Kangweon-Kyungki Math. Jour. **5** (1997), No. 2, pp. 107–112

# THE EQUIVALENCE OF TWO ALGEBARAIC $K$-THEORIES

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Abstract. For a ring $R$ with 1, the higher $K$-theory of Quillen is defined by the higher homotopy groups of the plus construction of the general linear group of $R$. On the other hand, the Volodin $K$-theory is defined by the higher homotopy groups of the Volodin space. In this paper we show that these two $K$-theories are equivalent. We show that the Volodin space is a homotopy fiber of the acyclic map from $BGL(R)$ to its plus construction.

## 0. Introduction

For a ring $R$ with 1, the Quillen's higher algebraic $K$-theory is defined by

$$K_i(R) = \pi_i BGL(R)^+ \quad \text{for } i \geq 1,$$

where $+$ means Quillen's plus construction with respect to the perfect normal subgroup $E(R)$. The premiere reason why we define $K$-theory in this way is that we can deal with $K$-theory *functorially* through this definition. There is another $K$-theory which is called Volodin $K$-theory. It is proved by Suslin that these two $K$-theories are equivalent. In this paper we prove this equivalence again. We here offer more transparent and more expositive way. In the proof we use a few homotopy theoretic techniques which are rather standard, but not easily found in the literature.

The Volodin space $X(R)$ is defined by $X(R) = \cup_\sigma BT^\sigma(R)$, where $T^\sigma(R)$ is a triangular matrix for a partial ordering $\sigma$ on $\mathbb{Z}_+$. In the main

Received January 21, 1997.

1991 Mathematics Subject Classification: 19D55.

Key words and phrases: triangular matrices, Volodin space, simplicial homotopy, semi- simplicity.

This work is supported by Inha University research fund(1996) and GARC-KOSEF.

theorem we show that $X(R) \to BGL(R) \to BGL(R)^+$ is a homotopy fibration sequence. The theorem implies that the Volodin $K$-theory $K_i^V(R) := \pi_{i-1}X(R)$ $(i \geq 3)$ is isomorphic to $K_i(R) = \pi_i BGL(R)^+$. One advantage of Volodin $K$-theory is that the Volodin space $X(R)$ is more concrete and more natural than $BGL(R)^+$. In the proof of the main theorem we use the acyclicity and the semi-simplicity of $X(R)$. We also exploit the comparison theorem of Serre spectral sequence.

## 1. Preliminaries

In this section we recall the definitions of simplicial sets and simplicial homotopy. We also recall the definition of Volodin spaces.

Definition 1.1. (Simplicial set) Let $\Delta$ be a category where objects are $\underline{n} = \{0, 1, \ldots, n\}$, $n \geq 0$, and a morphism $\underline{m} \to \underline{n}$ is a nondecreasing function. We call $\Delta$ the *simplicial category*. A *simplicial set* is a contravariant functor from $\Delta$ to the category of sets. A simplicial set $\Delta \to \underline{Sets}$ is usually denoted by $X_*$, and $X_n$ denotes $X(\underline{n})$ and $g^* : X_n \to X_m$ means $X(g : \underline{m} \to \underline{n})$. A *simplicial map* between two simplicial sets $f : X_* \to Y_*$ is a natural transformation from $X_*$ to $Y_*$.

There is another way to define simplicial sets. A simplicial set is defined to be a collection of sets $\{X_n\}_{n \geq 0}$ together with face maps $d_i : X_n \to X_{n-1}$ $(0 \leq i \leq n)$ and degeneracy maps $s_i : X_n \to X_{n+1}$ $(0 \leq i \leq n)$ satisfying 'simplicial relations(cf. [2])'. In this definition a simplicial map $f : X_* \to Y_*$ means a collection of maps $f_n : X_n \to Y_n$ which commute with face and degeneracy maps. For a simplicial set $X$, a simplex $x \in X_n$ is called *degenerate* if $x = s_i(y)$ for some $y \in X_{n-1}$ and $i$. Otherwise it is called *nondegenerate*.

Definition 1.2. For a simplicial set $X_*$ the *geometric realization* $|X_*|$ is defined to be the quotient of $\coprod_{n \geq 0} X_n \times \Delta^n$ by the following relations: Let $x \in X_n$ and $\lambda : \underline{m} \to \underline{n}$ be in $\Delta$. Then $(\lambda^* x, (t_0, \ldots, t_n)) \approx (x, \lambda_*(t_0, \ldots, t_m))$.

