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ON THE KNOTTED ELASTIC CURVES

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Abstract. According to the Bernoulli-Euler theory of elastic rods the bending energy of the wire is proportional to the total squared curvature of γ , which we will denote by $F(\gamma)=\int_\gamma k^2 ds$. If the result of J.Langer and D.Singer [3] extend to knotted elastic curve, then we obtain the following. Let $\{\gamma, M\}$ be a closed knotted elastic curve. If the curvature of γ is nonzero for everywhere, then γ lies on torus.

I. Introduction

Elastic curve (or elastica) and its generalizations have long been of interest in the context of elasticity theory. The elastica as a purely geometrical entity seems to have been largely ignored (for historical references concerning the classical elastica, we refer to the recent survey by Truesdel [2]).

Elastic curve is a mathematical model of Peano curve. And elastic energy (bending energy) is critical for $\mathcal{T}$ defined on regular curves. Euler was able to obtain a good qualitative description of all plane elastic curves. In fact, Peano curve not only has a curve but also knot.Thus elastic curve is not complete mathematical model of Peano curve. Here, in order to establish a mathematical model, consider the energy which is the sum of elastic energy and knotted energy. And define the curve its energy is critical.

II. Main theorem

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All curves, functions, vecterfields will be assumed C^∞ class. For 3-dimensional Euclidean space $\mathbf{R}^3$, Euclidean inner product will be denoted by $< \, , \, >$ and the Euclidean norm by $| \; |$.

Let $\gamma = \gamma(t) : [t_1, t_2] \to \mathbf{R}^3$ be a holomorphic C^∞-class curve. v will denote the velocity of γ and T, k will denote unite tangent vector and curvature of γ. (i.e, $v = |\frac{d\gamma}{dt}|, T = \frac{1}{v}\frac{d\gamma}{dt}, k = |\nabla_T T|$ where $\nabla_T = \frac{1}{v}\frac{d}{dt}$)

Define functional $\mathcal{F}$ by $\mathcal{F}(\gamma) = \int_{t_1}^{t_2} k^2 v dt$ and we called it elastic energy of γ.

Let M be a unit normal vector field along γ and $\{\gamma, M\}$ be a curve with unit normal vector field.

Definition 1. Let $\{\gamma, M\}$ be a curve with unit normal vectorfield and its domain is $[t_1, t_2]$. Define a function h on $[t_1, t_2]$ by $h =< \nabla_T M, L >$, and we called it a knot function of $\{\gamma, M\}$. Here $L = T \times M$, $\times$ is exterior product in $\mathbf{R}^3$. $h(t)$ be a quantity of knot of M at $\gamma(t)$.

Remark. If M is parallel to normal connection along to γ, then $h \equiv 0$.

Definition 2. $\{\gamma, M\}, v, h$ are the same notation as above.

(1) $\int_{t_1}^{t_2} h^2 v dt$ is called a knotted energy of $\{\gamma, M\}$.
(2) Let $\epsilon > 0$ be a constant. Define a functional $\mathcal{T}_\epsilon$ with respect to curve with unit normal vector field by

$$\mathcal{T}_\epsilon(\{\gamma, M\}) = \mathcal{T}(\gamma) + \epsilon \int_{t_1}^{t_2} h^2 v dt.$$

$\mathcal{T}_\epsilon(\{\gamma, M\})$ is called knotted elastic energy of coefficient ϵ of $\{\gamma, M\}$. Here, domain of $\mathcal{T}_\epsilon$ is the set of all curve with unit normal vector field.

Definition 3. Let $t_0 > 0, \phi \in \mathbf{R}/2\pi\mathbf{Z}$. $\{\gamma, M\}$ is called period t_0 if the following two conditions are satisfied.

(1) $\gamma(t + t_0) = \gamma(t)$.
(2) $M(t + t_0) = R(\phi)M(t)$ where $R(\phi) : T^2\mathbf{R}^3 \to T^\perp\mathbf{R}^3$ be a rotation of angle ϕ in each fiber of normal bundle $T^\perp\mathbf{R}^3$ along γ. (orientation is $R(\frac{\pi}{2})M(t) = L(t)$)

Let $C(t_0, \phi)$ be a set of all curves with unit normal vector field its period is t_0. Then we obtain a following lemma from the a first variation formula.

LEMMA 1. *$\{\gamma, M\} \in C(t_0, \phi)$ (its length $\int_0^{t_0} v dt$ is fixed) is critical point of $\mathcal{T}_\epsilon$ iff there exist real numbers μ, σ such that the following are satisfied.*

(1) $\nabla_T[2(\nabla_T)^2 T + (3k^2 - \mu + \epsilon h^2)T - 2\epsilon h R(\frac{\pi}{2})(\nabla_T T)] = 0.$
(2) $h(t) = \sigma.$

DEFINITION 4. Let $\{\gamma, M\}$ be a curve with unit normal vector field with velocity 1 (i.e. $v \equiv 1$). If (1) and (2) of the Lemma 1 are satisfied, then $\{\gamma, M\}$ is called a knotted elastic curve, σ is called a knot parameter of $\{\gamma, M\}$.

