A NOTE ON LIFTING TRANSFORMATION GROUPS

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Abstract. The purpose of this note is to compare two known results related to the lifting problem of an action of a topological group $G$ on a $G$-space $X$ to a covering space of $X$.

1. Introduction

For a $G$-space $X$ and a covering space $\tilde{X}_H$ of $X$ associated with a subgroup $H$ of $\pi_1(X, x_0)$, there exist some results related to the lifting problem of an action of $G$ on $X$ to an action of $G$ on $\tilde{X}_H$. In this note, we show that the result due to M. A. Armstrong [1] is equivalent to a minor modification of the result due to F. Rhodes [2] under some restricted conditions. Also, we briefly refer to a role of the evaluation map with respect to the lifting problem.

We shall assume throughout this note that $G$ is a locally path-connected topological group, that $X$ is a path-connected, locally path-connected, and locally simply connected $G$-space and that $p : \tilde{X}_H \to X$ is a covering projection associated with a subgroup $H$ of $\pi_1(X, x_0)$. Also, we use the following notations:

- $e$: the identity element of $G$.
- $\alpha * \beta$: the composition of two paths $\alpha$ and $\beta$.
- $f \circ g$: the composition of two functions $f$ and $g$.
- $i_X$: the identity function on a set $X$.
- $f_\#$: the homomorphism from $\pi_1(X, x_0)$ to $\pi_1(Y, f(x_0))$ induced by a map $f : X \to Y$.

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2. Preliminaries

For $g \in G$, let $\lambda$ be a path from $x_0$ to $gx_0$. Define $g_\ast : \pi_1(X, x_0) \to \pi_1(X, x_0)$ by $g_\ast([\alpha]) = [\lambda \ast g\alpha \ast \lambda^{-1}]$ for $[\alpha] \in \pi_1(X, x_0)$. It is clear that for every normal subgroup $H$ of $\pi_1(X, x_0)$, $g_\ast(H)$ is a normal subgroup which is independent of $\lambda$.

**Definition 2.1.** ([2]) A normal subgroup $H$ of $\pi_1(X, x_0)$ is said to be $G$-invariant if $g_\ast(H) = H$ for every $g \in G$.

**Definition 2.2.** ([2]) Given $g \in G$, a path order $g$, written by $(\alpha; g)$, with base point $x_0$ is a continuous function $\alpha : I \to X$ such that $\alpha(0) = x_0$ and $\alpha(1) = gx_0$.

**Lemma 2.3.** ([2]) Let $H$ be a subgroup of $\pi_1(X, x_0)$ and let $[\alpha; g]_H$ be the equivalence class of $(\alpha; g)$ under the equivalence relation

$$(\alpha; g) \sim (\beta; h) \text{ iff } g = h \text{ and } [\alpha \ast \beta^{-1}] \in H.$$If $H$ is $G$-invariant normal, then the set $\sigma_H(X, x_0, G)$ of equivalence classes forms a group under the rule of composition

$$[\alpha; g]_H \ast [\beta; h]_H = [\alpha \ast g\beta; gh]_H.$$**Lemma 2.4.** ([2]) Let $H$ be a subgroup of $\pi_1(X, x_0)$. If $\sigma_H(X, x_0, G)$ is a group, then we have a short exact sequence

$$0 \to \pi_1(X, x_0)/H \to \sigma_H(X, x_0, G) \to G \to 0,$$where $i([\alpha] \ast H) = [\alpha; e]_H$ and $j([\beta; g]_H) = g$.

From now on, $j$ always denotes the homomorphism defined in Lemma 2.4.

In [2], a basis of open nbds is defined for the set $\sigma_H(X, x_0, G)$ as follows. Given $[\alpha; g]_H$ and open nbds $U$ of $gx_0$ and $V$ of $e$, define $W_X([\alpha; g]_H, U, V)$ to be the set of classes $[\alpha \ast \beta; h]_H$ where $hg^{-1} \in V$ and $\beta$ is a path in $U$ from $gx_0$ to $hx_0$. Sets of the form of $W_X([\alpha; g]_H, U, V)$ constitute a basis for a topology on $\sigma_H(X, x_0, G)$.

