

## HARMONIC LITTLE BLOCH FUNCTIONS ON THE UPPER HALF-SPACE

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ABSTRACT. On the setting of the upper half-space of the euclidean  $n$ -space, we study some properties of harmonic little Bloch functions and we show that for a given harmonic little Bloch function  $u$ , there exists unique harmonic conjugates of  $u$  which are also little Bloch functions with appropriate norm bounds.

### 1. Introduction

The upper half-space  $H = H_n$  is the open subset of  $\mathbf{R}^n$  ( $n \geq 2$ ) given by

$$H = \{z = (z', z_n) \in \mathbf{R}^n : z_n > 0\},$$

where we have written a typical point  $z \in \mathbf{R}^n$  as  $z = (z', z_n)$ , with  $z' \in \mathbf{R}^{n-1}$  and  $z_n \in \mathbf{R}$ .

Given a harmonic function  $u$  on  $H$ , the functions  $v_1, \dots, v_{n-1}$  on  $H$  are called harmonic conjugates of  $u$  if

$$(1.1) \quad (v_1, \dots, v_{n-1}, u) = \nabla f$$

for some harmonic function  $f$  on  $H$ , where  $\nabla f$  denotes the gradient of  $f$ . If (1.1) holds, then  $v_1, \dots, v_{n-1}$  are partial derivatives of a harmonic function, so they are harmonic functions on  $H$ . Also (1.1) and the

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condition that  $f$  is harmonic is equivalent to the following “generalized Cauchy-Riemann equations”

$$D_k v_j = D_j v_k; D_n v_j = D_j u$$

$$\sum_{j=1}^{n-1} D_j v_j + D_n u = 0.$$

In particular,  $v$  is a harmonic conjugate of  $u$  if and only if  $u + iv$  is a holomorphic function on the upper half-plane  $H_2$ .

If  $u$  is harmonic function on  $H$ , then harmonic conjugates of  $u$  always exist. However they are far from unique. (If  $n > 2$ , then harmonic conjugates of a given harmonic function  $u$  may well differ more than a constant. We refer more on these to [3].)

A harmonic function  $u$  on  $H$  is called a harmonic Bloch function if

$$\|u\|_{\mathcal{B}} = \sup z_n |\nabla u(z)| < \infty,$$

where the supremum is taken over all  $z \in H$ . (Here we use the  $\mathbf{C}^n$  norm to calculate  $|\nabla u(z)|$ .) We let  $\tilde{\mathcal{B}}$  denote the collection of harmonic Bloch functions that vanish at  $z_0 = (0, 1)$ . Then we can show that  $\tilde{\mathcal{B}}$  is a Banach space under the Bloch norm  $\|\cdot\|_{\mathcal{B}}$ .

A harmonic Bloch function  $u$  is called a harmonic little Bloch function if it has the following vanishing condition;

$$(1.2) \quad \lim_{z \rightarrow \partial^\infty H} z_n |\nabla u(z)| = 0,$$

where  $\partial^\infty H$  denotes the union of  $\partial H$  and  $\{\infty\}$ . We let  $\tilde{\mathcal{B}}_0$  denote the set of all harmonic little Bloch functions on  $H$  vanishing at  $z_0$ . Then we can show that  $\tilde{\mathcal{B}}_0$  is also a Banach space under  $\|\cdot\|_{\mathcal{B}}$  from a straightforward computation. If we let  $C_0(H)$  denote the set of all continuous functions on  $H$  vanishing at  $\infty$ , then it is easy to show that the condition (1.2) is equivalent to the condition that the function  $u \in \tilde{\mathcal{B}}$  satisfies

$$(1.3) \quad z_n |\nabla u(z)| \in C_0(H).$$

In this paper we show some properties of harmonic little Bloch functions and we prove that for a given  $u \in \tilde{\mathcal{B}}_0$  there exist unique harmonic conjugates of  $u$  in  $\tilde{\mathcal{B}}_0$  with appropriate norm bounds. (In [4], there is a corresponding result to harmonic Bloch functions.)

