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DENSITY OF SEMIMARTINGALE DERIVEN BY CANONICAL STOCHASTIC DIFFERENTIAL EQUATION

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Abstract. The existence and the smoothness of density of random variables which are solutions of the canonical stochastic differential equation can be proved by simpler conditions in the Malliavin-Bismut method

0. Introduction

In this paper, we study the existence and the smoothness of densities for semimartingales which are defined by the solutions of canonical stochastic differential equation(canonical SDE) by, so called, the Malliavin-Bismut method.

Since P.Malliavin announced about the stochastic calculus in 1976, many Mathematicians; S.Watanabe, S.Kusuoka, and D.Stroock etc, studied the applications of Malliavin calculus to stochastic differential equations(SDE) for the continuous - type processes mainly. But, since J.M.Bismut studied it for the jump - type processes in 1983 and announced study for the calculus of boundary processes in 1984, there are not many persons who announced study for the jump - type model without R.Leandre [4](c.f. [1]). A little fortunately, we can meet a book[1] which is dealt with the existence and the regularity of density for the jump - type Markov processes. Therefore, we want to study the density for solution of canonical SDE.

The canonical SDE is, in general, defined by using the vector fields. Thus, in the study of densities of the solutions of canonical SDE, we

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think that the R.Léandre's method[4] is more good even though he used a special type *Lèvy* measure. But, in this study, will think only that how can be simplified the conditions of Malliavin-Bismut method [1] in the canonical SDE.

Let us think the SDE;

$$(*)\qquad \begin{aligned}\xi_t(x) = x + \sum_{j=1}^{m}\int_0^t v_j(\xi_s(x))dW_s^j + \int_0^t \mathcal{L}(\xi_{s-}(x))ds \\ + \int_0^t \int_{E_\alpha} \mathbf{c}_\alpha(x,z)\tilde{N}_\alpha(ds,dz),\end{aligned}$$

where $v_1, v_2, \cdots, v_m$ are C^∞-vector fields, $W_s = (W_s^1, W_s^2, \cdots, W_s^m)$ is a Brownian motion, $\tilde{N}_\alpha$ is a compensated Poisson point process and $\mathcal{L}$ is a generator of semigroup of probabilities. Let $\mathbb{B}(x)$ and $\mathbb{C}_\alpha(x,z)$ be the matrices defined by the coefficients of noise part and jump-part, respectively. If there exist two constants $\zeta, \theta > 0$ such that

$$|\mathbf{c}_\alpha(x,z) + x| \leq \zeta(1 + |x|^\theta),$$

for all $x \in \mathbb{R}^d$ and $z \in E_\alpha$ and there exists a Borel set $\Gamma_\alpha \subset \mathbb{R}^d \times E_\alpha$ such that for any $y \in \mathbb{R}^d$ and for the x - section $\Gamma_{\alpha,x} \subset \Gamma_\alpha$,

$$(\cup_{z\in\Gamma_{\alpha,x}}\{y|\mathbb{C}_\alpha y = 0\}) \cap \{y|\mathbb{B}y = 0\} = \{0\},$$

then the solution $\xi_t(x)$ of SDE (*) has a density $y \mapsto p_t(x,y)$ for all $x \in \mathbb{R}^d$ and $t \in (0,T]$.

Furthermore, for $x, y \in \mathbb{R}^d$, there exist two constants $\delta \geq 0, 0 < \epsilon \leq 1$ and two functions $f_\alpha(z)$ and $\rho_\alpha(z)$, which are defined by some conditions with a constant γ, such that

$$y^t\mathbb{C}_\alpha(x,z)y\rho_\alpha(z) \geq \gamma f_\alpha(z)\frac{|y|^2\epsilon}{1+|x|^\delta}$$

for all $z \in E_\alpha$, then $\xi_t(x)$ of the above SDE (*) has a smooth density $y \mapsto p_t(x,y)$.

Section I is the preliminaries part. In this section, we define the canonical SDE. In section II, we introduce the results of [1] to simplify the proofs of our results. In section III, we will deal with the existence and the regularity of density of process deriven by the canonical SDE.

I. Preliminaries

For a non-negative integer m, we denote by $C^m = C^m(\mathbb{R}^d;\mathbb{R}^d)$ the set of all maps from $\mathbb{R}^d$ into itself which are m-times continuously differentiable. In case $m = 0$, we denote it $C = C(\mathbb{R}^d;\mathbb{R}^d)$ which is the space of continuous maps from $\mathbb{R}^d$ into itself equipped with the compact uniform topology.