F. Rhodes [2] showed that, if $\sigma_H(X, x_0, G)$ is a group, it is a topological group with the topology just defined.
Definition 2.5. ([2]) Let \( \sigma_H(X, x_0, G) \) be a group. If there exists a continuous homomorphism \( \phi : G \to \sigma_H(X, x_0, G) \) such that \( j \circ \phi = i_G \), then the group \( \sigma_H(X, x_0, G) \) is said to admit a \textit{continuous split extension}.

Definition 2.6. We say that \( X \) admits a \textit{family of \( H \)-preferred paths} at \( x_0 \) if it is possible to associate with every element \( g \) of \( G \) a path \( k_g \) from \( gx_0 \) to \( x_0 \) such that \( [k_e] \in H \) and for every pair of elements \( g, h \), the paths \( k_g, k_h \) and \( k_{gh} \) associated with \( g, h \) and \( gh \) satisfy \( [g k_h * k_g * k_{gh}^{-1}] \in H \).

Definition 2.7. ([1]) Suppose that \( G \) also acts on a space \( Z \), and that \( f : Z \to X \) is a \( G \)-map which sends \( z_0 \) to \( x_0 \). If for every element \( g \) of \( G \), loop \( \alpha \) representing an element of \( H \) and path \( \gamma \) which joins \( z_0 \) to \( gz_0 \) in \( Z \), \( [(f \gamma) * g \alpha * (f \gamma^{-1})] \in H \), then \( H \) is said to be \((f, G)\)-\textit{invariant}.

3. Main Results

Lemma 3.1. Let \( H \) be a normal subgroup of \( \pi_1(X, x_0) \). If for every \( g \in G, \, g_*(H) \subset H \), then \( \sigma_H(X, x_0, G) \) is a group.

Proof. Assume \([\alpha_1; g]_H = [\alpha_2; g]_H \) and \([\beta_1; h]_H = [\beta_2; h]_H \). Then \([\alpha_1 \ast \alpha_2^{-1}], [\beta_1 \ast \beta_2^{-1}] \in H \). Since \( g^{-1} \alpha_2 \) is a path from \( g^{-1}x_0 \) to \( x_0 \), \([g^{-1} \alpha_2] = [g^{-1} \beta_2] \), \([g^{-1} \alpha_2^{-1}] \ast [g^{-1} \beta_2^{-1}] \ast [g^{-1} \alpha_2] \in H \). From this, we obtain

\[
[(\alpha_2 \ast g \beta_2)^{-1}] \ast [(\alpha_1 \ast \alpha_2^{-1}) \ast (\beta_1 \ast \beta_2^{-1})]^{-1} = [\alpha_1 \ast (g(\beta_1 \ast \beta_2^{-1}) \ast [g^{-1} \alpha_2^{-1}] \ast [g^{-1} \beta_2^{-1}] \ast [g^{-1} \alpha_2] \ast [g^{-1} \alpha_2]) \ast [g^{-1} \alpha_2] \ast H.
\]

Thus \( [(\alpha_1 \ast g \beta_1 \ast (\alpha_2 \ast g \beta_2)^{-1}] \in H \). This says that the binary operation is well defined. The other conditions for \( \sigma_H(X, x_0, G) \) to be a group is obvious. \( \square \)
Lemma 3.2. Let $H$ be a subgroup of $\pi_1(X, x_0)$. If there exists a path connected space $Z$, and an action of $G$ on $Z$, and a based $G$-map $f : (Z, z_0) \to (X, x_0)$ such that $f_\#(\pi_1(Z, z_0)) \subset H$, then $X$ admits a family of $H$-preferred paths at $x_0$. Furthermore, if $H$ is a normal subgroup of $\pi_1(X, x_0)$ such that $g_*(H) \subset H$ for all $g \in G$, then $\sigma_H(X, x_0, G)$ admits a continuous split extension.