## 2. Preliminary

In this section, we review some preliminary results from [1], [2]. First let's recall that  $b^2$  is the harmonic Bergman space consisting of all harmonic functions  $u$  on  $H$  such that

$$\|u\|_2 = \left( \int_H |u|^2 dV \right)^{1/2} < \infty$$

where  $dV$  denotes the Lebesgue volume measure on  $H$ , which we may write  $dz, dw$ , etc. By the mean value property and Jensen's inequality, one can easily verify that

$$(2.1) \quad |u(z)|^2 \leq \sigma_n^{-1} z_n^{-n} \|u\|_2^2$$

holds for all  $u \in b^2$  and for every  $z \in H$ . Here we use the notation  $\sigma_n$  for the volume of the unit ball of  $\mathbf{R}^n$ . It follows from inequality (2.1) that norm convergence in  $b^2$  implies uniform convergence on compact subsets of  $H$ . Thus,  $b^2$  is a Hilbert space. Inequality (2.1) also gives that, for each fixed  $z \in H$ , the map  $z \mapsto u(z)$  is a bounded linear functional on  $b^2$  and hence there exists a unique function  $R(z, \cdot) \in b^2$ , called the harmonic Bergman kernel, such that

$$u(z) = \int_H u(w) \overline{R(z, w)} dw$$

for all  $u \in b^2$ . It is known that  $R(z, w) = R(w, z)$  and that  $R(z, w)$  is real valued; thus we can remove the complex conjugate in the integral above. For this and related results see Chapter 8 of [1]. The explicit formula for the harmonic Bergman kernel is given by

$$R(z, w) = \frac{4}{n\sigma_n} \frac{n(z_n + w_n)^2 - |z - \bar{w}|^2}{|z - \bar{w}|^{n+2}}.$$

Here we use the notation  $\bar{w} = (w', -w_n)$  for  $w \in H$ . Note that if  $n = 2$ , then  $\bar{w}$  is the usual complex conjugate of  $w$ . From this explicit formula for  $R(z, w)$ , we can show that for each fixed  $z \in H$ , this harmonic Bergman kernel  $R(z, \cdot)$  is not even integrable. Since every bounded harmonic function on  $H$  is a harmonic Bloch function, one can not

expect that  $R(z, w)$  has a reproducing property for harmonic Bloch functions, however by modifying  $R(z, w)$  to  $\tilde{R}(z, w)$  by

$$\tilde{R}(z, w) = R(z, w) - R(z_0, w),$$

one can see that  $\tilde{R}(z, \cdot)$  is integrable on  $H$  and it satisfies the following reproducing property: If  $u \in \tilde{\mathcal{B}}$ , then for every  $z \in H$ ,

$$\begin{aligned} u(z) &= \int_H u(w) \tilde{R}(z, w) dw \\ &= -2 \int_H [D_{w_n} u(w)] w_n \tilde{R}(z, w) dw \\ (2.2) \quad &= -2 \int_H u(w) w_n D_{w_n} \tilde{R}(z, w) dw. \end{aligned}$$

From the explicit formula of  $\tilde{R}(z, w)$ , we can easily check that there is a positive constant  $C$  depending only on  $n$  such that

$$(2.3) \quad |\nabla_z \tilde{R}(z, w)| = |\nabla_z R(z, w)| \leq \frac{C(n)}{|z - \bar{w}|^{n+1}}$$

for all  $z, w \in H$ .

In [2], it is known that the Bloch norm is equivalent to the normal derivative norm: there are two positive constants  $c$  and  $C$  depending only on  $n$  such that

$$(2.4) \quad c\|u\|_{\mathcal{B}} \leq \|z_n D_{z_n} u\|_{\infty} \leq C\|u\|_{\mathcal{B}}$$

for all  $u \in \tilde{\mathcal{B}}$ .

