\lambda, \mu) = 1$.

Second, it is proved that $\delta_1 \sqcap \delta_2 \succ \delta_1$ from the following:

$$\begin{aligned}
\delta_1 \sqcap \delta_2(\lambda, \mu) &= inf\{\bigvee_{j,k}(\delta_1(\lambda_j, \mu_k) \wedge \delta_2(\lambda_j, \mu_k))\} \\
&\leq inf\{\bigvee_{j,k}(\delta_1(\lambda_j, \mu_k)\} \\
&= \delta_1(\lambda, \mu) \text{ (by FQP 2).}
\end{aligned}$$

Similarly, we have $\delta_1 \sqcap \delta_2 \succ \delta_2$.

Finally, if $\delta_1 \prec \delta$ and $\delta_2 \prec \delta$, then we have

$$\begin{aligned}
\delta_1 \sqcap \delta_2(\lambda, \mu) &= inf\{\bigvee_{j,k}(\delta_1(\lambda_j, \mu_k) \wedge \delta_2(\lambda_j, \mu_k))\} \\
&\geq inf\{\bigvee_{j,k} \delta(\lambda_j, \mu_k)\} \quad (\text{ since } \delta_i \prec \delta) \\
&= \delta(\lambda, \mu) \text{ (by FQP 2).}
\end{aligned}$$

It follows that $\delta_1 \sqcap \delta_2 \prec \delta$. □ □

Let (X,δ) be a fuzzy quasi-proximity space. For each $\lambda,\mu \in I^X$, we define $\delta^*(\lambda,\mu) = \delta \sqcap \delta^{-1}(\lambda,\mu)$. By the above theorem, we can easily prove that (X,δ^*) is a fuzzy proximity space.

Theorem 2.7. *Let $\lambda,\mu \in I^X$ be given in a fuzzy quasi-proximity space on (X,δ). Then for each $r \in (0,1]$, the followings are equivalent:*

(1) $\delta(\lambda,\mu) \geq r$,
(2) $\delta(c_{\delta^*}(\lambda,r), c_{\delta^*}(\mu,r)) \geq r$,
(3) $\delta(c_{\delta^{-1}}(\lambda,r), c_{\delta}(\mu,r)) \geq r$.

Proof. Since $\lambda \leq c_{\delta^*}(\lambda,r) \leq c_{\delta^{-1}}(\lambda,r)$ and $\mu \leq c_{\delta^*}(\mu,r) \leq c_{\delta}(\mu,r)$, we have (1) ⇒ (2) ⇒ (3) from the following:

$$\delta(\lambda,\mu) \leq \delta(c_{\delta^*}(\lambda,r), c_{\delta^*}(\mu,r)) \leq \delta(c_{\delta^{-1}}(\lambda,r), c_{\delta}(\mu,r)).$$

We will show that (3) ⇒ (1). Suppose that there exists $\lambda,\mu \in I^X$ such that

$$\delta(\lambda,\mu) < r \leq \delta(c_{\delta^{-1}}(\lambda,r), c_{\delta}(\mu,r)).$$

Since $\delta(\lambda,\mu) < r$, by (FQP3), there exists $\rho \in I^X$ such that

$$\delta(\lambda,\rho) < r, \quad \delta(\tilde{1}-\rho,\mu) < r.$$

Since $\delta(\tilde{1}-\rho,\mu) < r$, we have $c_\delta(\mu,r) \leq \rho$. Since $\delta(\lambda,\rho) < r$ and $c_\delta(\mu,r) \leq \rho$, we have $\delta(\lambda, c_\delta(\mu,r)) < r$. Again, since $\delta(\lambda, c_\delta(\mu,r)) < r$, there exists $\nu \in I^X$ such that

$$\delta(\lambda,\nu) < r, \quad \delta(\tilde{1}-\nu, c_\delta(\mu,r)) < r.$$

Now, since $\delta^{-1}(\nu,\lambda) = \delta(\lambda,\nu) < r$, we have $c_{\delta^{-1}}(\lambda,r) \leq \tilde{1}-\nu$. Hence $\delta(c_{\delta^{-1}}(\lambda,r), c_\delta(\mu,r)) < r$. It is a contradiction. □ □

Definition 2.8. Let (X,δ_1) and (Y,δ_2) be fuzzy quasi-proximity spaces. A function $f:(X,\delta_1)\to(Y,\delta_2)$ is a *fuzzy quasi-proximity map* if it satisfies $\delta_1(\mu,\nu) \leq \delta_2(f(\mu),f(\nu))$, for every $\mu,\nu \in I^X$.