Proof. For each $g \in G$, choose a path $\gamma_g$ in $Z$ which joins $g z_0$ to $z_0$ and let $k_g = f \gamma_g$. By hypothesis, $[k_e] = [f \gamma_e] = f_\#([\gamma_e]) \in H$. If $g, h \in G$, then $g \gamma_h \ast \gamma_g \ast \gamma_g^{-1} \ast \gamma_h^{-1}$ is a loop at $z_0$. Since $f_\#(\pi_1(X, x_0)) \subset H$, $[g k_h \ast k_g \ast k_g^{-1} \ast k_h^{-1}] \in H$. Thus $\{k_g|g \in G\}$ is a collection of $H$-preferred paths at $x_0$. Now, assume that $g_*(H) \subset H$ for all $g \in G$. By Lemma 3.1, $\sigma_H(X, x_0, G)$ is a group. Define $\phi : G \to \sigma_H(X, x_0, G)$ by $\phi(g) = [k_g^{-1} \ast g]_H$. Since $\{k_g|g \in G\}$ is a family of $H$-preferred paths,

$$\phi(g_1 g_2) = [k_{g_1, g_2}^{-1} \ast g_1 g_2]_H = [k_{g_1}^{-1} \ast g_1 k_{g_2}^{-1} \ast g_1 g_2]_H = [k_{g_1}^{-1} \ast g_1]_H \ast [k_{g_2}^{-1} \ast g_2]_H = \phi(g_1) \ast \phi(g_2).$$

This shows that $\phi$ is a splitting homomorphism. Let $W_X([k_g^{-1} \ast g]_H, U, V)$ be a basis element containing $[k_g^{-1} \ast g]_H$. Choose an open nbd $V_1$ of $e$ such that $V_1 \subset V$ and for any $h_1 \in V_1, h_1 g x_0 \in U$. Also, choose an open nbd $V_2$ of $e$ such that for all $h_2 \in V_2, h_2 g z_0 \in f^{-1}(U)$. Let $V'$ be the path component of $V_1 \cap V_2$ which contains $e$, let $g' \in V' g$ and let $c : I \to V g$ be a path which joins $g$ and $g'$. Then the map $g : I \to Z$, defined by $\gamma(s) = c(s) z_0$ is a path in $f^{-1}(U)$ which joins $g z_0$ to $g' z_0$, and hence $f \gamma$ is a path in $U$ joining $g z_0$ to $g' x_0$. Since $[k_g^{-1} \ast (f \gamma) \ast k_g] = f_\#([\gamma_g^{-1} \ast \gamma \ast \gamma_g')] \in H$, we have $[k_{g'}^{-1} \ast g']_H = [k_g^{-1} \ast (f \gamma); g']_H \in W_X([k_g^{-1} \ast g]_H, U, V)$ and hence $\phi(V' g) \subset W_X([k_g^{-1} \ast g]_H, U, V)$. Consequently, $\phi$ is continuous. □

Theorem 3.3. Let $H$ be a normal subgroup of $\pi_1(X, x_0)$ and let $Z$ and $f$ be the same as in Lemma 3.2. If

(i) $H$ is $(f, G)$-invariant and
(ii) $f_\#(\pi_1(Z, z_0)) \subset H$,

then $\sigma_H(X, x_0, G)$ is a group which admits a continuous split extension. Furthermore, $g_*(H) = H$ for every $g \in G$. 


Proof. By Lemma 3.2, there exists a family \( \{k_g\} | g \in G \) of \( H \) preferred paths at \( x_0 \). Let \( g \in G \) and \( [\alpha] \in H \). Since for every \( g \in G \), 
\[
g_*([\alpha]) = [k^{-1}_g * g\alpha * k_g] = [(f\gamma^{-1}_g) * g\alpha * (f\gamma_g)] \in H \]
by (i), we have \( g_*(H) \subset H \). By Lemma 3.1 and Lemma 3.2, \( \sigma_H(X, x_0, G) \) is a group which admits a continuous split extension.

To show that \( H \subset g_*(H) \), let \( [\alpha] \in H \). Since \( g\gamma^{-1}_g * \gamma_g \) is a loop in \( Z \) based at \( z_0 \), \( [g\gamma^{-1}_g * k_g] = f_#([g\gamma^{-1}_g * \gamma_g]) \in H \) by (ii). Let \( \beta = g\gamma^{-1}_g * k_g \). Then

\[
[\alpha] = [\beta^{-1} * (\beta * \alpha * \beta^{-1}) * \beta] \\
= [k^{-1}_g * g(k^{-1}_g * g^{-1}(\beta * \alpha * \beta^{-1}) * k^{-1}_g) * k_g] \\
= g_*([k^{-1}_g * g^{-1}(\beta * \alpha * \beta^{-1}) * k^{-1}_g]) \\
= (g_* \circ g_*^{-1})([\beta * \alpha * \beta^{-1}])
\]

\( \in g_*(H) \).

\[\square\]

Lemma 3.4. Let \( \sigma_H(X, x_0, G) \) be a group. Then \( X \) admits a family of \( H \)-preferred paths at \( x_0 \) if and only if the short exact sequence in Lemma 2.4 splits.