### 3. Main results

In this section we prove the main results of this paper. For  $u \in \tilde{\mathcal{B}}_0$  and for  $j = 1, 2, \dots, n-1$ , set

$$(3.1) \quad A_j[u](z) = -2 \int_H [D_{w_j} u(w)] w_n \tilde{R}(z, w) dw$$

for every  $z \in H$ . Then by taking the integration by parts with respect to  $w_j$ -axis, we easily get

$$(3.2) \quad A_j[u](z) = 2 \int_H u(w) w_n D_{w_j} \tilde{R}(z, w) dw$$

for  $z \in H$ . Below we show that for each  $j = 1, 2, \dots, n-1$ ,  $A_j$  maps  $\tilde{\mathcal{B}}_0$  into  $\tilde{\mathcal{B}}_0$  boundedly. To do so we need the following lemma.

LEMMA 3.1. *There is a constant  $C$  depending only on  $n$  such that*

$$\int_H \frac{1}{|z - \bar{w}|^{n+1}} dw \leq \frac{C}{z_n}$$

for all  $z \in H$ .

*Proof.* Fix  $z \in H$ . Then we have

$$(3.3) \quad \int_H \frac{1}{|z - \bar{w}|^{n+1}} dw \leq \int_0^\infty \frac{1}{(z_n + w_n)^2} \int_{\mathbf{R}^{n-1}} \frac{(z_n + w_n)}{|z - \bar{w}|^n} dw' dw_n.$$

From the Poisson integral theory, we know the inner integral of (3.3) equals  $n\sigma_n/2$ . (See [1] and [3] for details on this.) Hence, after applying change of variable  $w_n \mapsto z_n w_n$ , we see that

$$\int_H \frac{1}{|z - \bar{w}|^{n+1}} dw \leq \frac{C}{z_n}.$$

This completes the proof. □ □

We are now ready to show one of the main results of this paper. Here and for the rest of this paper,  $C$  denotes the constant depending only on  $n$  which varies from line to line.

THEOREM 3.2. *For each  $j = 1, 2, \dots, n - 1$ , the map  $A_j$  is bounded and linear from  $\tilde{\mathcal{B}}_0$  into  $\tilde{\mathcal{B}}_0$ .*

*Proof.* Fix  $j$ . The linearity of the map  $A_j$  is clear and let  $u \in \tilde{\mathcal{B}}_0$ . Because  $\tilde{R}(z_0, w) = 0$ , we have  $A_j[u](z_0) = 0$ . By passing the Laplacian  $\Delta_z$  through the integral in (3.1), we easily see that  $A_j[u]$  is harmonic on  $H$  since  $\tilde{R}(z, w)$  is harmonic as a function of  $z$ . Note that

$$\begin{aligned} z_n |\nabla A_j[u](z)| &= 2z_n \left| \int_H [D_{w_j} u(w)] w_n \nabla_z \tilde{R}(z, w) dw \right| \\ &\leq C z_n \|u\|_{\mathcal{B}} \int_H \frac{1}{|z - \bar{w}|^{n+1}} dw \\ &\leq C \|u\|_{\mathcal{B}}, \end{aligned}$$

where we used the estimate (2.3) and the Lemma 3.1. This shows that  $A_j[u] \in \tilde{\mathcal{B}}$  with  $\|A_j[u]\|_{\mathcal{B}} \leq C\|u\|_{\mathcal{B}}$ . Therefore it remains to show the vanishing property (1.3) of  $A_j[u]$ . Note that

$$z_n |\nabla A_j[u](z)| \leq Cz_n \int_H |D_{w_j} u(w)| w_n \frac{1}{|z - \bar{w}|^{n+1}} dw$$

for some constant  $C$ . Note also that  $w_n |D_{w_j} u(w)| \in C_0(H)$ , because  $u \in \tilde{\mathcal{B}}_0$ . Let  $\epsilon > 0$ . Then there is a compact set  $K$  in  $H$  such that  $|w_n D_{w_j} u(w)| < \epsilon$  on  $H \setminus K$ . Therefore we have

$$(3.4) \quad \begin{aligned} z_n |\nabla A_j[u](z)| &\leq \epsilon C \int_{H \setminus K} \frac{z_n}{|z - \bar{w}|^{n+1}} dw \\ &\quad + C \|u\|_{\mathcal{B}} \int_K \frac{z_n}{|z - \bar{w}|^{n+1}} dw. \end{aligned}$$