Let $0 < \delta \leq 1$. We denote by $C_b^{m+\delta} = C_b^{m+\delta}(\mathbb{R}^d;\mathbb{R}^d)$ the set of all $v \in C^m$ such that derivatives $D^\alpha v$ are bounded and uniformly $\delta-$*Hölder* continuous for any α with $|\alpha| \leq m$. Let $\tilde{C} = \tilde{C}(\mathbb{R}^d \times \mathbb{R}^d;\mathbb{S}_+)$, where $\mathbb{S}_+$ is the space of $d \times d$ - matrices. We define the subspace $\tilde{C}_b^{m+\delta} = \tilde{C}_b^{m+\delta}(\mathbb{R}^d \times \mathbb{R}^d;\mathbb{S}_+)$ of $\tilde{C}$ similarly. For details, see [3].

Let $(\Omega, \mathcal{F}, P)$ be a probability space where the filtration $\mathcal{F}_t; t \in [0, \infty)$ of sub - σ - field of $\mathcal{F}$ is defined. Let $X(x,t), t \geqq 0$ be a family of $\mathbb{R}^d$ - valued stochastic process with spatial parameter $x \in \mathbb{R}^d$ defined on $(\Omega, \mathcal{F}, P)$. If $X(x,t)$ is continuous in x for each t a.s., we can regard it as a C-valued process. We denote it sometimes by $X(t) = X(x,t), t \geq 0$.

Let $X(x,t)$ be a cadlag semimartingale with values in C. We define the point process $N(t,E)$ over $[0,\infty) \times C$ associated with $X(t)$ by

$$\text{(I-1)} \qquad N((s,t],E) = \sum_{s<r\leq t} \mathcal{X}_E(\Delta X(r)), \Delta X(s) = X(s) - X(s-),$$

where E is a Borel subset of C excluding 0. Then there exists a unique predictable process $\hat{N}(t,E)$ which is called the compensator such that

$$\tilde{N}(t,E) = N(t,E) - \hat{N}(t,E)$$

is a localmartingale. For a bounded Borel subset U of C, consider a C - valued semimartingale $X(x,t)$ which is represented as;

(I-2)

$$\begin{aligned} X(x,t) &= X_c(x,t) + X_d(x,t) \\ &= M^c(x,t) + B^c(x,t) + \int_U v(x)\tilde{N}(t,dv) + \int_{U^c} v(x)N(t,dv), \end{aligned}$$

where $M^c(x,t)$ is a continuous localmartingale for any x, $B^c(x,t)$ is a continuous predictable process of bounded variation for any x, and the integral form

$$\int_U v(x)\tilde{N}(t,dv)$$

is a discontinuous localmartingale part of $X(x,t)$ for any x.

Let $A_t, t \in [0,\infty)$ be a continuous increasing process adapted to the filtration $\mathcal{F}_t$ such that $A_0 = 0$ a.s. Then there exist predictable processes $\alpha^{ij}(x,y,t)$ and $\beta^i(x,t)$, and for the compensator $\hat{N}(t,E)$, there exists a predictable measure-valued process $\nu_t(E)$ satisfying

$$\langle M^{c,i}(x,t), M^{c,j}(y,t)\rangle = \int_0^t \alpha^{ij}(x,y,s)dA_s,$$

$$B^{c,i}(x,t) = \int_0^t \beta^i(x,s)dA_s,$$

and

$$\hat{N}(t,E) = \int_0^t \nu_s(E)dA_s.$$

The system (α,β,ν) is called the characteristic of semimartingale $X(x,t)$ with respect to A_t.

Let $X(x,t), t \geq 0$ be a C -valued semimartingale equipped with the characteristic (α,β,ν). We introduce a condition;

Condition (A). For a positive predictable process $K_t, t \geq 0$ satisfying

$$\int_0^T K_t dA_t < \infty \text{ a.s. for any } T > 0,$$

(i) $\alpha(x,y,t)$ is a continuous $\tilde{C}_b^{1+1}$ - valued process satisfying

$$\|\alpha(t)\|\tilde{}_{1+1} \leq K_t \text{ a.s.}$$

(ii) $\beta(x,t)$ is a continuous C_b^{0+1} - valued process satisfying

$$\|\beta(t)\|_{0+1} \leq K_t \text{ a.s.}$$

(iii) The measure $\nu_t(\cdot)$ is supported by C_b^{1+1}. Further, there exists a Borel set $U \subset C_b^{1+1}$ such that for some constant $c > 0$, $\|\nu\|_{1+1} \leq c$ for all $v \in U$, and