We may think of a simplicial set $X_*$ as a CW-complex $|X_*|$ together with a decomposition into simplices.

We can deal with homotopy theory in the category of simplicial sets.

DEFINITION 1.3. Let $\Delta_n^m = \{f : \underline{n} \to \underline{m} \mid f \text{ preserves the order}\} = \mathrm{Hom}_\Delta(\underline{n}, \underline{m})$. Then $\Delta_*^m$ becomes a simplicial set. It has clearly one nondegenerate $m$-simplex so that $|\Delta_*^m| \cong \Delta^m$. $\Delta_*^m$ is called *standard $m$-simplex.* We may regard $p$-simplex of $\Delta_*^m$ as a $(p+1)$-tuple $(a_0, a_1, \dots, a_p)$ of integers such that $0 \leq a_0 \leq \cdots \leq a_p \leq m$.

We have $|\Delta_*^1| \cong I = [0,1]$. The set $\{u_n : \underline{n} \to \underline{1} \mid u_n(i) = 0 \text{ for all } i,\ n \in \mathbb{Z}_+\}$ realizes $0 \in [0,1]$ and $\{v_n : \underline{n} \to \underline{1} \mid v_n(i) = 1 \text{ for all } i,\ n \in \mathbb{Z}_+\}$ realizes $1 \in [0,1]$.

Now we define a homotopy between two simplicial maps. Let $f, g : X_* \to Y_*$ be simplicial maps. Then we say $f$ is *homotopic* to $g$ if there exists a simplicial map (which is called simplicial homotopy) $H : \Delta_*^1 \times X_* \to Y_*$ such that $H|_{0 \times X_*} = f$ and $H|_{1 \times X_*} = g$. With this definition, we can execute the homotopy theory in the category of simplicial sets, just as we do in the category of topological spaces. We use this in the proof of Lemma 2.1.

## 2. An isomorphism between $K_i(R)$ and $K_i^V(R)$

For a ring $R$ the higher algebraic $K$-theory is defined by

$$K_i(R) = \pi_i BGL(R)^+ \text{ for } i \geq 1.$$

Let $\sigma$ be a partial ordering on $\{1, \dots, n\}$. Define the tringular matrix $T_n^\sigma(R)$ by $\{M \in GL_n(R) \mid M_{ij} = \delta_{ij} \text{ unless } i \overset{\sigma}{<} j\}$. Let $T^\sigma(R) = \varinjlim T_n^\sigma(R)$. Define the Volodin space $X(R)$ by

$$X(R) = \cup_\sigma BT^\sigma(R).$$

The Volodin $K$-theory is defined to be $K_i^V(R) = \pi_{i-1} X(R)$ for $i \geq 3$. In this section we show that two $K$-theories (Volodin $K$-theory and Quillen's $K$-theory) are equivalent. It suffices to show that $X(R) \to BGL(R) \to BGL(R)^+$ is a homotopy fibration (Theorem 2.3).

Let $W(St_n(R), \{T_n^\sigma(R)\})$ be the geometric realization of the simplicial set whose $p$-simplices are the sequences $(g_0, \dots, g_p)$ of elements of $St_n(R)$ such that all $g_i^{-1} g_j$ lie in the same $T_n^\sigma(R)$ and the face maps are omittings and the degeneracy maps are repeatings. Note that

$$W(St_n(R), \{T_n^\sigma(R)\}) \cong \cup_\sigma B(St_n(R), T_n^\sigma(R), *),$$

where $B(St_n(R), T_n^\sigma(R), *)$ is a two-sided bar construction. Let $W_n = W(St_n(R), \{T_n^\sigma(R)\})$. The group $St_n(R)$ acts on $W$ by left multiplication. This action is free and the corresponding quotient (coinvariant) space is $\cup_\sigma B(*, \{T_n^\sigma(R)\}, *) = \cup_\sigma BT_n^\sigma(R)$ which equals $X_n(R)$. It is known that $W_n$ is simply connected, so we have $\pi_1 X_n(R) = St_n(R)$. Hence we have $\pi_1 X(R) = St_n(R)$. We first prove the semi-simplicity of $X(R)$ by using simplicial homotopy. A space $X$ is said to be semi-simple if $\pi_1 X$ acts trivially on $\pi_i X$ for $i \geq 2$.

