ON THE NORMAL BUNDLE OF A SUBMANIFOLD IN A KÄHLER MANIFOLD

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Abstract. We show that the normal bundle of a Lagrangian submanifold in a Kähler manifold has a symplectic structure and provide the equivalent conditions for the normal bundle of such to be Kähler.

1. Preliminaries

We consider a submanifold $\tilde{M}$ of a Riemannian manifold $(M^{2n}, g)$. A Riemannian metric $G$ is induced on $\tilde{M}$ and there is also a metric $G^\perp$ induced on each fiber of the normal bundle $N\tilde{M}$ of $\tilde{M}$. We call by $D$ the Riemannian connection of $(\tilde{M}, G)$. The normal connection $D^\perp$ and its curvature tensor $R^\perp$ are defined as usual (in the sense of [3]). It is well known (Refer to [1]) that on the normal bundle $N\tilde{M}$, there is a naturally induced metric $\tilde{g}$, called the Sasaki metric. This metric structure was determined, by Recziegel [4], in an invariant manner:

$$\tilde{g}(\tilde{X}, \tilde{Y}) = G(\pi_*\tilde{X}, \pi_*\tilde{Y}) + G^\perp(K\tilde{X}, K\tilde{Y})$$

where $\pi_*$ is the differential of the projection map $\pi : N\tilde{M} \to \tilde{M}$ of the normal bundle $N\tilde{M}$ and $K : TN\tilde{M} \to N\tilde{M}$ is the connection map. Note that both the mappings $\pi_*$ and $K$ are onto and fiber-preserving linear transformations.

We call the kernels of the mappings $\pi_*$ and $K$ the vertical subspace $VNM$ and the horizontal subspace $HN\tilde{M}$, respectively. Then, the
vertical subspace and the horizontal subspace are orthogonal in the Sasaki metric and the vertical space is tangent to the fiber. Moreover, there is a decomposition:

$T_{\tilde{x}}N\tilde{M} = H_{\tilde{x}}N\tilde{M} \oplus V_{\tilde{x}}N\tilde{M}$

for each $\tilde{x} = (x, \xi) \in N\tilde{M}$.

Given a tangent vector field $X$ and a normal vector field $\eta$ to $\tilde{M}$, there are defined the horizontal lift $X^H \in HN\tilde{M}$ and the vertical lift $\eta^V \in VN\tilde{M}$ such that

$$\pi^*X^H = X, \quad KX^H = 0, \quad \pi^*\eta^V = 0, \quad K\eta^V = \eta.$$  

Thus, we have that at the point $\tilde{x} = (x, \xi)$

$$\tilde{g}(X^H, Y^H)_{\tilde{x}} = G(\pi_*X^H, \pi_*Y^H)_x = G(X, Y)_x$$

(1) $$\tilde{g}(X^H, \eta^V)_x = G(\pi_*X^H, \pi_*\eta^V)_x + G^\perp(KX^H, K\eta^V)_x = 0$$

Now, since we can write $\tilde{X} = (\pi_*X)^H + (K\tilde{X})^V$ for a tangent vector $\tilde{X}$ to $N\tilde{M}$, it is enough to consider various combinations of horizontally and vertically lifted vector fields.

We will need the following lemmas. Proofs are routine and we omit them here.

**Lemma 1.1** ([1]). Let $X$ and $Y$ be tangent vector fields, and $\eta$ and $\zeta$ normal vector fields of $\tilde{M}$. Then, at each point $(x, \xi)$ of the normal bundle $N\tilde{M}$, the Lie brackets are:

$$[\eta^V, \zeta^V] = 0, \quad [X^H, \eta^V] = (D_X^\perp \eta)^V, \quad \pi_*[X^H, Y^H] = [X, Y], \quad K[X^H, Y^H] = -R^X_{XY}\xi.$$  

By definition, $R^X_{XY}\eta$ is a normal vector field of $\tilde{M}$. For any normal vector field $\zeta$, we may compute the inner product $g(R^X_{XY}\eta, \zeta)$. We define the adjoint $\tilde{R}_{\eta\zeta}X$ by the equality $G(\tilde{R}_{\eta\zeta}X, Y) = g(R^X_{XY}\eta, \zeta)$. The covariant derivatives with respect to the Riemannian connection $\nabla$ of the Sasaki metric $\tilde{g}$ on $N\tilde{M}$ are easily computed. And, we have
Lemma 1.2 ([1]). Let $X$ and $Y$ be tangent vector fields, and $\eta$ and $\zeta$ normal vector fields of $\tilde{M}$. Then, at each point $(x, \xi)$ of the normal bundle $N\tilde{M}$,