Let $l > 0, \phi \in \mathbf{R}/2\pi\mathbf{Z}$ and $\mathcal{U}C(l, \phi)$ be the set of velocity 1 of γ in $C(l, \phi)$ and $\{\gamma, M\}$ be an element of $\mathcal{U}C(l, \phi)$.

Define

$$T_{\{\gamma,M\}}C(l,\phi) = \left\{ (\Lambda, f) \mid \begin{array}{l} \Lambda \text{ is vector field of period} \\ l \text{ along } \gamma,\ f \text{ is a function} \\ \text{of period } l \end{array} \right\},$$

$$T_{\{\gamma,M\}}\mathcal{U}C(l,\phi) = \left\{ (\Lambda, f) \mid \begin{array}{l} (\Lambda, f) \in T_{\{}\gamma, M\}C(l,\phi) \\ \text{i.e.} < \nabla_T \Lambda, T >= 0 \end{array} \right\}.$$

Then the following lemma is satisfied.

LEMMA 2. *For the variation $\{\gamma, M\}_\lambda$ of $\{\gamma, M\}$ in $\mathcal{U}C(l, \phi)$,*
$\{\gamma, M\}_\lambda = \{\gamma_\lambda, M_\lambda\}$ $(-\lambda_0 < \lambda < \lambda_0, \{\gamma, M\}_0 = \{\gamma, M\})$
$(\frac{\partial \gamma_l}{\partial \lambda}|_{\lambda=0}, < \frac{\partial M_\lambda}{\partial \lambda}, L_\lambda > |_{\lambda=0}) \in T_{\{\gamma,M\}}\mathcal{U}C(l, \phi)$. Left side is called a variational vector field of variation $\{\gamma, M\}_\lambda$

Conversely, for any $(\Lambda, f) \in T_{\{\gamma,M\}}\mathcal{U}C(l, \phi)$, there exist a variation $\{\gamma, M\}_\lambda$ of $\{\gamma, M\}$ in $\mathcal{U}C(l, \phi)$ such that $\frac{\partial \gamma_l}{\partial \lambda}|_{\lambda=0} = \Lambda$, $< \frac{\partial M_\lambda}{\partial \lambda}, L_\lambda > |_{\lambda=0} = f$.

In the above Lemma 2, $T_{\{\gamma,M\}}\mathcal{U}C(l, \phi)$ is tangent space of $\mathcal{U}C(l, \phi)$ at $\{\gamma, M\}$.

Lemma 3. *Let $\{\gamma, M\}$ be a knotted elastic curve and also variational vector field of $\{\gamma, M\}_\lambda$. Then*

$$\frac{d^2}{d\lambda^2}|_{\lambda=0}\mathcal{T}_\epsilon(\{\gamma, M\}_\lambda) = \int_0^l < \mathcal{T}_{\{\gamma,M\}}(\Lambda, f), (\Lambda, f) > ds$$

where

$$\begin{aligned}\mathcal{T}_{\{\gamma,M\}}(\Lambda, f) = (&p[\nabla_T\{2(\nabla_T)^3\Lambda + (3k^2 - \mu + \epsilon\sigma^2)\nabla_T\Lambda \\ &- 2\epsilon\sigma R(\frac{\pi}{2})((\nabla_T)^2\Lambda - < (\nabla_T)^2\Lambda, T > T) \\ &- 2\epsilon(< \nabla_T\Lambda, R(\frac{\pi}{2})(\nabla_T T) > + Tf)R(\frac{\pi}{2})(\nabla_T T)\}] \\ &- 2\epsilon(T < \nabla_T\Lambda, R(\frac{\pi}{2})(\nabla_T T) > + T^2 f))\end{aligned}$$

($p : T_{\{\gamma,M\}}C(l,\phi) \to T_{\{\gamma,M\}}\mathcal{U}C(l,\phi)$ is an orthogonal projection with respect to L^2-inner product)

We consider eigenvalue problem $\mathcal{T}_{\{\gamma,M\}}(\Lambda, f) = p(\Lambda, f), \phi \in \mathbf{R}$, we can obtain the following theorem.