Proof. \((\Rightarrow)\) Define \( \phi : G \to \sigma_H(X, x_0, G) \) by \( \phi(g) = [\alpha^{-1}_g; g]_H \), where \( \alpha_g \) is an \( H \)-preferred path associated with \( g \). Clearly, \( j \circ \phi = i_G \). Let \( g, h \in G \). Since \( [g\alpha_h * \alpha_g * \alpha^{-1}_h] \in H \), we have \( \phi(gh) = [\alpha^{-1}_{gh}; gh]_H = [\alpha^{-1}_g * \alpha^{-1}_h; gh]_H = [\alpha^{-1}_g; g] * [\alpha^{-1}_h; h]_H = \phi(g) * \phi(h) \). Thus \( \phi \) is a splitting homomorphism.

\((\Leftarrow)\) Let \( \phi : G \to \sigma_H(X, x_0, G) \) be a splitting homomorphism. Then \( \phi(e) = [c_{x_0}; e]_H \), where \( c_{x_0} \) is the constant path at \( x_0 \). For each \( g \in G \), let \( \phi(g) = [\alpha_g; g]_H \). Since \( [\alpha_{gh}; gh]_H = \phi(gh) = \phi(g) * \phi(h) = [\alpha_g * g\alpha_h; gh]_H \), we have \( [\alpha_g * g\alpha_h * \alpha^{-1}_{gh}] \in H \). Therefore, \( \{\alpha^{-1}_g | g \in G\} \) is a collection of \( H \)-preferred paths at \( x_0 \).

\[\square\]

Theorem 3.5. Let \( H \) be a normal subgroup of \( \pi_1(X, x_0) \). If \( g_*(H) \subset H \) for every \( g \in G \) and \( \sigma_H(X, x_0, G) \) admits a continuous split extension, then the action of \( G \) lifts to an action of \( G \) on \( \tilde{X}_H \).
Proof. Define \( \tilde{\mu} : \sigma_H(X, x_0, G) \times \tilde{X}_H \to \tilde{X}_H \) by \( \tilde{\mu}([\alpha; g]_H, \omega) = [\alpha \ast g \omega] \) for \( [\alpha; g]_H \in \sigma_H(X, x_0, G) \) and \( \omega \in \tilde{X}_H \). Then \( \tilde{\mu} \) is a well-defined action of \( \sigma_H(X, x_0, G) \) on \( \tilde{X}_H \). (see Proposition 2 of [2]) By hypothesis, there exists a continuous homomorphism \( \phi : G \to \sigma_H(X, x_0, G) \) such that \( j \circ \phi = i_G \). Let \( \mu \) be the composition of

\[
G \times \tilde{X}_H \xrightarrow{\phi \times \tilde{\mu}} \sigma_H(X, x_0, G) \times \tilde{X}_H \xrightarrow{\tilde{\mu}} \tilde{X}_H.
\]

Clearly, \( \mu \) covers the action of \( G \) on \( X \). Let \( \phi(g) = [\alpha_g; g]_H \) for \( g \in G \). By Lemma 3.4, \( \{\alpha_g^{-1} : g \in G\} \) is a family of \( H \)-preferred paths. Thus for \( g_1, g_2 \in G \) and \( \omega \in \tilde{X}_H \),

\[
\mu(g_1 g_2, \omega) = [\alpha_{g_1} g_2 \ast (g_1 g_2) \omega = [\alpha_{g_1} \ast g_1 \alpha_{g_2} \ast (g_1 g_2) \omega = [\alpha_{g_1} \ast g_1 (\alpha_{g_2} \ast g_2 \omega) = \mu(g_1, [\alpha_{g_2} \ast g_2 \omega]) = \mu(g_1, \mu(g_2, \omega)).
\]

Since \( \mu(e, \omega) = \omega \) for all \( \omega \in \tilde{X}_H \), we conclude that \( \mu \) is an action of \( G \) on \( \tilde{X}_H \). \( \square \)

Now, let \( E : G \to X \) be the evaluation map define by \( E(g) = gx_0 \) for \( g \in G \).

