COMPLEX BORDISM OF CLASSIFYING SPACES OF THE DIHEDRAL GROUP

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Abstract. In this paper, we study the $BP_\ast$-module structure of $BP_\ast(BG)$ mod $(p, v_1, \cdots)^2$ for non-abelian groups of the order $p^3$. We know $grBP_\ast(BG) = BP_\ast \otimes H(H_\ast(BG); Q_1) \oplus BP_\ast/(p, v_1) \otimes ImQ_1$. The similar fact occurs for $BP_\ast$-homology $grBP_\ast(BG) = BP_\ast s^{-1}H(H_\ast(BG); Q_1) \oplus BP_\ast/(p, v)s^{-1}H^{odd}(BG)$ by using the spectral sequence $E_2^{s,t} = Ext_{BP_\ast}(BP_\ast(BG), BP^s) \Rightarrow BP^*(BG)$.

0. Introduction

Let $G$ be a finite group. By $G$-$U$-manifold we mean a weakly complex manifold with a free $G$-action preserving its weakly complex structure. The group of bordism classes of closed $G$-$U$-manifolds is isomorphic to the complex bordism group $M_\ast(BG)$ of the classifying spaces $BG$. If $S$ is a Sylow $p$-subgroup of $G$, the inclusion map induces a splitting epimorphism $MU_\ast(BS) \Rightarrow MU_\ast(BG)$. Hence we need know first for $p$-group $G$. Moreover Quillen isomorphism $MU_\ast(-)_{(p)} \cong MU_\ast(p) \otimes_{BP_\ast} BP_\ast(-)$ shows that we need to know only $BP_\ast(BG)$. When $G$ is a cyclic or quaternion group, giving dimensional filtration the graded group $grBP_\ast(BG) = BP_\ast \otimes H_\ast(BG)$ since $H^{even}(BG) = 0$ [M]. By Johnson-Wilson [3], $grBP_\ast(BG)$ is given for an elementary abelian $p$-groups using arguments to generalize Kunneth formula. In this paper we determine $BP_\ast$-module structure of $grBP_\ast(BG)$ mod $(p, v_1, \cdots)^2$ for non-abelian groups of the order $p^3$. For $p = 2$, the new

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group is the dihedral group $D_4$. The bordism group $grBP_*(BD_{2q}), q \neq 2$, was studied by Kamata-Minami [5]. Recall the Milnor primitive operation $Q_0 = \beta, Q = p^1\beta - \beta p^1(= S_q^2S_q^1 - S_q^1S_q^2$ for $p = 2)$. For the above group, we can extend the operation $Q_1$ on $H_*(BG)$ so that $Q_1|_{H_{even}(BG)} = 0$. By Tezuka-Tagita [7], we know $grBP_*(BG) = BP_* \otimes H(H_*(BG); Q_1) \oplus BP_*/(p, v_1) \otimes ImQ_1$ since $d_{2p-1} = v_1 \otimes Q_1$ is the only non zero differential in the Atiyah-Hirzebruch spectral sequence. The similar fact occurs for $BP_*$-homology $grBP_*(BG) = BP_* s^{-1} \otimes H(H_*(BG); Q_1) \oplus BP_*/(p, v_1)s^{-1}H_{odd}(BG)$ where $s^{-1}$ is the descending degree one map, by using the spectral sequence $E_2^{*,*} = Ext_{BP_*}(BP_*(BG), BP_*) \Rightarrow BP_*(BG)$. In particular, generators and relations are given explicitly for $BP_*(BD)$ in the last section.

1. Bordism and cobordism

Assume always that $G$ is a $p$-group. Let us write by $H^*$ (resp. $H_{\mathbb{Z}/p}$, $H^{even}$, $H^{odd}$) the cohomology $H^*(BG)$ (resp. $H^*(BG; \mathbb{Z}/p)$, $H^{even}(BG)$, $H^{odd}(BG)$). In this section we consider only groups which satisfy the following assumption.

**Assumption 1.1.** $pH_{\mathbb{Z}/p} = 0$ hence $H^{odd} \subset H_{\mathbb{Z}/p}^{odd}$, moreover $Q_1/H^{odd}$ is injective.

Since $Q_1/H^{odd}$ is injective, we can define $Q_1|H^{even} = 0$.