\[
\tilde{\nabla}_\eta V \zeta^V = 0, \quad \tilde{\nabla}_{X \eta} \zeta^V = (D_{X\eta} \zeta)^V + \frac{1}{2}(\tilde{R}_{\xi \eta} X)^H, \\
\tilde{\nabla}_\eta Y^H = \frac{1}{2}(\tilde{R}_{\xi \eta} Y)^H, \quad \tilde{\nabla}_{X \eta} Y^H = (D_X Y)^H - \frac{1}{2}(R_{X \eta} X)^V.
\]

2. Main results

Let us consider a Lagrangian submanifold $L$ of a Kähler manifold $(M^{2n}, J, g)$. On the normal bundle $NL$ of $L$, we define $\tilde{J}$ by

\[
\tilde{J}X^H = (JX)^V
\]

and

\[
\tilde{J}\xi^V = (J\xi)^H
\]

for any tangent vector field $X$ and normal vector field $\xi$ to $L$. Then, it is easy to see that $\tilde{J}$ is an almost complex structure on $NL$.

Thus, we have

Proposition 2.1. $(NL, \tilde{J}, \tilde{g})$ is an almost Hermitian manifold.

Proof. By the argument above, it remains to show that $\tilde{J}$ is compatible with the Sasaki metric $\tilde{g}$ on $NL$. And, this follows immediately from the compatibility of $J$ with $g$ and (1). □ □

Let us denote by $\tilde{\nabla}$ and $\nabla$ the Riemannian connection of $\tilde{g}$ and $g$, respectively. Let $G$ be the induced metric on $L$, $D$ its Riemannian connection, and $\widetilde{R}$ the curvature tensor on $L$.

We now present our main theorems.

Theorem 2.2. Let $L$ be a Lagrangian submanifold of a Kähler manifold $(M^{2n}, J, g)$. Then, $(NL, \tilde{J}, \tilde{g})$ is a symplectic manifold.
Proof. By Proposition 2.1, it remains to show that the fundamental 2-form $\Omega$ defined by $\Omega(\tilde{X}, \tilde{Y}) = \tilde{g}(\tilde{X}, \tilde{J}\tilde{Y})$ is non-degenerate and closed. The non-degeneracy of $\Omega$ is immediate since $\tilde{g}$ is positive definite and $\tilde{J}$ is non-singular at each point.

In order to show that $\Omega$ is closed, we first observe that

$$\Omega(X^H, Y^H) = \tilde{g}(X^H, \tilde{J}Y^H) = g(\pi_*X^H, \pi_*\tilde{J}Y^H) + g(KX^H, K\tilde{J}Y^H) = 0,$$  

$$\Omega(X^H, \eta^V) = \tilde{g}(X^H, \tilde{J}\eta^V) = g(\pi_*X^H, \pi_*\tilde{J}\eta^V) + g(KX^H, K\tilde{J}\eta^V) = g(X, J\eta),$$

and likewise,

$$\Omega(\eta^V, \zeta^V) = 0.$$

We recall here the coboundary formula

$$3d\Phi(X, Y, Z) = X\Phi(Y, Z) + Y\Phi(Z, X) + Z\Phi(X, Y) - \Phi([X, Y], Z) - \Phi([Y, Z], X) - \Phi([Z, X], Y).$$

Using Lemma 1.1 and (2) - (4), we compute

$$3d\Omega(X^H, Y^H, Z^H) = -\Omega([X^H, Y^H], Z^H) - \Omega([Y^H, Z^H], X^H) - \Omega([Z^H, X^H], Y^H)$$

$$= \Omega((R_{XY}^\perp \xi)^V, Z^H) + \Omega((R_{YZ}^\perp \xi)^V, X^H) + \Omega((R_{ZX}^\perp \xi)^V, Y^H)$$

But, from the Gauss-Weingarten equations, we have

$$\nabla_X J\xi = D_X J\xi + \sigma(X, J\xi)$$
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and

\[ J\nabla_X \xi = -JA_\xi X + JD_X^\perp \xi. \]

We compare tangential parts of these using Kähler condition and get

(5) \[ JD_X^\perp \xi = D_X J\xi. \]

Continuing our computation, using this and the Bianchi identity, we get

\begin{align*}
3d\Omega(X^H, Y^H, Z^H) &= -g(R_{XY} J\xi, Z) - g(R_{YZ} J\xi, X) - g(R_{ZX} J\xi, Y) \\
&= g(R_{XY} Z, J\xi) + g(R_{YZ} X, J\xi) + g(R_{ZX} Y, J\xi)
\end{align*}