$$\nu_t(U^c) \leq K_t, \int_U \|v\|_{1+1}^2 \nu_t(dv) \leq K_t.$$

Let $\xi_t, t \geq 0$ be an $\mathbb{R}^d$ - valued cadlag process satisfying Condition (A) adapted to $(\mathcal{F}_t)$. Then we can define the *Itô* integrals and the *Stratonovich* integrals, respectively;

$$\int_s^t X(\xi_{r-}, dr), \int_s^t X(\xi_r, \circ dr).$$

Let $v(x)$ be a *Lipschitz* continuous vector field. Then by Condition (A)-(iii), the possible infinite sum

$$\sum_{s \leq t} [e^{\Delta X(s)}(x) - x - \Delta X(x, s)]$$

is absolutely convergent a.s.. Therefore, we can define the canonical integral of a cadlag semimartingale ξ_t based on the vector field - valued semimartingale $X(t)$ as following;

$$\text{(I-3)} \qquad \begin{aligned} \int_s^t X(\xi_r, \diamond dr) &= \int_s^t X_c(\xi_r, \circ dr) + \int_s^t X_d(\xi_{r-}, dr) \\ &\quad + \sum_{s \leq r \leq t} [e^{\Delta X(r)}(\xi_{r-}) - \xi_{r-} - \Delta X(\xi_{r-}, r)], \end{aligned}$$

where the first part and the second part of the right hand side are *Stratonovich* integral and *Itô* integral, respectively.

Let $X(x, t), t \geq 0$ be a C - valued semimartingale whose characteristic satisfy Condition (A). Consider a canonical SDE which is represented by

$$\text{(I-4)} \qquad \xi_t(x) = x + \int_0^t X(\xi_s(x), \diamond ds),$$

where $0 \leq s \leq t$. It is known that the equation (I-4) has a unique solution $\xi_t(x), t \geq 0$ for any $s \leq t, x \in \mathbb{R}^d$.

If we represent the canonical SDE by using the C^∞ - vector field

$v(x)$, then (I-4) can be represented as following;

$$\text{(I-5)}\quad \begin{aligned} \xi_t(x) = x &+ \int_0^t X_c(\xi_s(x), \circ ds) \\ &+ \int_0^t \int_U [e^v(\xi_{s-}(x)) - \xi_{s-}(x)] \tilde{N}(ds, dv) \\ &+ \int_0^t \int_U [e^v(\xi_{s-}(x)) - \xi_{s-}(x) - v(\xi_{s-})] \nu_s(dv) ds \\ &+ \int_0^t \int_{U^c} [e^v(\xi_{s-}(x)) - \xi_{s-}(x)] N(ds, dv). \end{aligned}$$

II. Malliavin calculus for process with jumps

In this section, we introduce the results which are in [1], because in section III, we use this method to prove our results.

Let $(\Omega, \mathcal{F}, P), t \in [0, T]$ be a filtered probability space endowed with;

(i) a standard m-dimensional Brownian motion $W = (W^i)_{i \le m}$;
(ii) for $1 \le \alpha \le A$, a Poisson random measure

$$N_\alpha = N_\alpha(\omega; dt, dz) \text{ on } [0, T] \times E_\alpha,$$

where E_α is an open subset of $\mathbb{R}^{\beta\alpha}$ with infinite Lebesgue measure. The compensator $\hat{N}_\alpha$ of N_α is of the form;

$$\hat{N}_\alpha(dt, dz) = G_\alpha(dz) dt,$$

where G_α denotes Lebesgue measure on E_α. The compensated Poisson measure $\tilde{N}_\alpha$ is given by

$$\tilde{N}_\alpha(dt, dz) = N_\alpha(dt, dz) - G_\alpha(dz) dt;$$

(iii) the random elements (W^i, N_α) are independent.