**Lemma 1.2.** $grBP_*(BG) \cong BP_* \otimes H(H^*; Q_1) + BP_*/(p, v_1) \otimes ImQ_1$.

**Proof.** Consider Atiyah-Hirzebruch spectral sequence

$$E_2^{*,*} = H^*(BG; BP_*) = BP_* \otimes H \implies BP_*(BG).$$

The first nonzero differential is $d_{2p-1} = v_1 \otimes Q_1$, hence we get that $E_{2p}$ is isomorphic the righthandside of the module in the lemma. Since $KerQ_1 = ImQ_1 + H(H^*; Q_1)$ is even dimensionally generated, and so is $E_{2p}$. Therefore $E_{2p} \cong E_\infty$. □ □

Given $\mathbb{Z}_{(p)}$-module $A$, let us write by $FA$ the $\mathbb{Z}_{(p)}$-free module generated by $\mathbb{Z}_{(p)}$-module generators of $A$. Let $F(x)$ be a generator which corresponds $x$ in $A$. 

Theorem 1.3. There is a $BP^*$-module isomorphism

$$BP^*(BG) \cong BP^* \otimes (FH(H^*;Q_1) + FImQ_1)/R$$

where $R$ is generated, modulo $(p, v_1, \cdots)^2$, $\sum_{n=0}^\infty v_nF(Q_nQ_1^{-1}(x)) = 0$ for $i = 0, 1$, and $x \in KerQ_1$.

Proof. If $\overline{x_1} \in ImQ_1$, then there is a relation $v_1x_1 + v_2x_2 + \cdots = 0$ form Lemma 1.2, for $\rho(x_1) = \overline{x_1}$ where $\rho : BP \to HZ(p)$ is the Thom map. From Lemma 2.1 there is $y \in HZ/p^*$ such that $Q_n(y) = \rho(X_n)$, and $y = Q_1^{-1}x_1$. Since $BP^*(BG) \otimes_{BP^*} Z(p) = H^{even}$ we have the relation in the lemma. For $x_0 \in ImQ_0$, we also have the relation by the same arguments. □ □

Now we consider the bordism theory. We also write by $H_*$ the homology $H_*(BG)$. Since $H_*$ is torsion module, there is an isomorphism

$$H_1s^{-1}H_*, \quad \text{for } * \geq 2,$$

where $s^{-1}$ is the operation descending degree one. Note that if $px = 0$, $s^{-1}x = Q_0^{-1}x$ for $x \in H^*$.

Consider the spectral sequence

$$E_{s,t}^2 = H_s(BG; BP_s) \Rightarrow BP^*(BG)$$

Lemma 1.5. $E_{s,t}^{2p} \cong BP_s s^{-1}H(H^*;Q_1) + BP_s/(p, v_1)s^{-1}H^{odd}$.

Proof. First note $HZ/P_* = Hom(HZ/p^*; Z/p)$. Hence we can define the dual operation $Q_1s$ in $HZ/p_*$. Since $Q_1Q_0 = -Q_0Q_1$. We see easily

$$Q_1s^{-1}(ImQ_1) = s^{-1}H^{odd}.$$  

The first non zero differential in (1.4) is $d_{2p-1} = v_1 \otimes Q_1$. Hence we get the lemma. □ □

We use here arguments by Ravenel and Johnson-Wilson [3]. Recall the universal coefficient spectral sequence

$$E_{s,t}^{2*} = Ext_{BP_*}(BP_s(BG), BP^*) \Rightarrow BP^*(BG).$$

Given $BP_*$-filtration in $BP_*(BG)$, we can construct spectral sequence

$$G_{s,t}^{*,*} = Ext_{BP_*}(grBP_*(BG), BP^*) \Rightarrow E_{s,t}^{*,*}.$$  

Then it is easily seen
Lemma 1.8 [3, Lemma 6.5]. \( \text{Ext}_{BP_*}(BP_*/(p^k), BP^*) \cong sBP^*/(p^k) \), 
\( \text{Ext}_{BP_*}(BP_*/(p,v_1), BP^*) \cong s^{2p}BP_*/(p,v_1) \).

