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ON THE NORMAL BUNDLE OF A SUBMANIFOLD IN A KÄHLER MANIFOLD

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Abstract. We show that the normal bundle of a Lagrangian submanifold in a Kähler manifold has a symplectic structure and provide the equivalent conditions for the normal bundle of such to be Kähler.

1. Preliminaries

We consider a submanifold $\tilde{M}$ of a Riemannian manifold (M^{2n}, g). A Riemannian metric G is induced on $\tilde{M}$ and there is also a metric $G^{\perp}$ induced on each fiber of the normal bundle $N\tilde{M}$ of $\tilde{M}$. We call by D the Riemannian connection of $(\tilde{M}, G)$. The normal connection $D^{\perp}$ and its curvature tensor $R^{\perp}$ are defined as usual (in the sense of [3]). It is well known (Refer to [1]) that on the normal bundle $N\tilde{M}$, there is a naturally induced metric $\tilde{g}$, called the Sasaki metric. This metric structure was determined, by Recziegel [4], in an invariant manner:

$$\tilde{g}(\tilde{X}, \tilde{Y}) = G(\pi_* \tilde{X}, \pi_* \tilde{Y}) + G^{\perp}(K\tilde{X}, K\tilde{Y})$$

where π_* is the differential of the projection map $\pi : N\tilde{M} \to \tilde{M}$ of the normal bundle $N\tilde{M}$ and $K : TN\tilde{M} \to N\tilde{M}$ is the connection map. Note that both the mappings π_* and K are onto and fiber-preserving linear transformations.

We call the kernels of the mappings π_* and K the vertical subspace $VN\tilde{M}$ and the horizontal subspace $HN\tilde{M}$, respectively. Then, the

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vertical subspace and the horizontal subspace are orthogonal in the Sasaki metric and the vertical space is tangent to the fiber. Moreover, there is a decomposition:

$$T_{\tilde{x}}N\tilde{M} = H_{\tilde{x}}N\tilde{M} \oplus V_{\tilde{x}}N\tilde{M}$$

for each $\tilde{x} = (x, \xi) \in N\tilde{M}$.

Given a tangent vector field X and a normal vector field η to $\tilde{M}$, there are defined the horizontal lift $X^H \in HN\tilde{M}$ and the vertical lift $\eta^V \in VN\tilde{M}$ such that

$$\pi_* X^H = X, KX^H = 0, \pi_* \eta^V = 0, \text{ and } K\eta^V = \eta.$$

Thus, we have that at the point $\tilde{x} = (x, \xi)$

$$\begin{aligned} \tilde{g}(X^H, Y^H)_{\tilde{x}} &= G(\pi_* X^H, \pi_* Y^H)_x = G(X, Y)_x \\ \tilde{g}(X^H, \eta^V)_{\tilde{x}} &= G(\pi_* X^H, \pi_* \eta^V)_x + G^{\perp}(KX^H, K\eta^V)_{\xi} = 0 \qquad (1) \\ \tilde{g}(\eta^V, \zeta^V)_{\tilde{x}} &= G^{\perp}(K\eta^V, K\zeta^V)_{\xi} = G^{\perp}(\eta^V, \zeta^V)_{\xi} \end{aligned}$$

Now, since we can write

$$\tilde{X} = (\pi_* \tilde{X})^H + (K\tilde{X})^V$$

for a tangent vector $\tilde{X}$ to $N\tilde{M}$, it is enough to consider various combinations of horizontally and vertically lifted vector fields.

We will need the following lemmas. Proofs are routine and we omit them here.

Lemma 1.1 ([1]). *Let X and Y be tangent vector fields, and η and ζ normal vector fields of $\tilde{M}$. Then, at each point (x, ξ) of the normal bundle $N\tilde{M}$, the Lie brackets are:*

$$[\,\eta^V, \zeta^V] = 0, \qquad [X^H, \eta^V] = (D^{\perp}_X \eta)^V,$$
$$\pi_*[X^H, Y^H] = [X, Y], \qquad K[X^H, Y^H] = -\,R^{\perp}_{XY}\xi.$$

By definition, $R^{\perp}_{XY}\eta$ is a normal vector field of $\tilde{M}$. For any normal vector field ζ, we may compute the inner product $g(R^{\perp}_{XY}\eta, \zeta)$. We define the adjoint $\hat{R}_{\eta\zeta}X$ by the equality $G(\hat{R}_{\eta\zeta}X, Y) = g(R^{\perp}_{XY}\eta, \zeta)$. The covariant derivatives with respect to the Riemannian connection $\tilde{\nabla}$ of the Sasaki metric $\tilde{g}$ on $N\tilde{M}$ are easily computed. And, we have