For each $x \in \mathbb{R}^d$, consider a SDE of the form;

$$\text{(II-1)}\quad \begin{aligned} \xi_t(x) = x &+ \int_0^t a(\xi_{s-}(x)) ds + \int_0^t b(\xi_{s-}(x)) dW_s \\ &+ \sum_\alpha \int_0^t \int_{E_\alpha} c_\alpha(\xi_{s-}(x), z) \tilde{N}_\alpha(ds, dz) \end{aligned}$$

with the coefficients of the following;

$$a = (a^i)_{1 \le i \le d} : \mathbb{R}^d \to \mathbb{R}^d,$$
$$b = (B^{ij})_{1 \le i \le d, 1 \le j \le m} : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^m,$$
$$c_\alpha = (c^i_\alpha)_{1 \le i \le d} : \mathbb{R}^d \times E \to \mathbb{R}^d.$$

We make two assumptions as following;

ASSUMPTION (A-r). (i) a and b are r-times differentiable with bounded derivatives of all order between 1 and r.
(ii) c_α is r-times differentiable on $\mathbb{R}^d \times E$, and

$$c_\alpha(0, \cdot) \in \cap_{2 \le p < \infty} L^p(E_\alpha, G_\alpha)$$
$$\sup_x |D^n_{x^n} c_\alpha(x, \cdot)| \in \cap_{2 \le p < \infty} L^p(E_\alpha, G_\alpha) \text{ for } 1 \le n \le r.$$
$$\sup_{x,z} |D^{n+q}_{x^n z^q} c_\alpha(x, z)| < \infty \text{ for } 1 \le n + q \le r \text{ and } q \ge 1.$$

We put as following

$$B(x) = b(x) b(x)^t$$

$$C_\alpha(x, z) = \begin{cases} (I + D_x c_\alpha)^{-1} (D_z c_\alpha)(D_z c_\alpha)^t ((I + D_x c_\alpha)^{-1})^t, \\ \quad \text{if } I + D_x c_\alpha \text{ is invertible,} \\ 0, \text{ otherwise.} \end{cases}$$

Here, I is the $d \times d$ - identity matrix. We also set;

$$N_x = \text{ kernel of } B(x) \text{ in } \mathbb{R}^d,$$
$$N^\alpha_{xz} = \text{ kernel of } C_\alpha(x, z) \text{ in } \mathbb{R}^d.$$

ASSUMPTION (B). For each $\alpha = 1, 2, \cdots, A$, there is a Borel subset Γ_α of $\mathbb{R}^d \times E_\alpha$ such that, if $\Gamma_{\alpha,x}$ is the x-section of Γ_α in E_α and if

$$W^\alpha_x = \begin{cases} \cup_{z \in \Gamma_{\alpha,x}} N^\alpha_{x,z} & \text{if } G_\alpha(\Gamma_{\alpha,x}) = \infty, \\ \mathbb{R}^d, & \text{if } G_\alpha(\Gamma_{\alpha,x}) < \infty, \end{cases}$$

then

$$(\cap_\alpha W^\alpha_x) \cap N_x = \{0\} \text{ for all } x \in \mathbb{R}^d.$$

Proposition II-1. *Under (A-3) and (B), the variables* $\xi_t(x)$ *of (II-1) have a density* $y \mapsto p_t(x,y)$ *for all* $x \in \mathbb{R}^d, t \in (0,T]$.

To get the regularity of density, we define two functions;

Definition II-1. A measurable function $f_\alpha : E_\alpha \to [0,\infty)$ is called (ζ,θ)-broad, where ζ and θ are two positive numbers, if the following integral is finite;

$$\int_0^\infty s^{\zeta-1} exp[-\theta \int_{E_\alpha} (1-e^{-sf(z)})dz]ds < \infty. \tag{II-2}$$

Definition II-2. For each α, we define a function $\rho : E_\alpha \to [0,\infty)$ having the following properties;
(i) $\rho \in C_b^\infty$,
(ii) $\rho_\alpha(z) \to 0$ as $z \to \partial(E_\alpha)$ (boundary of E_α),
(iii) $|D^r_{z^r}\rho_\alpha| \in L^1(E_\alpha, G_\alpha)$ for all $r \in \mathbb{N}$.

Assumption (SB). There exist two constants $\epsilon > 0, \delta \geq 0$, and for all $\alpha = 1,2,\cdots,A$, a (ζ,θ)-broad function $f_\alpha(z)$ and the function $\rho_\alpha(z)$ meeting in Definition II-2, such that for all $x,y \in \mathbb{R}^d$,

$$y^t B(x) y + \sum_\alpha inf_{z:f_\alpha(z)>0} \frac{\rho_\alpha(z)}{f_\alpha(z)} y^t C_\alpha(x,z) y \geq |y|^2 \frac{\epsilon}{1+|x|^\delta}.$$

Assumption (SC). There is a constant $\zeta > 0$ such that

$$|det(I + D_x c_\alpha(x,z))| \geq \zeta,$$

identically.

Then we get the smoothness of density.