Using the above definition, we can easily prove the following lemma.

Lemma 2.9. *If* $f : (X, \delta_1) \to (Y, \delta_2)$ *is a fuzzy quasi-proximity map, then:*

(a) $f : (X, \delta_1^{-1}) \to (Y, \delta_2^{-1})$ *is a fuzzy quasi-proximity map,*
(b) $f : (X, \delta_1^*) \to (Y, \delta_2^*)$ *is a fuzzy quasi-proximity map.*

We will investigate relationships between fuzzy topological spaces and fuzzy quasi-proximity spaces.

Theorem 2.10. *If a function* $f : (X, \delta_1) \to (Y, \delta_2)$ *is a fuzzy quasi-proximity map, then:*

(a) $f : (X, \mathcal{T}_{\delta_1}) \to (Y, \mathcal{T}_{\delta_2})$ *is a gp-map,*
(b) $f : (X, \mathcal{T}_{\delta_1^{-1}}) \to (Y, \mathcal{T}_{\delta_2^{-1}})$ *is a gp-map,*
(c) $f : (X, \mathcal{T}_{\delta_1^*}) \to (Y, \mathcal{T}_{\delta_2^*})$ *is a gp-map.*

Proof. (a). Suppose that f is not a gp-map. Then there exists $\lambda \in I^Y$ such that $\mathcal{T}_{\delta_2}(\lambda) > \mathcal{T}_{\delta_1}(f^{-1}(\lambda))$. Hence there exists $r \in I$ such that $\mathcal{T}_{\delta_2}(\lambda) > r > \mathcal{T}_{\delta_1}(f^{-1}(\lambda))$. Since $\mathcal{T}_{\delta_2}(\lambda) > r$, for some $c > r$, then

$$\lambda = i_{\delta_2}(\lambda, c) = \bigvee\{\rho \mid \delta_2(\rho, \tilde{1} - \lambda) < c\}.$$

Since f is a fuzzy quasi-proximity map, by Lemma 1.2, we have

$$\begin{aligned} f^{-1}(\lambda) &= \bigvee\{f^{-1}(\rho) \mid \delta_2(\rho, \tilde{1} - \lambda) < c\} \\ &\leq \bigvee\{f^{-1}(\rho) \mid \delta_1(f^{-1}(\rho), \tilde{1} - f^{-1}(\lambda)) < c\} \\ &\leq i_{\delta_1}(f^{-1}(\lambda), c). \end{aligned}$$

So, by Theorem 2.3 (ii), we have $i_{\delta_1}(f^{-1}(\lambda), c) = f^{-1}(\lambda)$. It follows that $\mathcal{T}_{\delta_1}(f^{-1}(\lambda)) \geq c > r$. It is a contradiction.

(b) and (c) are easy from Lemma 2.9 and (a). □ □

3. Initial fuzzy quasi-proximity structures

Now we will prove the existence of initial fuzzy quasi-proximity structures.

Definition 3.1. Let $(X_i, \delta_i)_{i\in\triangle}$ be a family of fuzzy quasi-proximity spaces. Let X be a set and, for each $i \in \triangle$, $f_i : X \to X_i$ a function. The *initial structure* δ is the coarsest fuzzy quasi-proximity on X with respect to which for each $i \in \triangle$, f_i is a fuzzy quasi-proximity map.

Theorem 3.2. *(Existence of initial structures) Let $(X_i, \delta_i)_{i\in\triangle}$ be a family of fuzzy quasi-proximity spaces. Let X be a set and, for each $i \in \triangle$, $f_i : X \to X_i$ a mapping. Define the function $\delta : I^X \times I^X \to I$ on X by*

$$\delta(\lambda, \mu) = inf\{\bigvee_{j,k} inf_{i\in\triangle}\delta_i(f_i(\lambda_j), f_i(\mu_k))\},$$

where for every finite families $(\lambda_j), (\mu_k)$ such that $\lambda = \bigvee_{j=1}^{n} \lambda_j$ and $\mu = \bigvee_{k=1}^{m} \mu_k$. Then:

1. *δ is the coarsest fuzzy quasi-proximity on X with respect to which for each $i \in \triangle$, f_i is a fuzzy quasi-proximity map.*
2. *A map $f : (Y, \delta') \to (X, \delta)$ is a fuzzy quasi-proximity map iff each $f_i \circ f : (Y, \delta') \to (X_i, \delta_i)$ is a fuzzy quasi-proximity map.*

Proof. (1). First, we will show that δ is a fuzzy quasi-proximity on X.