Lemma 1.2 ([1]). *Let X and Y be tangent vector fields, and η and ζ normal vector fields of $\tilde{M}$. Then, at each point (x, ξ) of the normal bundle $N\tilde{M}$,*

$$\tilde{\nabla}_{\eta^V}\zeta^V = 0\,, \qquad \tilde{\nabla}_{X^H}\zeta^V = (D^{\perp}_X\zeta)^V + \frac{1}{2}(\hat{R}_{\xi\zeta}X)^H\,,$$
$$\tilde{\nabla}_{\eta^V}Y^H = \frac{1}{2}(\hat{R}_{\xi\eta}Y)^H\,, \qquad \tilde{\nabla}_{X^H}Y^H = (D_XY)^H - \frac{1}{2}(R^{\perp}_{XY}\xi)^V\,.$$

2. Main results

Let us consider a Lagrangian submanifold L of a Kähler manifold (M^{2n}, J, g). On the normal bundle NL of L, we define $\tilde{J}$ by

$$\tilde{J}X^H = (JX)^V$$

and

$$\tilde{J}\xi^V = (J\xi)^H$$

for any tangent vector field X and normal vector field ξ to L. Then, it is easy to see that $\tilde{J}$ is an almost complex structure on NL.

Thus, we have

Proposition 2.1. *$(NL, \tilde{J}, \tilde{g})$ is an almost Hermitian manifold.*

Proof. By the argument above, it remains to show that $\tilde{J}$ is compatible with the Sasaki metric $\tilde{g}$ on NL. And, this follows immediately from the compatibility of J with g and (1). □ □

Let us denote by $\tilde{\nabla}$ and ∇ the Riemannian connection of $\tilde{g}$ and g, respectively. Let G be the induced metric on L, D its Riemannian connection, and $\underline{R}$ the curvature tensor on L.

We now present our main theorems.

Theorem 2.2. *Let L be a Lagrangian submanifold of a Kähler manifold (M^{2n}, J, g). Then, $(NL, \tilde{J}, \tilde{g})$ is a symplectic manifold.*

Proof. By Proposition 2.1, it remains to show that the fundamental 2-form Ω defined by $\Omega(\tilde{X}, \tilde{Y}) = \tilde{g}(\tilde{X}, \tilde{J}\tilde{Y})$ is non-degenerate and closed. The non-degeneracy of Ω is immediate since $\tilde{g}$ is positive definite and $\tilde{J}$ is non-singular at each point.

In order to show that Ω is closed, we first observe that

$$\begin{aligned} \Omega(X^H, Y^H) &= \tilde{g}(X^H, \tilde{J}Y^H) \\ &= g(\pi_* X^H, \pi_* \tilde{J}Y^H) + g(KX^H, K\tilde{J}Y^H) \\ &= 0\,, \end{aligned} \tag{2}$$

$$\begin{aligned} \Omega(X^H, \eta^V) &= \tilde{g}(X^H, \tilde{J}\eta^V) \\ &= g(\pi_* X^H, \pi_* \tilde{J}\eta^V) + g(KX^H, K\tilde{J}\eta^V) \\ &= g(X, J\eta), \end{aligned} \tag{3}$$

and likewise,

$$\Omega(\eta^V, \zeta^V) = 0. \tag{4}$$

We recall here the coboundary formula

$$\begin{aligned} 3d\Phi(X, Y, Z) = X\Phi(Y, Z) + Y\Phi(Z, X) + Z\Phi(X, Y) \\ - \Phi([X, Y], Z) - \Phi([Y, Z], X) - \Phi([Z, X], Y). \end{aligned}$$

Using Lemma 1.1 and (2) - (4), we compute

$$\begin{aligned} 3d\Omega(X^H, Y^H, Z^H) &= -\Omega([X^H, Y^H], Z^H) - \Omega([Y^H, Z^H], X^H) \\ &\quad - \Omega([Z^H, X^H], Y^H) \\ &= \Omega((R^{\perp}_{XY}\xi)^V, Z^H) + \Omega((R^{\perp}_{YZ}\xi)^V, X^H) \\ &\quad + \Omega((R^{\perp}_{ZX}\xi)^V, Y^H) \end{aligned}$$

But, from the Gauss-Weingarten equations, we have

$$\nabla_X J\xi = D_X J\xi + \sigma(X, J\xi)$$

and

$$J\nabla_X \xi = -JA_\xi X + JD^\perp_X \xi \,.$$

We compare tangential parts of these using Kähler condition and get

$$JD^\perp_X \xi = D_X J\xi \,. \tag{5}$$

Continuing our computation, using this and the Bianchi identity, we get

$$\begin{aligned} 3d\Omega(X^H, Y^H, Z^H) &= -\, g(\underline{R}_{XY} J\xi, Z) - g(\underline{R}_{YZ} J\xi, X) - g(\underline{R}_{ZX} J\xi, Y) \\ &= g(\underline{R}_{XY} Z, J\xi) + g(\underline{R}_{YZ} X, J\xi) + g(\underline{R}_{ZX} Y, J\xi) \\ &= 0 \end{aligned} \tag{6}$$