**Lemma 3.6.** If \( N \) is a \( G \)-invariant subgroup of \( \pi_1(G, e) \) such that \( E^#(N) \subset H \), then the map

\[
E^R_# : \sigma_N(G, e, G) \to \sigma_H(X, x_0, G),
\]

defined by \( E^R_#([\gamma; g]_N) = [E\gamma; g]_H \) for \( [\gamma; g]_N \in \sigma_N(G, e, G) \), is a continuous homomorphism.

**Proof.** Clearly, \( E^R_# \) is a well-defined homomorphism. Now, let \( [\gamma; g]_N \in \sigma_N(G, e, G) \) and let \( W_X([w\gamma; g]_H, U, V) \) be an open neighborhood of \( [E\gamma; g]_H \). Since \( E \) is continuous, there exists an open neighborhood \( U' \) of \( g \) such that \( E(U') \subset U \). Let \( V' \) be an open neighborhood
of \( e \) such that \( V'g \subset U' \cap Vg \). Then for any \( h \in V'g \) and any path \( \gamma' \) in \( U' \) from \( g \) to \( h \), \( h \in Vg \) and \( E\gamma' \) is a path in \( U \) from \( gx_0 \) to \( hx_0 \). This means that

\[
E^R_\#(W_G([\gamma;g]_N,U',V')) \subset W_X([E\gamma;g]_H,U,V).
\]

Thus, \( E^R_\# \) is continuous. \( \Box \)

**Lemma 3.7.** Let \( N \) be a \( G \)-invariant subgroup of \( \pi_1(G,e) \) such that \( E^R_\#(N) \subset H \). If \( \sigma_N(G,e,G) \) admits a continuous split extension, then \( \sigma_H(X,x_0,G) \) admits a continuous split extension.

**Proof.** Consider the following commutative diagram

\[
\begin{array}{ccc}
\sigma_N(G,e,G) & \xrightarrow{j'} & G \\
E^R_\# & \downarrow{i_G} & \\
\sigma_H(X,x_0,G) & \xrightarrow{j} & G
\end{array}
\]

where \( j'([\gamma;g]_N) = g \) for \( [\gamma;g]_N \in \sigma_G(G,e,G) \).

By hypothesis, there exists a continuous homomorphism \( \phi' : G \to \sigma_N(G,e,G) \) such that \( j' \circ \phi' = i_G \). Let \( \phi = E^R_\# \circ \phi' \). By Lemma 3.6, \( \phi \) is a continuous homomorphism. Since \( j \circ \phi = j \circ (E^R_\# \circ \phi') = j' \circ \phi' = i_G \), \( \sigma_H(X,x_0,G) \) admits a continuous split extension. \( \Box \)

**Lemma 3.8.** If \( \pi_1(G,e) = N \), then \( \sigma_N(G,e,G) \) admits a continuous split extension.

**Proof.** By hypothesis, \( j' : \sigma_N(G,e,G) \to G \) is an isomorphism. Let \( \phi' = (j')^{-1} \). For \( g \in G \), let \( \phi'(g) = [\alpha_g;g]_H \) and let \( W([\alpha_g;g]_H,U,V) \) be an open nbhd of \( [\alpha_g;g]_H \). Without loss of generality, we may assume that \( U \) is path connected. For \( h \in Vg \), choose a path \( \gamma \) in \( U \) from \( gx_0 \) to \( hx_0 \). Since \( \phi' \) is an isomorphism, \( [\alpha_h;h]_H = [\alpha_g \ast \gamma;h]_H \in W([\alpha_g;g]_H,U,V) \), and hence \( \phi'(Vg) \subset W([\alpha_g;g]_H,U,V) \). This implies that \( \phi' \) is continuous. \( \Box \)

**Corollary 3.9.** Let \( H \) be a \( G \)-invariant normal subgroup of \( \pi_1(X,x_0) \). If \( E^R_\#(\pi_1(G,e)) \subset H \), then the action of \( G \) on \( X \) lifts to an action of \( G \) on \( \tilde{X}_H \).
References


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