Theorem 4. *Let γ be a circle with radius 1 and $\{\gamma, M\}$ be a knotted elastic curve. Then the eigenvalue of Jacobi operator $\mathcal{T}_{\{\gamma,M\}}$ has the following properties.*

(1) *The eigenvalue of $\mathcal{T}_{\{\gamma,M\}}$ is positive whenever $0 \leq \epsilon^2\sigma^2 < 3$.*
(2) *There exists a non-trivial eigenvector its eigen value 0 whenever $\epsilon^2\sigma^2 = m^2 - 1 (2 \leq m$, m is integer).*
(3) *There exists a negative eigenvalue whenever $\epsilon^2\sigma^2 > 3$.*

The curvature k and torsion τ of elastic curve are represented by elliptic function. By J. Langer and D. Singer [3], for every elastic curve γ in $\mathbf{R}^3$ there is naturally associated to γ a cylindrical coordinate system (r, θ, z) on $\mathbf{R}^3$, the restrictions to γ of the coordinate fields $\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}\frac{\partial}{\partial z}$ being expressible in terms of k, τ, T, N, B. Thus we can see the following theorem.

Theorem 5 [J. Langer and D. Singer][3]. *Let γ be a closed elastic curve. Then γ lies on embedded tori of revolution.*

If the proof of the above theorem extend to a knotted elastic curve, then we obtain the following result.

THEOREM 6. *Let $\{\gamma, M\}$ be a closed knotted elastic curve. Suppose that, the curvature k of γ is nonzero for everywhere. Then γ lies on torus.*

Proof. Let $\{\gamma, M\}$ be a knotted elastic curve. First of all, find the curvature k and torsion τ. Application of the Frenet formula for (1) of [Lemma 1] leads to the ordinary differential equation for k, τ. Solution of the equation is concretely represented by Jacobi function. In this representation, k and τ are periodic functions with the same period and if k is constant then τ is also constant. Secondly, we will construct a cylindrical coordinate system. If k is constant, then γ is a straight line or a circle. Suppose that k is not constant.

If $\gamma = \gamma(s)$ be a curve with velocity 1 and its curvature is positive at every point, Then Λ is a vector field along γ. Λ extends to a killing vector field on $\mathbf{R}^3$ iff Λ satisfies the following.

$$< \nabla_T \Lambda, T >= 0 \tag{a}$$

$$< (\nabla_T)^2 \Lambda, N >= 0 \tag{b}$$

$$< (\nabla_T)^3 \Lambda - \frac{k_s}{k} (\nabla_T)^2 \Lambda + k^2 \nabla_T \Lambda, B >= 0 \tag{c}$$

where T, N, B form the Frenet frame for γ. Put

$$J_0 = 2(\nabla_T)^2 T + (3k^2 - \mu + \epsilon\sigma^2)T - 2\epsilon\sigma R(\frac{\pi}{2})(\nabla_T T)$$

$$H = \epsilon\sigma T + kB$$

$$J_1 = H - \frac{< J_0, H >}{|J_0|^2} J_0 \ \ (\mu, \sigma \text{ are constant in Lema 1}).$$

Then J_0, H, J_1 is also killing along γ. Let $\overline{J}_0, \overline{J}_1$ be the extension of J_0, J_1 on $\mathbf{R}^3$. In (1) of Lemma 1, we can see $\overline{J}_0$ is constant vector field. Thus $\overline{J}_1$ is a rotation field perpendicular to $\overline{J}_0$. By the above statement we obtain a cylindrical coordinate (r, θ, z). It satisfies $\frac{\partial}{\partial z} = \frac{1}{|\overline{J}_0|}\overline{J}_0$, $\frac{\partial}{\partial \theta} = c\overline{J}_1$ where c is a positive constant. Setting $\gamma(s) = (r(s), \theta(s), z(s))$ one then obtains

$$r(s) = c|J_1(s)|$$

$$\theta_s(s) = \frac{< T, \frac{\partial}{\partial \theta} >}{|\frac{\partial}{\partial \theta}|^2} = \frac{< T, J_1(s) >}{c|J_1(s)|^2}$$

$$z_s(s) =< T, \frac{1}{|J_0(s)|} J_0(s) >$$

where J_0, J_1 are vector fields along γ and components of T, N, B are represented by curvature and torsion of γ.

If $\{\gamma, M\}$ is periodic, then r,z are also periodic and curve of γ in rz-plane is a simple closed curve. Thus every closed knotted elastic curve lies on torus of revolution. Since curvature and torsion of knotted elastic curve are periodic function with some period, γ is periodic iff $\Delta z = 0$ (i.e. $\frac{\Delta\theta}{2\pi} \in \mathbf{Q}$) □ □

THEOREM 7 [J.LANGER AND D.SINGER][3]. *For any closed elastic curve* $-\pi \leq \Delta\theta \leq 0, \frac{\Delta\theta}{2\pi} \in \mathbf{Q}$ *and conversely for any* ψ, *such that* $-\pi \leq \psi \leq 0, \frac{\psi}{2\pi} \in \mathbf{Q}$ *there exists a unique closed elastic curve such that* $\Delta\theta = \psi$.

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