Likewise, we compute, using Lemma 1.2 and (2) - (4),

$$\begin{aligned} 3d\Omega(X^H, Y^H, \eta^V) &= X^H \Omega(Y^H, \eta^V) + Y^H \Omega(\eta^V, X^H) \\ &\quad - \Omega([X^H, Y^H], \eta^V) - \Omega([Y^H, \eta^V], X^H) \\ &\quad - \Omega([\eta^V, X^H], Y^H) \\ &= \tilde{g}(\tilde{\nabla}_{X^H} Y^H, (J\eta^V)^H) + \tilde{g}(Y^H, \tilde{\nabla}_{X^H} (J\eta)^H) \\ &\quad + \tilde{g}(\tilde{\nabla}_{Y^H} \eta^V, (JX)^V) + \tilde{g}(\eta^V, \tilde{\nabla}_{Y^H} (JX)^V) \\ &\quad - \Omega([X,Y]^H - (R^\perp_{XY}\xi)^V, \eta^V) \\ &\quad - \Omega((D^\perp_Y \eta)^V, X^H) + \Omega((D^\perp_X \eta)^V, Y^H) \\ &= \tilde{g}((D_X Y)^H, (J\eta)^H) + \tilde{g}(Y^H, (D_X J\eta)^H) \\ &\quad + \tilde{g}((D^\perp_Y \eta)^V, (JX)^V) + \tilde{g}(\eta^V, (D^\perp_Y JX)^V) \\ &\quad - \Omega([X,Y]^H, \eta^V) - \Omega((D^\perp_Y \eta)^V, X^H) \\ &\quad + \Omega((D^\perp_X \eta)^V, Y^H) \\ &= \tilde{g}((D_X Y)^H, (J\eta)^H) + \tilde{g}(\eta^V, (D^\perp_Y JX)^V) \\ &\quad - \tilde{g}([X,Y]^H, (J\eta)^H) \\ &= \tilde{g}((D_X Y)^H, \tilde{J}\eta^V) + \tilde{g}(\tilde{J}\eta^V, -(D_X Y)^H) \\ &\quad - \tilde{g}([X,Y]^H, \tilde{J}\eta^V) \\ &= 0 \end{aligned} \tag{7}$$

and

$$\begin{aligned}3d\Omega(X^H,\eta^V,\zeta^V) &= \eta^V\Omega(\zeta^V,X^H)+\zeta^V\Omega(X^H,\eta^V)\\ &\quad -\Omega([X^H,\eta^V],\zeta^V)-\Omega([\eta^V,\zeta^V],X^H)\\ &\quad -\Omega([\zeta^V,X^H],\eta^V)\\ &= \tilde{g}(\tilde{\nabla}_{\eta^V}\zeta^V,\tilde{J}X^H)+\tilde{g}(\zeta^V,\tilde{\nabla}_{\eta^V}(JX)^V)\\ &\quad +\tilde{g}(\tilde{\nabla}_{\zeta^V}X^H,(J\eta)^H)+\tilde{g}(X^H,\tilde{\nabla}_{\zeta^V}(J\eta)^H)\\ &\quad -\Omega((D^{\perp}_X\eta)^V,\zeta^V)+\Omega((D^{\perp}_X\zeta)^V,\eta^V)\\ &= \frac{1}{2}\tilde{g}((\hat{R}_{\xi\zeta}X)^H,(J\eta)^H)+\frac{1}{2}\tilde{g}((\hat{R}_{\xi\zeta}J\eta)^H,X^H)\\ &= 0 \end{aligned} \tag{8}$$

Finally,

$$3d\Omega(\eta^V,\zeta^V,\delta^V)=0 \tag{9}$$

is trivial. This completes our proof. □ □

THEOREM 2.3. *Let L be a Lagrangian submanifold of a Kähler manifold (M^{2n},J,g). Then, the followings are equivalent:*

(1) *NL is Kähler.*
(2) *L has flat normal connection.*
(3) *L is flat.*

Proof. We compute the Nijenhuis torsion.

$$\begin{aligned}[\tilde{J},\tilde{J}](X^H,Y^H) &= -[X^H,Y^H]+[\tilde{J}X^H,\tilde{J}Y^H]\\ &\quad -\tilde{J}[(JX)^V,Y^H]-\tilde{J}[X^H,(JY)^V]\\ &= -[X^H,Y^H]+(JD^{\perp}_YJX)^H-(JD^{\perp}_XJY)^H\\ &= -[X,Y]^H+(R^{\perp}_{XY}\xi)^V+(JD^{\perp}_YJX-JD^{\perp}_XJY)^H\end{aligned}$$