Therefore from Lemma 1.2, Lemma 1.5 and Lemma 1.8, we get

\[(1.9) \quad \text{Ext}_{BP_*}((E_2^{2p}, \text{ in Lemma 1.5}), BP^*) \cong \text{gr}BP^*/(BG) \]

If \( E_2^{2p} \neq E_\infty = \text{gr}BP^*/(BG) \) in (1.4), there is an element in \( \text{gr}BP^*/(BG) \) which does not correspond \( G_\infty \) and \( G_2 \), and this makes a contradiction to (1.6). Hence (1.6), (1.7) and \( E_2^{2p} \) in (1.4) all collapse.

Theorem 1.10. There is a \( BP_* \)-module isomorphism

\[ BP_*(BG) \cong BP_* \otimes Fs^{-1}(H(H^*;Q_1) + H^{\text{odd}})/R \]

where the relation \( R \) is generated, modulo \((p,v_1,\ldots)^2\), by

\[ \sum v_n s^{-1}Q_0 F(Q_n s^{-1}Q^{-1} s^{-1}(x)) = 0 \]

for \( i = 0,1, x \in (H(H^*;Q_1) + H^{\text{odd}}) \).

2. \( Q_2 \)-operation

We give examples 2.1 – 2.3 satisfying Assumption 1.1.

2.1. \( G = \mathbb{Z}/p \times \mathbb{Z}/p \). The cohomology \( H^{\text{even}} = \mathbb{Z}/p[y_1,y_2] \) and \( H^{\text{odd}} = H^{\text{even}}e \) where \( |y_1| = 2, |e| = 3 \) and \( Q_1e = y_1^p y_2 - y_1 y_2^p \).

2.2. \( G \) is a non abelian \( p \)-group of the order \( p^3 \). Then \( G \) is isomorphic to one of \( D, Q, E, M \) (see Lewis [6] or [7]). The cohomology \( H^{\text{even}} \) is generated by elements \( c_1, \ldots, c_2, y_1, y_2 \), and \( H^{\text{odd}} \) is generated as an \( H^{\text{even}} \)-module by \( e \) (resp. 0, \( d_1 \) and \( d_2, e \)) for \( D \) (resp. \( Q, E, M \)). Then we can take ring generators such that the \( Q_1 \)-operation is given by \( Q_1 e = c_2 y_2 \) (resp. 0, \( Q_1 d_i = c_2 y_i, Q_1 e = c_p y_2^p \)). Hence Assumption 1.1 is satisfied for these cases.

2.3. The semi-dihedral groups \( SD_2 \). \( H\mathbb{Z}/2^* \) is detected by \( (D, Q) \) (see [2]). Hence we get the assumption.
3. Description of $BP_*(BD)$

In this section we write down $BP_*(BD)$ more explicitly. Recall $D = \langle a, b | a^4 = b^2 = 1, [a, b] = a^2 \rangle$. The cohomology is given (1, 6, 8).

\[(3.1)\]
\[
\begin{align*}
H^{\text{even}} &= (\tilde{\mathbb{Z}}/2[y_1, y_2]/(y_1^2 + y_1 y_2)) \otimes \tilde{\mathbb{Z}}/4[C_2] \\
H^{\text{odd}} &= (\mathbb{Z}/2[y_1, y_2, c_2]/(y_1^2 + y_1 y_2)e) \\
HZ/2^* &= \mathbb{Z}/2[x_1, x_2, u]/(x_1^2 + x_1 x_2)
\end{align*}
\]

Where $\tilde{\mathbb{Z}}/s[x]$ means $\mathbb{Z}[x]/(sx)$ and where $x_i^2 = y_i, C = u^2$ and $e = x_2 u$ in $HZ/2^*$. Since $Q_0 u = ux_2, Q_1 e = y_2 c_2$. Hence we get

\[(3.2)\]
\[
H(H^*; Q_1) = (\tilde{\mathbb{Z}}/2[y_2] \oplus \tilde{\mathbb{Z}}/4[c_2]) \otimes \wedge(y_1).
\]

From Lemma 1.5 and Theorem 1.10, we have

\[(3.3)\]
\[
gr BP_*(BD) = BP_*(1) \oplus BP_*/2s^{-1}\{y_1^i, y_2^i, y_1 c_2^j\}
+ BP/4s^{-1}\{c_2^j\} \oplus BP_*/(2, v_1)s^{-1}\{y_1^i s_2^j, y_2^i c_2^j e\}.
\]