(FQP1). Since $\delta_i(f_i(\lambda), \tilde{0}) = 0$ for all $\lambda \in I^X$, it is clear.

(FQP2). For any $\lambda, \mu, \nu \in I^X$, we will show that

$$\delta(\lambda, \mu \vee \nu) \le \delta(\lambda, \mu) \vee \delta(\lambda, \nu).$$

Suppose that there exist $\lambda, \mu, \nu \in I^X$ such that

$$c = \delta(\lambda, \mu \vee \nu) > \delta(\lambda, \mu) \vee \delta(\lambda, \nu).$$

There are finite families $(\lambda_j), (\lambda'_m), (\mu_k)$ and (ν_l) such that $\lambda = \bigvee \lambda_j = \bigvee \lambda'_m$, $\mu = \bigvee \mu_k$ and $\nu = \bigvee \nu_l$ with

$$c > \bigvee_{j,k}(inf_{i\in\triangle}\delta_i(f_i(\lambda_j), f_i(\mu_k))), \quad c > \bigvee_{m,l}(inf_{i\in\triangle}\delta_i(f_i(\lambda'_m), f_i(\nu_l))).$$

It follows that $\lambda = \bigvee_{j,m}(\lambda_j \wedge \lambda'_m)$ and $\mu \vee \nu = (\bigvee \mu_k) \vee (\bigvee \nu_l)$.

Since $inf_{i\in\triangle}\delta_i(f_i(\lambda_j), f_i(\mu_k)) \geq inf_{i\in\triangle}\delta_i(f_i(\lambda_j \wedge \lambda'_m), f_i(\mu_k))$ and $inf_{i\in\triangle}\delta_i(f_i(\lambda'_m), f_i(\nu_l)) \geq inf_{i\in\triangle}\delta_i(f_i(\lambda_j \wedge \lambda'_m), f_i(\nu_l))$,

$$\begin{aligned} c &> (\bigvee_{j,k} inf_{i\in\triangle}\delta_i(f_i(\lambda_j), f_i(\mu_k))) \vee (\bigvee_{m,l} inf_{i\in\triangle}\delta_i(f_i(\lambda'_m), f_i(\nu_l))) \\ &\geq (\bigvee_{j,k} inf_{i\in\triangle}\delta_i(f_i(\lambda_j \wedge \lambda'_m), f_i(\mu_k))) \\ &\quad \vee (\bigvee_{m,l} inf_{i\in\triangle}\delta_i(f_i(\lambda_j \wedge \lambda'_m), f_i(\nu_l))) \\ &\geq \delta(\lambda, \mu \vee \nu) = c. \end{aligned}$$

It is a contradiction.

Similarly, we have $\delta(\lambda \vee \rho, \mu) \leq \delta(\lambda, \mu) \vee \delta(\rho, \mu)$.

(FQP3). If for any $\lambda, \mu \in I^X$, $\delta(\lambda, \mu) < r$, we will show that there exists $\rho \in I^X$ such that $\delta(\lambda, \rho) < r$ and $\delta(\tilde{1} - \rho, \mu) < r$.

If for any $\lambda, \mu \in I^X$, $\delta(\lambda, \mu) < r$, then there are finite families $(\lambda_j), (\mu_k)$ such that $\lambda = \bigvee_{j=1}^p \lambda_j$, $\mu = \bigvee_{k=1}^q \mu_k$ with

$$\delta(\lambda, \mu) \leq \bigvee_{j,k} inf_{i\in\triangle}(\delta_i(f_i(\lambda_j), f_i(\mu_k)) < r$$

i.e., for all j, k, $inf_{i\in\triangle}(\delta_i(f_i(\lambda_j), f_i(\mu_k)) < r$. It follows that for any j, k, there exists an $i_{jk} \in \triangle$ such that $\delta_{i_{jk}}(f_{i_{jk}}(\lambda_j), f_{i_{jk}}(\mu_k)) < r$. Since $\delta_{i_{jk}}$ is a fuzzy quasi-proximity on $X_{i_{jk}}$, by (FQP3), there exists $\rho_{jk} \in I^{X_{i_{jk}}}$ such that $\delta_{i_{jk}}(f_{i_{jk}}(\lambda_j), \rho_{jk})) < r$ and $\delta_{i_{jk}}(\tilde{1} - \rho_{jk}, f_{i_{jk}}(\mu_k)) < r$.