LEMMA 2.1. *$X(R)$ is semi-simple.*

*Proof.* We show that $\pi_1 X(R) = St(R)$ acts trivially on $W = \varinjlim W(St_n(R), \{T_n^\sigma(R)\})$ up to homotopy, that is, for each $x \in St(R)$, the map $x : W \to W$ is homotopic to the identity map. It suffices to show that the canonical inclusion $u_n : W_n \to W_{n+1}$ is homotopic to $y \cdot u_n$ for all $y \in St_{n+1}(R)$.

The simplicial homotopy between $u_n$ and $y \cdot u_n$ is defined by:

$$\begin{aligned} &H : \Delta^1_{s+t} \times (W_n)_{s+t} \to (W_{n+1})_{s+t}, \\ &H((0, \dots, 0, 1, \dots, 1), (x_0, \dots, x_{s+t})) \\ &\quad = (x_0, \dots, x_s, y \cdot x_{s+1}, \dots, y \cdot x_{s+t}), \end{aligned}$$

where $(0, \dots, 0, 1, \dots, 1)$ has $s+1$ 0's and $t$ 1's. This homotopy is well-defined, since whenever $x_i^{-1} x_j$ belongs to $T_n^\sigma(R)$ for some $\sigma$, $x_i^{-1}(y x_j)$ also belongs to $T_{n+1}^{\bar\sigma}(R)$, where $\bar\sigma$ is an arbitrary extension of $\sigma$ to $\{1, \dots, n+1\}$. □ □

Suslin proved in [9] that $X(R)$ is acyclic by showing that the canonical inclusion $X_n(R) \to X(R)$ induces the zero map on homology. Now we prove the following lemma.

LEMMA 2.2. *For a ring $R$ with 1, the Volodin space $X(R)$ is a homotopy fiber of $BSt(R) \to BSt(R)^+$.*

*Proof.* The composite $X(R) \to BSt(R) \to BSt(R)^+$ is null-homotopic by the following fact: any map from an acyclic space to a nilpotent space is null-homotopic. This fact can be seen by considering Postnikov decomposition and noting that $X(R)$ is acyclic and $BSt(R)^+$ is simply connected ($St(R)$ is perfect). By the homotopy lifting property of the fibration there exists a map $\alpha : X(R) \to F$ such that the diagram

$$\begin{array}{ccc} X(R) & \longrightarrow & BSt(R) \\ {\scriptstyle\alpha}\downarrow & & \| \\ F & \longrightarrow & BSt(R) \end{array}$$

commutes. We show that $\alpha$ is an equivalence. We have a diagram of filteration sequences

$$\begin{array}{ccccc} \widetilde{X}(R) & \longrightarrow & X(R) & \longrightarrow & BSt(R) \\ {\scriptstyle\beta}\downarrow & & {\scriptstyle\alpha}\downarrow & & \| \\ \widetilde{F} & \longrightarrow & F & \longrightarrow & BSt(R) \end{array}$$

Since $X(R)$ is a semi-simple (Lemma 2.1), $\pi_1 BSt(R) = St(R)$ acts trivially on $\pi_i X(R)$, so does on $H_i \widetilde{X}(R)$. $St(R)$ also acts trivially on $H_i \widetilde{F}$, since it acts through $\pi_1 BSt(R)^+$ which is trivial. Now by the comparison theorem of Serre spectral sequence (cf.[4], p 355), the map $\beta : \widetilde{X}(R) \to \widetilde{F}$ is a homology equivalence. Hence $\beta$ is a homotopy equivalence, since $\widetilde{X}(R)$ and $\widetilde{F}$ are simply connected. Thus $\alpha$ is a homotopy equivalence by the five lemma. □ □

We now prove our main theorem.

THEOREM 2.3. *$X(R)$ is a homotopy fiber of $BGL(R) \to BGL(R)^+$.*

*Proof.* We first prove that $X(R)$ is a homotopy fiber of $BE(R) \to BE(R)^+$. Consider the diagram of fibration sequences:

$$\begin{array}{ccccc} G & \longrightarrow & K_1(R) & \longrightarrow & F \\ \downarrow & & \downarrow & & \downarrow \\ X(R) & \longrightarrow & BE(R) & \longrightarrow & BE(R)^+ \\ \downarrow & & \downarrow & & \downarrow \\ H & \longrightarrow & BGL(R) & \longrightarrow & BGL(R)^+ \end{array}$$

By the same argument as above, the map $X(R) \to H$ is a homotopy equivalence. □ □

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