We will construct $D-U$-manifolds which represent elements in (3.3). Before doing this, we see how these generators in $HZ_*$ are defined. Consider the extension

\[(3.4)\]
\[
0 \to \langle a \rangle = \mathbb{Z}/4 \to D \to \langle b \rangle = \mathbb{Z}/2 \to 0
\]

and induced spectral sequence (see Lewis p.510 [6]). The action $b^*$ on $H^*(BZ/4) \cong \tilde{\mathbb{Z}}/4[u]$ is given by $b^* u = 3u = -u$. Let us write $T = (1 - b^*)$ and $N = (1 + b^*)$. Then

\[(3.5)\]
\[
E^2_{0,*} = H_*/\text{Im}T = \begin{cases} 
\mathbb{Z}/4\{s^{-1}u^i\} & \text{if } i \mid 2 \\
\mathbb{Z}/2\{s^{-1}u^i\} & \text{otherwise}
\end{cases}
\]

\[
E^2_{2j+1,*} = \text{Ker}T/\text{Im}N = \begin{cases} 
\mathbb{Z}/2\{s^{-1}u^i\} & \text{if } i \mid 2 \\
\mathbb{Z}/2\{s^{-1}2u^i\} & \text{otherwise}
\end{cases}
\]

\[
E^2_{2j+2,*} = \text{Ker}N/\text{Im}T = \begin{cases} 
\mathbb{Z}/2\{s^{-1}2u^i\} & \text{if } i \mid 2 \\
\mathbb{Z}/2\{s^{-1}u^i\} & \text{otherwise}
\end{cases}
\]
By the universal coefficient theorem and (3.1) this spectral sequence collapses (confer Lewis p.510).

The elements $s^{-1}u$, $s^{-1}u^2 \in E^2_{0,*}$ corresponds $s^i y_1$, $s^{-1}e_2$, the element $s^{-1}2u \in E^2_{1,1}$ corresponds $s^{-1}e$, and $s^{-1}u \in E^2_{2j,2}$ corresponds $s^{-1}(y_1y_2^j)$. Moreover $1 \in E^2_{2j-1,0}$ corresponds $s^{-1}y_2^j$.

We define a $D$-$U$-manifold

$$(3.7) \quad X(j, i) = (S^{2j-1} \times D/ < a >) \times < b > S^{2i-1}$$

where $D$ acts on $S^{2j-1} \times D/ < a >$ by

$$\begin{align*}
&a(z, 0) = (iz, 0) \\
&a(z, 1) = (-iz, 1)
\end{align*}$$

$$\begin{align*}
&b(z, 0) = (z, 1) \\
&b(z, 1) = (z, 0)
\end{align*}$$

identifying $(z, n) \in S^{2j-1} \times \mathbb{Z}/2 \subset C^j \times \mathbb{Z}/2$, and where $b$ acts on $S^{2i-1}$ by $b(z) = (-z)$ in $C^i$. Then we get the map

$$(3.8) \quad \xi : X(j, i)/D \to BD.$$  

The fibering

$$S^{2j-1}/ < a > \twoheadrightarrow X(j, i)/D \twoheadrightarrow S^{2i-1}/ < b >$$

induces the spectral sequence

$$(3.9) \quad H_*(S^{2i-1}/ < b >; H_*(S^{2j-1}/ < a >)) \Rightarrow H_*(X(j, i)/D).$$

The map $\xi$ in (4.8) induces the map of spectral sequences (3.9) to (3.5). Then the fundamental class of $X(j, i)$ is represented in $E_\infty$ in (3.9) by the nonzero element of right up side. Hence we know that $X(2j, 0) = s^{-1}e^j$, $X(2j - 1, 0) = s^{-1}y_1 e_2^{j-1}$, $X(0, i) = s^{-1}y_2^i$, and for $ij > 0$, $X(2j, i) = s^{-1}e c_2^{j-1} y_1 y_2^{i-1} = s^{-1}e c_2^{j-1} y_1^i$, $X(2j - 1, i) = s^{-1}e c_2^{j-1} y_2^{i-1}$.

The only element which is not expressed by $X(j, i)$ is $s^{-1}y_1^j$ for $j \geq 2$. Note that there is a homomorphism $\lambda$ in $D$ such that $\lambda : b \leftrightarrow ab$, $\lambda : a \leftrightarrow a^3$. Then $s^{-1}y_2 = s^{-1}y_2 + s^{-1}y_1$. Take $X'(0, i) = M/ < ab > \times S^{2i-1}$ and this manifold represents $s^{-1}y_1^j + s^{-1}y_2^j$. 