Proposition II-2. *Let* $r \geq 3$ *be the number of differentiation which is of (A-r). Assume (SC) and either*
(i). (A-(r+d+3)) and (SB) with $\theta \leq t$, *and* $[t/\theta]\zeta > 2d(r+d+1)$, *or*
(ii). (A-(r+3)) and (SB) with $\theta \leq t$, *and* $[t/\theta]\zeta > 2d^2(r+1)$.
Then, the density $y \mapsto p_t(x,y)$ *of the random variable* $\xi_t(x)$ *of (II-1) exists and is of class* C^r.

III. Density in canonical stochastic differential equation

Consider a canonical SDE of the form;

$$\text{(III-1)} \qquad d\xi_t(x) = \sum_{j=1}^{m} v_j(\xi_t(x)) \diamond dZ_t^j$$

deriven by a vector field - valued semimartingale

$$\text{(III-2)} \qquad X_t(x) = \sum_{j=1}^{m} v_j(x) Z_t^j,$$

where $Z_t = (Z_t^1, Z_t^2, \cdots, Z_t^m)$ is an $\mathbb{R}^m$ - valued semimartingale and $v_1, v_2, \cdots, v_m$ are the smooth complete vector fields on $\mathbb{R}^d$. By the solution of (III-1), we define a cadlag semimartingale $\xi_t(x); t \geq 0$ which values in $\mathbb{R}^d$ adapted to $\mathcal{F}_t = \sigma(Z_s; s \leq t)$ satisfying ;
(III-3)

$$\begin{aligned}
\xi_t(x) &= x + \sum_{j=1}^{m} \int_0^t v_j(\xi_s(x)) \diamond dZ_s^j \\
&= x + \sum_{j=1}^{m} \int_0^t v_j(\xi_s(x)) \circ dZ_c^j(s) + \sum_{j=1}^{m} \int_0^t v_j(\xi_{s-}(x)) dZ_d^j(s) \\
&+ \sum_{0<s\leq t} [exp(\sum_{j=1}^{m} \Delta Z_s^j v_j)(\xi_{s-}(x)) - \xi_{s-}(x) - \sum_{j=1}^{m} \Delta Z_s^j v_j(\xi_{s-}(x))].
\end{aligned}$$

If we use the Markov semimartingale $Z_t = (Z_t^1, Z_t^2, \cdots, Z_t^m)$ of the form ;

$$Z_t^j = W_t^j + b^j t + \int_{E_\alpha} z^j \tilde{N}_\alpha((0,t], dz), j = 1, 2, \cdots, m,$$

where $W_t = (W_t^1, W_t^2, \cdots, W_t^m)$ is a Brownian motion, and the compensator $\hat{N}_\alpha((0,t], dz)$ is of the form

$$\hat{N}_\alpha(ds, dz) = G_\alpha(dz) ds,$$

where $G_\alpha(dz)$ is a Lebesgue measure. Then the solution of canonical SDE (III-3) can be represented as;
(III-4)

$$\begin{aligned}\xi_t(x) = x &+ \sum_{j=1}^{m}\int_0^t v_j(\xi_s(x))dW_s^j + \int_0^t \mathcal{L}(\xi_{s-}(x))ds \\ &+ \int_0^t\int_{E_\alpha}[exp(\sum_{j=1}^{m} z^j v_j)(\xi_{s-}(x)) - \xi_{s-}(x)]\tilde{N}_\alpha(ds,dz),\end{aligned}$$

where

$$\begin{aligned}\mathcal{L}(x) &= \mathcal{A}(x) \\ &\quad + \int_{E_\alpha}[exp(\sum_{j=1}^{m} z^j v_j)(x) - x - \sum_{j=1}^{m} z^j v_j(x)]G_\alpha(dz), \\ \mathcal{A}(x) &= (1/2)\sum_{j=1}^{m} v_j^2(x) + v_0(x).\end{aligned}$$

Form the equation (III-4), we put as

$$\mathbf{c}_\alpha(x,z) = exp(\sum_{j=1}^{m} z^j v_j)(x) - x,$$

and

$$\tilde{\mathbf{c}}_\alpha(x,z) = \mathbf{c}_\alpha(x,z) + x.$$

Then we know that $D_x\tilde{\mathbf{c}}_\alpha(x,z)$ is invertible.

PROPOSITION III-1. *Let us consider a matrix linear differential equation;*

$$\text{(III-5)} \qquad \begin{aligned}&\frac{d}{dt}X_t = A(t)X_t, \\ &X_0 = I.\end{aligned}$$

Then there exists a solution X_t of (III-5) and $det(X_t) \neq 0$.