(6)

Likewise, we compute, using Lemma 1.2 and (2) - (4),

\begin{align*}
3d\Omega(X^H, Y^H, \eta^V) &= X^H \Omega(Y^H, \eta^V) + Y^H \Omega(\eta^V, X^H) \\
&\quad - \Omega([X^H, Y^H], \eta^V) - \Omega([Y^H, \eta^V], X^H) \\
&\quad - \Omega([\eta^V, X^H], Y^H) \\
&= \tilde{g}(\tilde{\nabla}_{X^H} Y^H, (J\eta^V)^H) + \tilde{g}(Y^H, \tilde{\nabla}_{X^H} (J\eta)^H) \\
&\quad + \tilde{g}(\tilde{\nabla}_{Y^H} \eta^V, (JX)^V) + \tilde{g}(\eta^V, \tilde{\nabla}_{Y^H} (JX)^V) \\
&\quad - \Omega([X, Y]^H - (R^\perp_X \xi)^V, \eta^V) \\
&\quad - \Omega((D^\perp_X \eta)^V, X^H) + \Omega((D^\perp_X \eta)^V, Y^H) \\
&= \tilde{g}((D_X Y)^H, (J\eta)^H) + \tilde{g}(\eta^V, (D_X J\eta)^H) \\
&\quad + \tilde{g}((D^\perp_X \eta)^V, (JX)^V) + \tilde{g}(\eta^V, (D^\perp_X JX)^V) \\
&\quad - \Omega([X, Y]^H, \eta^V) - \Omega((D^\perp_X \eta)^V, X^H) \\
&\quad + \Omega((D^\perp_X \eta)^V, Y^H) \\
&= \tilde{g}((D_X Y)^H, (J\eta)^H) + \tilde{g}(\eta^V, (D^\perp_X JX)^V) \\
&\quad - \tilde{g}([X, Y]^H, (J\eta)^H) \\
&= \tilde{g}((D_X Y)^H, \tilde{J}\eta^V) + \tilde{g}(\tilde{J}\eta^V, -(D_X Y)^H) \\
&\quad - \tilde{g}([X, Y]^H, \tilde{J}\eta^V)
\end{align*}

(7) \[ = 0 \]
and
\[3d\Omega(X^H, \eta^V, \zeta^V) = \eta^V \Omega(\zeta^V, X^H) + \zeta^V \Omega(X^H, \eta^V) \]
\[-\Omega([X^H, \eta^V], \zeta^V) - \Omega([\eta^V, \zeta^V], X^H) \]
\[-\Omega([\zeta^V, X^H], \eta^V) \]
\[= \tilde{g}(\tilde{\nabla}_{\eta^V} \zeta^V, \tilde{J}X^H) + \tilde{g}(\zeta^V, \tilde{\nabla}_{\eta^V} (JX)^V) \]
\[+ \tilde{g}(\tilde{\nabla}_{\zeta^V} X^H, (J\eta)^H) + \tilde{g}(X^H, \tilde{\nabla}_{\zeta^V} (J\eta)^H) \]
\[-\Omega((D^H_X \eta)^V, \zeta^V) + \Omega((D^H_X \zeta)^V, \eta^V) \]
\[= \frac{1}{2} \tilde{g}((\tilde{R}_{\xi \zeta} X)^H, (J\eta)^H) + \frac{1}{2} \tilde{g}((\tilde{R}_{\xi \zeta} J\eta)^H, X^H) \]
\[= 0 \quad (8)\]

Finally,
\[3d\Omega(\eta^V, \zeta^V, \delta^V) = 0 \quad (9)\]
is trivial. This completes our proof.

**Theorem 2.3.** Let \(L\) be a Lagrangian submanifold of a Kähler manifold \((M^{2n}, J, g)\). Then, the followings are equivalent:

1. \(NL\) is Kähler.
2. \(L\) has flat normal connection.
3. \(L\) is flat.