Put

$$\rho_j = \bigvee_{k=1}^q f_{i_{jk}}^{-1}(\rho_{jk}), \quad \rho = \bigwedge_{j=1}^p \rho_j.$$

Since $f_{i_{jk}}(f_{i_{jk}}^{-1}(\rho_{jk})) \leq \rho_{jk}$, then

$$\begin{aligned} \delta(\lambda_j, \rho_j) &\leq \bigvee_{k=1}^q \delta_{i_{jk}}(f_{i_{jk}}(\lambda_j), f_{i_{jk}}(f_{i_{jk}}^{-1}(\rho_{jk}))) \\ &\leq \bigvee_{k=1}^q \delta_{i_{jk}}(f_{i_{jk}}(\lambda_j), \rho_{jk}) < r. \end{aligned}$$

Using (FQP2) and Remark 1, we have $\delta(\lambda, \rho) < r$.

In a similar way, by the definition of δ, for all $k = 1, ..., q$,

$$\delta(f^{-1}_{i_{jk}}(\tilde{1} - \rho_{jk}), \mu_k) \leq \delta_{i_{jk}}(\tilde{1} - \rho_{jk}, \mu_k) < r,$$

because $f_{i_{jk}}(f^{-1}_{i_{jk}}(\tilde{1} - \rho_{jk})) \leq \tilde{1} - \rho_{jk}$. By Remark 1, we have $\delta(\tilde{1} - \rho_j, \mu_k) < r$. By (FQP2), we have $\delta(\tilde{1} - \rho, \mu_k) < r$ and $\delta(\tilde{1} - \rho, \mu) < r$.

(FQP4). We will show that if $\lambda \not\leq \tilde{1} - \mu$, then $\delta(\lambda, \mu) = 1$.

If $\lambda \not\leq \tilde{1} - \mu$, then, for every finite families $(\lambda_j), (\mu_k)$ such that $\lambda = \bigvee \lambda_j$ and $\mu = \bigvee \mu_k$, there exist j_0, k_0, x_0 such that $\lambda_{j_0}(x_0) + \mu_{k_0}(x_0) > 1$. It follows that, for all $i \in \triangle$,

$$f_i(\lambda_{j_0})(f_i(x_0)) + f_i(\mu_{k_0})(f_i(x_0)) \geq \lambda_{j_0}(x_0) + \mu_{k_0}(x_0) > 1.$$

Since for each $i \in \triangle$, δ_i is a fuzzy quasi-proximity on X_i, we have $\delta_i(\lambda_{j_0}, \mu_{k_0}) = 1$. Hence $\delta(\lambda, \mu) = 1$.

Second, from the definition of δ, since

$$\begin{aligned}\delta(\lambda, \mu) &= inf\{\bigvee_{j,k} inf_{i\in\triangle}\delta_i(f_i(\lambda_j), f_i(\mu_k))\}\\ &\leq inf\{\bigvee_{j,k} \delta_i(f_i(\lambda_j), f_i(\mu_k))\}\\ &= \delta_i(f_i(\lambda), f_i(\mu)) \text{ (by FQP 2)},\end{aligned}$$

for each $i \in \triangle$, $f_i : (X, \delta) \to (X_i, \delta_i)$ is a fuzzy quasi-proximity map.

If $f_i : (X, \delta') \to (X_i, \delta_i)$ is a fuzzy quasi-proximity map, then, for every $i \in \triangle$, since

$$\begin{aligned}\delta(\lambda, \mu) &= inf\{\bigvee_{j,k} inf_{i\in\triangle}\delta_i(f_i(\lambda_j), f_i(\mu_k))\}\\ &\geq inf\{\bigvee_{j,k} \delta'(\lambda_j, \mu_k)\}\\ &= \delta'(\lambda, \mu) \text{ (by FQP 2)},\end{aligned}$$

we have $\delta'(\lambda, \mu) \leq \delta(\lambda, \mu), \ \forall \lambda, \mu \in I^X$.