So, using (5), we have

$$\begin{aligned}[\tilde{J},\tilde{J}](X^H,Y^H) &= -[X^H,Y^H]+(R^{\perp}_{XY}\xi)^V-(D_YX-D_XY)^H\\ &= -[X,Y]^H+(R^{\perp}_{XY}\xi)^V-[Y,X]^H\\ &= (R^{\perp}_{XY}\xi)^V\end{aligned} \tag{10}$$

$$\begin{aligned}
[\tilde{J},\tilde{J}](X^H,\zeta^V) &= -[X^H,\zeta^V]+[(JX)^V,(J\zeta)^H] \\
&\quad -\tilde{J}[(JX)^V,\zeta^V]-\tilde{J}[X^H,(J\zeta)^H] \\
&= -(D^\perp_X\zeta)^V-(D^\perp_{J\zeta}JX)^V-\tilde{J}([X,J\zeta]^H-(R^\perp_{XJ\zeta}\xi)^V) \\
&= -(D^\perp_X\zeta)^V-(D^\perp_{J\zeta}JX)^V-(J[X,J\zeta])^V+(JR^\perp_{XJ\zeta}\xi)^H \\
&= -(\nabla_X\zeta)^V-(A_\zeta X)^H-(\nabla_{J\zeta}JX)^V-(A_{JX}J\zeta)^V \\
&\quad -(J\nabla_X J\zeta)^V+(J\nabla_{J\zeta}X)^V+(JR^\perp_{XJ\zeta}\xi)^H \\
&= -(\nabla_X\zeta)^V-(A_\zeta X)^H-(\nabla_{J\zeta}JX)^V-(A_{JX}J\zeta)^V \\
&\quad +(\nabla_X\zeta)^V+(\nabla_{J\zeta}JX)^V+(JR^\perp_{XJ\zeta}\xi)^H \\
&= -(A_\zeta X)^V-(A_{JX}J\zeta)^V+(JR^\perp_{XJ\zeta}\xi)^H \qquad (11)
\end{aligned}$$

Again, using (5) and the Kähler condition, we see that

$$\begin{aligned}
J[X,J\zeta] &= -\nabla_X\zeta-\nabla_{J\zeta}JX \\
&= -D^\perp_X\zeta-D^\perp_{J\zeta}JX+A_\zeta X+A_{JX}J\zeta.
\end{aligned}$$

But, since $J[X,J\zeta]$ is normal to L, we have

$$A_\zeta X+A_{JX}J\zeta=0.$$

Thus, we see, from (11), that

$$\begin{aligned}
[\tilde{J},\tilde{J}](X^H,\zeta^V) &= (JR^\perp_{XJ\zeta}\xi)^H \\
&= (\underline{R}_{XJ\zeta}J\xi)^H \qquad (12)
\end{aligned}$$

and

$$\begin{aligned}
[\tilde{J},\tilde{J}](\eta^V,\zeta^V) &= -[\eta^V,\zeta^V]+[(J\eta)^H,(J\zeta)^H] \\
&\quad -\tilde{J}[(J\eta)^H,\zeta^V]-\tilde{J}[\eta^V,(J\zeta)^H] \\
&= [(J\eta)^H,(J\zeta)^H]-\tilde{J}(D^\perp_{J\eta}\zeta)^V+\tilde{J}(D^\perp_{J\zeta}\eta)^V \\
&= [J\eta,J\zeta]^H-(R^\perp_{J\eta J\zeta}\xi)^V-(JD^\perp_{J\eta}\zeta)^H+(JD^\perp_{J\zeta}\eta)^H \\
&= [J\eta,J\zeta]^H-(R^\perp_{J\eta J\zeta}\xi)^V-(D_{J\eta}J\zeta)^H+(D_{J\zeta}J\eta)^V \\
&= -(R^\perp_{J\eta J\zeta}\xi)^V \qquad (13)
\end{aligned}$$

In view of (10), (12), and (13), we conclude that $[\tilde{J}, \tilde{J}]$ vanishes if and only if L has flat normal connection.

This together with the equations (6) - (9) shows that NL is Kähler if and only if L has flat normal connection.

Moreover, using (11), we have

$$R_{XY} J\xi = J R^{\perp}_{XY} \xi,$$

from which we easily see that L is flat if and only if NL has flat normal connection. □ □

In the proof of the previous theorem, we have also shown

Corollary 2.4. *$\tilde{J}$ on NL is integrable if and only if L has flat connection.*

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