**Proof.** We compute the Nijenhuis torsion.

\([\tilde{J}, \tilde{J}](X^H, Y^H) = - [X^H, Y^H] + [\tilde{J}X^H, \tilde{J}Y^H] \]
\[- \tilde{J}[(JX)^V, Y^H] - \tilde{J}[X^H, (JY)^V] \]
\[= - [X^H, Y^H] + (JD^H_Y X - JD^H_X Y)^H \]
\[= - [X, Y]^H + (R_{XY}^Y \xi)^V + (JD^H_Y JX - JD^H_X JY)^H \]

So, using (5), we have

\([\tilde{J}, \tilde{J}](X^H, Y^H) = - [X^H, Y^H] + (R_{XY}^Y \xi)^V - (D_Y X - D_X Y)^H \]
\[= - [X, Y]^H + (R_{XY}^Y \xi)^V - [Y, X]^H \]
\[= (R_{XY}^Y \xi)^V \quad (10)\]
\[ [\tilde{J}, \tilde{J}](X^H, \zeta^V) = - [X^H, \zeta^V] + [(JX)^V, (J\zeta)^H] \]
\[ - \tilde{J}[(JX)^V, \zeta^V] - \tilde{J}[X^H, (J\zeta)^H] \]
\[ = - (D_{\tilde{X}}\zeta)^V - (D_{\tilde{J}\zeta}JX)^V - \tilde{J}([X, J\zeta]^H - (R_{X, J\zeta}^\perp \xi)^V) \]
\[ = - (D_{\tilde{X}}\zeta)^V - (D_{\tilde{J}\zeta}JX)^V - (J[X, J\zeta])^V + (JR_{X, J\zeta}^\perp \xi)^H \]
\[ = - (\nabla_X\zeta)^V - (A\zeta X)^H - (\nabla_{J\zeta}JX)^V - (A_{JX} J\zeta)^V \]
\[ - (J\nabla_X J\zeta)^V + (J\nabla_{J\zeta}X)^V + (JR_{X, J\zeta}^\perp \xi)^H \]
\[ = - (\nabla_X\zeta)^V - (A\zeta X)^H - (\nabla_{J\zeta}JX)^V - (A_{JX} J\zeta)^V \]
\[ + (\nabla_X\zeta)^V + (\nabla_{J\zeta}JX)^V + (JR_{X, J\zeta}^\perp \xi)^H \]
\[ = - (A\zeta X)^V - (A_{JX} J\zeta)^V + (JR_{X, J\zeta}^\perp \xi)^H \]

(11)

Again, using (5) and the Kähler condition, we see that
\[ J[X, J\zeta] = - \nabla_X\zeta - \nabla_{J\zeta} JX \]
\[ = - D_{\tilde{X}}\zeta - D_{\tilde{J}\zeta} JX + A\zeta X + A_{JX} J\zeta. \]

But, since \( J[X, J\zeta] \) is normal to \( L \), we have
\[ A\zeta X + A_{JX} J\zeta = 0. \]

Thus, we see, from (11), that
\[ [\tilde{J}, \tilde{J}](X^H, \zeta^V) = (JR_{X, J\zeta}^\perp \xi)^H = (R_{X, J\zeta} J\zeta)^H \]

(12)

and
\[ [\tilde{J}, \tilde{J}](\eta^V, \zeta^V) = - [\eta^V, \zeta^V] + [(J\eta)^H, (J\zeta)^H] \]
\[ - \tilde{J}[(J\eta)^H, \zeta^V] - \tilde{J}[\eta^V, (J\zeta)^H] \]
\[ = [(J\eta)^H, (J\zeta)^H] - \tilde{J}(D_{\tilde{J}\eta} \zeta)^V + \tilde{J}(D_{\tilde{J}\eta} J\zeta)^V \]
\[ = [J\eta, J\zeta]^H - (R_{J\eta, J\zeta} \xi)^V - (JD_{\tilde{J}\eta} \zeta)^H + (JD_{\tilde{J}\eta} J\zeta)^H \]
\[ = [J\eta, J\zeta]^H - (R_{J\eta, J\zeta} \xi)^V - (DJ_{\eta} J\zeta)^H + (D_{J\zeta} J\eta)^V \]

(13)
\[ = - (R_{J\eta, J\zeta} \xi)^V \]
In view of (10), (12), and (13), we conclude that $[\tilde{J}, \tilde{J}]$ vanishes if and only if $L$ has flat normal connection.

This together with the equations (6) - (9) shows that $NL$ is Kähler if and only if $L$ has flat normal connection.

Moreover, using (11), we have

$$R_{XY}J\xi = JR_{XY}^\perp\xi,$$

from which we easily see that $L$ is flat if and only if $NL$ has flat normal connection.

□ □

In the proof of the previous theorem, we have also shown

**Corollary 2.4.** $\tilde{J}$ on $NL$ is integrable if and only if $L$ has flat connection.

**References**


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