Let  $I$  and  $II$  denote, respectively, the two integrals of (3.4). Then from Lemma 3.1, we get

$$I \leq \int_H \frac{z_n}{|z - \bar{w}|^{n+1}} dw \leq C.$$

Notice that

$$II \leq C(n, K) \frac{z_n}{1 + |z|^{n+1}}$$

for some constant  $C(n, K)$  depending only on  $n$  and  $K$  and notice also that the function  $z \mapsto z_n/(1 + |z|^{n+1})$  is in  $C_0(H)$ . This shows that  $z_n |\nabla A_j[u](z)| \in C_0(H)$  and the proof is complete.  $\square$   $\square$

Now we are ready to prove that for a given  $u \in \tilde{\mathcal{B}}_0$ , there are unique harmonic conjugates  $v_1, v_2, \dots, v_{n-1}$  of  $u$  which are also in  $\tilde{\mathcal{B}}_0$ . The proof of the following theorem is quite similar to the proof of the theorem given in [4] for harmonic Bloch functions, however we give the proof of it for the reader's convenience.

**THEOREM 3.3.** *For each  $u \in \tilde{\mathcal{B}}_0$ , there exist unique harmonic conjugates  $v_1, \dots, v_{n-1}$  of  $u$  on  $H$  such that  $v_j \in \tilde{\mathcal{B}}_0$  for each  $j$ . Moreover, there exists a positive constant  $C$  such that  $\|v_j\|_{\mathcal{B}} \leq C\|u\|_{\mathcal{B}}$  for each  $j$ .*

*Proof.* Let  $u \in \tilde{\mathcal{B}}_0$ . For each  $j = 1, \dots, n - 1$ , set  $v_j = A_j[u]$ . Then by Theorem 3.2, we know each  $v_j \in \tilde{\mathcal{B}}_0$  and  $\|v_j\|_{\mathcal{B}} \leq C\|u\|_{\mathcal{B}}$  for each  $j$ . Now from the explicit formula of  $\tilde{R}$ , we can check easily that for  $j, k = 1, 2, \dots, n - 1$ ,

$$D_{z_k} D_{w_j} \tilde{R}(z, w) = -D_{z_j} D_{z_k} R(z, w) = D_{z_j} D_{w_k} \tilde{R}(z, w),$$

$$(3.5) \quad D_{z_n} D_{w_j} \tilde{R}(z, w) = -D_{z_j} D_{w_n} \tilde{R}(z, w).$$

Note that

$$D_{z_n} D_{w_n} \tilde{R}(z, w) = D_{z_n}^2 R(z, w).$$

Therefore by differentiating through the integral in (3.2), we have  $D_k v_j = D_j v_k$ . Similary from (3.1) and (3.5), we get  $D_n v_j = D_j u$ . Furthermore, from (2.2) and (3.2) we also have

$$\sum_{j=1}^{n-1} D_j v_j(z) + D_n u(z) = -2 \int_H u(w) w_n \Delta_z R(z, w) dw = 0$$

for all  $z \in H$  since  $R(z, w)$  is harmonic on  $H$  as a function of  $z$ . Therefore  $v_1, \dots, v_{n-1}, u$  satisfy the generalized Cauchy-Riemann equations and it follows that  $v_1, \dots, v_{n-1}$  are harmonic conjugates of  $u$ .

To complete the proof of this theorem, we only need to show the uniqueness part. Suppose that  $u_1, \dots, u_{n-1}$  are also harmonic conjugates of  $u$  satisfying  $u_j \in \tilde{\mathcal{B}}_0$  for each  $j$ . Then by (2.4) we have

$$\|v_j - u_j\|_{\mathcal{B}} \leq C \|z_n D_{z_n} (v_j - u_j)\|_{\infty}$$

for each  $j$ . Because  $D_{z_n} (v_j - u_j) = D_j (u - u) = 0$ , we have  $\|v_j - u_j\|_{\mathcal{B}} = 0$  and so  $v_j = u_j$  for each  $j$ . This completes the proof.  $\square \quad \square$

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