Proof. If we put

$$\varphi_t(x,t) = \tilde{\mathbf{c}}_\alpha(x(t),z),$$

we get that

$$\begin{aligned}\frac{d}{dt}(\frac{\partial}{\partial x^i}\varphi_t(x,z)) &= \frac{\partial}{\partial x^i}(\frac{d}{dt}\varphi_t(x,z)) \\ &= \sum_{j=1}^{m} z^j \sum_{k=1}^{d} \frac{\partial}{\partial x^k} v_j(\varphi_t(x,z)) \frac{\partial}{\partial x^i}\varphi_t^k(x,z),\end{aligned}$$

where $\frac{\partial}{\partial x^i}\varphi_t^k(x,z)$ is an $d \times d$ - matrix. Therefore, if we think an equation

$$\text{(III-6)} \qquad \begin{aligned}&\frac{d}{dt} D_x \tilde{\mathbf{c}}_\alpha(x(t),z) = V(t) D_x \tilde{\mathbf{c}}_\alpha(x(t),z), \\ &D_x \tilde{\mathbf{c}}_\alpha(x(0),z) = I,\end{aligned}$$

where $V(t)$ is an $d \times d$ - matrix, then (III-6) is a matrix linear differential equation. □ □

Thus from the Proposition III-1, we see that $D_x \tilde{\mathbf{c}}_\alpha(x,z)$ is invertible. On the other hand, if we put

$$\text{(III-7)} \qquad \mathbb{B}(x) = (a^{ik}(x))_{d\times d},$$

where

$$(a^{ik}(x))_{d\times d} = \sigma_{d\times m}(x)(\sigma_{d\times m}(x))^t$$

and

$$\sigma_{d\times m}(x) = \begin{pmatrix} v_1^1(x) & v_2^1(x) & \cdots & v_m^1(x) \\ v_1^2(x) & v_2^2(x) & \cdots & v_m^2(x) \\ \cdots & & & \\ v_1^d(x) & v_2^d(x) & \cdots & v_m^d(x) \end{pmatrix}_{d\times m},$$

then we see that

$$\mathbb{B}(\tilde{\mathbf{c}}_\alpha(x,z)) = (D_z \mathbf{c}_\alpha(x,z))(D_z \mathbf{c}_\alpha(x,z))^t.$$

Thus we put as following;

(III-8) $$\mathbb{C}_\alpha(x,z) = (D_x\tilde{\mathbf{c}}_\alpha(x,z))^{-1}\mathbb{B}(\tilde{\mathbf{c}}_\alpha(x,z))[(D_x\tilde{\mathbf{c}}_\alpha(x,z)^{-1}]^t.$$

We make two assumptions;

Assumption ($\mathbb{A}$). There exists two constants $\zeta, \theta > 0$ such that

$$|\tilde{\mathbf{c}}_\alpha(x,z)| \leq \zeta(1+|x|^\theta)$$

for all $x \in \mathbb{R}^d$ and $z \in E_\alpha$.

Assumption ($\mathbb{B}$). There is a Borel subset $\Gamma_\alpha = \{(x,z)\} \subset \mathbb{R}^d \times E_\alpha$ such that for any $y \in \mathbb{R}^d$ and for the x -section $\Gamma_{\alpha,x} \subset \Gamma_\alpha$, if $G_\alpha(\Gamma_{\alpha,x}) = \infty$,

$$(\cup_{z\in\Gamma_{\alpha,z}}\{y|\mathbb{C}_\alpha(x,z)y=0\}) \cap \{y|\mathbb{B}(x)y=0\} = \{0\},$$

if $G_\alpha(\Gamma_{\alpha,x}) < \infty$,

$$\mathbb{R}^d \cap \{y|\mathbb{B}(x)y=0\} = \{0\}.$$

Then we get the existence theorem of density.

Theorem III-1. *Under* ($\mathbb{A}$) *and* ($\mathbb{B}$), *the solution* $\xi_t(x)$ *of (III-4) has a density* $y \mapsto p_t(x,y)$ *for all* $x \in \mathbb{R}^d$ *and* $t \in (0,T]$.