(2). Necessity of the composition condition is clear since the composition of fuzzy quasi-proximity maps is a fuzzy quasi-proximity map.

Conversely, suppose f is not a fuzzy quasi-proximity map. Then there exists $\lambda, \mu \in I^Y$ such that

$$\delta'(\lambda, \mu) > \delta(f(\lambda), f(\mu)).$$

Therefore there are finite families $(\lambda'_j), (\mu'_k)$ such that
$f(\lambda) = \bigvee_{j=1}^{p} \lambda'_j,\ f(\mu) = \bigvee_{k=1}^{q} \mu'_k$, and

$$\delta'(\lambda, \mu) > \bigvee_{j,k} inf_{i \in \triangle} \delta_i(f_i(\lambda'_j), f_i(\mu'_k)).$$

It follows that for any j, k, there exists an $i_{jk} \in \triangle$ such that

$$\delta_{i_{jk}}(f_{i_{jk}}(\lambda'_j), f_{i_{jk}}(\mu'_k)) < \delta'(\lambda, \mu).$$

On the other hand, $f_i \circ f$ is a fuzzy quasi-proximity map. For any j, k, by Lemma 1.2, since $f_i(f(f^{-1}(\lambda'_j))) \leq f_i(\lambda'_j)$,

$$\delta'(f^{-1}(\lambda'_j), f^{-1}(\mu'_k)) \leq \delta_{i_{jk}}(f_{i_{jk}}(\lambda'_j), f_{i_{jk}}(\mu'_k)).$$

Since $\lambda \leq f^{-1}(f(\lambda)) = \bigvee_{j=1}^{p} f^{-1}(\lambda'_j)$, we have

$$\begin{aligned}\delta'(\lambda, \mu) &\leq \bigvee_{j,k} \delta'(f^{-1}(\lambda'_j), f^{-1}(\mu'_k)) \text{ (by FQP 2 and Lemma 1.2)}\\ &\leq \bigvee_{j,k} \delta_{i_{jk}}(f_{i_{jk}}(\lambda'_j), f_{i_{jk}}(\mu'_k))\\ &< \delta'(\lambda, \mu).\end{aligned}$$

It is a contradiction. □ □

By the above theorem, we can define subspaces and products in the obvious way.

Definition 3.3. Let (X, δ) be a fuzzy quasi-proximity and A be a subset of X. The pair (A, δ_A) is said to be a *subspace* of (X, δ) if it is endowed with the initial fuzzy quasi-proximity structure with respect to the inclusion map.

Definition 3.4. Let X be the product $\prod_{i\in\triangle} X_i$ of the family $\{(X_i, \delta_i) \mid i \in \triangle\}$ of fuzzy quasi-proximity spaces. An initial fuzzy quasi-proximity structure $\delta = \otimes\delta_i$ on X with respect to all the projections $\pi_i : X \to X_i$ is called the *product fuzzy quasi-proximity structure* of $\{\delta_i \mid i \in \triangle\}$, and $(X, \otimes\delta_i)$ is called the *product fuzzy quasi-proximity space.*

Using Theorem 3.2, we have the following corollary.

Corollary 3.5. *Let $(X_i, \delta_i)_{i\in\triangle}$ be a family of fuzzy quasi-proximity spaces. Let $X = \prod_{i\in\triangle} X_i$ be a set and, for each $i \in \triangle$, $\pi_i : X \to X_i$ a mapping. The structure $\delta = \otimes\delta_i$ on X is defined by*

$$\delta(\lambda, \mu) = inf\{\bigvee_{j,k} inf_{i\in\triangle}\delta_i(\pi_i(\lambda_j), \pi_i(\mu_k))\},$$

where for every finite families $(\lambda_j), (\mu_k)$ such that $\lambda = \bigvee_{j=1}^{n} \lambda_j$ and $\mu = \bigvee_{k=1}^{m} \mu_k$. Then:

(1) *δ is the coarsest fuzzy quasi-proximity on X with respect to which for each $i \in \triangle$, π_i is a fuzzy quasi-proximity map.*
(2) *A map $f : (Y, \delta') \to (X, \delta)$ is a fuzzy quasi-proximity map iff each $\pi_i \circ f : (Y, \delta') \to (X_i, \delta_i)$ is a fuzzy quasi-proximity map.*

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