Proof. To prove by showing that Assumptions ($\mathbb{A}$) and ($\mathbb{B}$) imply Assumptions (A) and (B) of Section II, we put the coefficients of (II-1) into the coefficients of (III-4) as following;

(III-9) $$\begin{aligned} a(x) &= \mathcal{L}(x), \\ b(x) &= (v_j^i(x))_{1\leq i\leq d, 1\leq j\leq m}, \\ c_\alpha(x,z) &= \mathbf{c}_\alpha(x,z), \end{aligned}$$

Then because the component functions $v_j^i(x)$ of the vector fields v_j are C^∞ - functions with bounded variations for all order ≥ 1, the components $v_j^i(x)$ are of linear growth of all order ≥ 1. Further, because $\mathbf{c}_\alpha(x,z)$ is an exponential function with respect to z,

$$\begin{aligned} D_z\mathbf{c}_\alpha(x,z) &= (\frac{\partial}{\partial z^1}\mathbf{c}_\alpha(x,z), \cdots, \frac{\partial}{\partial z^m}\mathbf{c}_\alpha(x,z)) \\ &= (v_1(\tilde{\mathbf{c}}_\alpha), \cdots, v_m(\tilde{\mathbf{c}}_\alpha)), \end{aligned}$$

whose all components are vector fields. Thus, form the Assumption ($\mathbb{A}$), we can choose $\zeta', \theta' > 0$ such that

$$|D^k_{z^k}\mathbf{c}_\alpha(x,z)| \le \zeta'(1+|x|^{\theta'}) \text{ for } k \ge 1.$$

Thus, we can choose a function $\eta \in \cap_{2\le p<\infty} L^p(E_\alpha, G_\alpha)$ such that

$$|D^n_{x^n}\mathbf{c}_\alpha(x,z)| \le \zeta''|\eta(z)|(1+|x|^{\theta''}), n \ge 1,$$

for $\zeta'', \theta'' > 0$. Thus we can get,

$$|D^{n+k}_{x^n z^k}\mathbf{c}_\alpha(x,z)| \le \zeta(1+|x|^{\theta}), n+k \ge 1, k \ge 1.$$

Therefore we see that the coefficients (a, b, c_α) of (III-9) satisfy the conditions (i) and (ii) of Assumption (A-3) for SDE (II-1).

On the other hand, it is easy to see that Assumption ($\mathbb{B}$) implies Assumption (B) for SDE (II-1). □ □

Remark. This theorem is general a little in some sense. Therefore, let us think the subcases. If $Rank(\mathbb{B}(x)) = d$, then we can get the same result. See Corollary. Even though $Rank(\mathbb{B}(x)) < d, \xi_t(x)$ of (III-4) can have the density. We will explain this case by Example - (ii), which is dealt the case $d = 2$, i.e., in $\mathbb{R}^2$. To get the density, it must be $Rank(\mathbb{B}(x))(or Rank(\mathbb{C}_\alpha(x,z))) = d/2$, because of $Rank(\mathbb{B}(x)) = Rank(\mathbb{C}_\alpha(x,z))$.

Corollary. *If $Rank(\mathbb{B}(x)) = d$ or $Rank(\mathbb{C}_\alpha(x,z)) = d$, then the solution $\xi_t(x)$ of (III-4) has a density $y \mapsto p_t(x,y)$ for all $x \in \mathbb{R}^d$ and $t \in (0,T]$.*

Proof. If $Rank(\mathbb{B}(x)) = d$, then we see that, for any $y \in \mathbb{R}^d$,

$$\{y|\mathbb{B}(x)y = 0\} = \{0\}.$$

Therefore, from the setting (III-7), we see that Assumption ($\mathbb{B}$) is satisfied. Similarly, when $Rank(\mathbb{C}_\alpha(x,z)) = d$, we can get the same result. □ □

EXAMPLE. In $\mathbb{R}^2 = \{\mathbf{x} = (x^1, x^2)\}$, let $Z_t = (Z_t^1, Z_t^2)$ be a *Lèvy* process;

$$Z_t^j = W_t^j + \int_{E_\alpha} z^j \tilde{N}_\alpha((0,t], dz), j = 1, 2,$$

where $W_t = (W_t^1, W_t^2)$ is a Brownian motion. Then the deriving process $X_t(\mathbf{x})$ is a *Lèvy* process of the form;

$$X_t(\mathbf{x}) = v_1(\mathbf{x})Z_t^1 + v_2(\mathbf{x})Z_t^2.$$

Thus if we think the canonical SDE;

$$d\xi_t(\mathbf{x}) = \sum_{j=1}^{2} v_j(\xi_t(\mathbf{x})) \diamond dZ_t^j,$$

we get the solution $\xi_t(\mathbf{x})$ satisfying;

$$\begin{aligned} \xi_t(\mathbf{x}) &= \mathbf{x} + \sum_{j=1}^{2} \int_0^t (\xi_s(\mathbf{x}))dW_t^j + \int_0^t \mathcal{L}(\xi_{s-}(\mathbf{x}))ds \\ &+ \int_0^t \int_{E_\alpha} [exp(\sum_{j=1}^{2} z^j v_j)(\xi_{s-}(\mathbf{x})) - \xi_{s-}(\mathbf{x})]\tilde{N}_\alpha(ds, dz). \end{aligned} \tag{III-10}$$

Let us check the conditions of existence of density for $\xi_t(\mathbf{x})$ of (III-10).

(i). If $Rank(\mathbb{B}(x)) = 2$, then for all $y = (y^1, y^2) \in \mathbb{R}^2$,

$$\{y|\mathbb{B}(x)y = 0\} = \{(0,0)\}.$$

Thus from the Condition $(\mathbb{B}), \xi_t(x)$ has a density.

(ii). If $Rank(\mathbb{B}(x)) = 1$, then $Rank(\mathbb{C}_\alpha(x,z)) = 1$ also, and we get

$$\{y|\mathbb{B}(x)y = 0\} = \{(0, y_2)\} \text{ or } \{(y_1, 0)\}.$$

Therefore, to get the density, it must be held that for all $z \in \Gamma_{\alpha,x}$,

$$\{y|\mathbb{C}_\alpha(x,z)y = 0\} = \{(y_1, 0)\} \text{ or } \{(0, y_2)\},$$

respectively. □

To get the smoothness of density, we make an assumption which implies Assumption (SB) of Section II, for the functions f_α and ρ_α which are defined by Definition II-1 and Definition II-2, respectively.

ASSUMPTION ($\mathbb{SB}$). There exist $\epsilon > 0, \delta \geq 0, (\zeta, \theta)$-broad function $f_\alpha(z)$ and the function $\rho_\alpha(z)$ such that for all $x, y \in \mathbb{R}^d$,

$$y^t\mathbb{B}(x)y + inf_{z;f_\alpha(z)>0}\frac{\rho_\alpha(z)}{f_\alpha(z)}y^t\mathbb{C}_\alpha(x,z)y \geq |y|^2\frac{\epsilon}{1+|x|^\delta}. \tag{III-11}$$

Then now, we get the following smoothness result;

THEOREM III-2. *Assume that* ($\mathbb{A}$). *If for* $x, y \in \mathbb{R}^d$, *there exist two constants* $\delta \geq 0, 0 < \epsilon \leq 1$, *and two functions* $f_\alpha(z) \in L^1(E_\alpha, G_\alpha)$ *which is satisfying*

$$0 \leq f_\alpha(z) \leq 1 \text{ and } \int_{E_\alpha} f_\alpha(z)dz = \gamma \leq \epsilon \tag{III-12}$$

and $\rho_\alpha(z)$ *which is of Definition II-2 such that*

$$y^t\mathbb{C}_\alpha(x,z)y\rho_\alpha(z) \geq \gamma f_\alpha(z)\frac{|y|^2}{1+|x|^\delta} \text{ for all } z \in E_\alpha, \tag{III-13}$$

then $\xi_t(x)$ *of (III-4) has a smooth* (C^∞) *density* $y \to p_t(x,y)$.

Proof. From the setting (III-7) and (III-8), we see that $\mathbb{C}_\alpha(x,z)$ and $\mathbb{B}(x)$ are $d \times d$ - symmetric nonnegative matrices, and that we can choose $\epsilon' > 0$ such that

$$y^t\mathbb{B}(x)y \geq |y|^2\epsilon' \geq \frac{\epsilon'|y|^2}{1+|x|^\delta} \tag{III-14}$$

for all $x, y \in \mathbb{R}^d$. Thus from (III-13) and (III-14), if we take again $\epsilon \leq (1/2)min(\epsilon, \epsilon')$, we see that ($\mathbb{SB}$), which imply (SB) of Section II, is satisfied.

On the other hand, since $D_x\tilde{\mathbf{c}}_\alpha(x,z)$ is invertible, there exists a constant $\zeta > 0$ such that

$$|det(D_x\tilde{\mathbf{c}}_\alpha(x,z))| \geq \zeta$$

identically. Thus we see that (SC) of Section II is satisfied also. Therefore, we can get the same result as Proposition II-2. □ □

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