Next consider relations \( \sum v_n Q_n^* Q_k^{-1}(x) = 0 \). First consider the case \( x = X(0, i) \). Since \( s^{-1}y_2 = Q_0^* y_2 \), we see \( Q_0^{-1} s^{-1}y_2^i = y_2^i \). The \( Q_n^* \)-operation acts

\[
Q_n^* y_2^i = \sum <y_2^i, Q_n x_2 y_2^k > x_2 y_2^k, \quad \text{where recall} \quad x_2 = y_2
\]

Therefore we have

\[
(3.10) \quad \sum v_n X(0, i - p^n + 1) = 0, \quad \sum v_n X'(0, i - p^n + 1) = 0
\]

This relation is well known and also given by the relation in \( BP_*(BZ/2) \) and \([2]\) the product of the formal group law in \( BP_* \)-theory (for example, see \([4],[5]\)).

When \( x = X(2j, 0) \), the fact \( Q_0^{-1} s^{-1}(c_2^i) = 0 \) induces only trivial relation. As for \( x = X(2j - 1, 0) \), the formula

\[
Q_n^* c_2^j y_1 = \sum <c_2^j y_1, Q_n c_2^k x_1 > c_2^k x_1 = 0 \quad \text{for} \quad n \geq 1
\]

follows the relation

\[
(3.11) \quad 2X(2j - 1, 0) = 0.
\]

At last we consider the case \( ij > 0 \). Since \( s^{-1}y_2^i c_2^j e = c_2^j y_2^i u \) (see (3.1)), we get

\[
(3.12) \quad Q_n^* c_2^j y_2^i e = \sum <c_2^j y_2^i e, Q_n c_2^k y_2^j u > c_2^k y_2^j u = \sum <c_2^j y_2^i e, c_2^k y_2^j Q_n u > c_2^k y_2^j u = \sum <c_2^{j-k} y_2^{-i} e, Q_n u > c_2^k y_2^j u.
\]

**Lemma 3.13.** There are polynomials \( F_n(u, y_2) \) such that \( Q_n u = f_n(u, y_2) u x_2 \) and \( f_{n+1} = u f_n^2 + y_2 f_n^2 + (\partial f_n/\partial u)^2 y_2 u^2 \).
Proof. At first recall $Q_0u = ux_2$. $Q_1$-action is

$$Q_1u = Sq^2Q_0u + Q_0Sq^2u = Sq^2(ux_2) = u^2x_2 + ux_2^3 = ux(u + x_2^2).$$

By the induction on $n \geq 1$, we see

$$Q_{n+1}u = (Sq^{n+1}Q_n + Q_nSq^{n+1})u = Sq^{n+1}Q_nu = Sq^{n+1}(uxu), \quad \text{where } |uxu| = 2^n+1 + 1,$$

$$= xu^2f^2 + x^3uf^2 + x^2u^2Sq^{[n+1]}f.$$ 

If $f_n = \sum \lambda_iu^iy^j_2$, then

$$Sq^{[j]}f_n = \sum \lambda_iu(ux_2)u^{2(i-1)}_2y^j_2 = ux_2(\partial f_n/\partial u).$$

Therefore $Q_{n+1}u = ux_2(u^2f_n^2 + x^2f_n^2 + x^2u^2(\partial f/\partial u)^2). \quad \square \quad \square$

Let us write $f_n = \sum f_n, u^iy^j$. Then we get

$$Q_{n*}c^j_2y^i_2e = \sum <c^k_2y^j_2e, \sum f_n, u^iy^j_22^n-1-te > c^{i-1}_2y^{e-1}_2u$$

$$= \sum f_n, 2t, c^j_2y^i_2(2^n-1-2t)u.$$

Hence we have the relation

$$(3.14) \quad \sum v_n(\sum f_n, 2t)X(j - t, i + 2t + 1 - 2^n)) = 0.$$

Next consider the relation such that $v_1X(j, i) + \cdots = 0$. If $Q_{1*}w = c^j_2y^i_2u$, then

$$c^j_2y^i_2u = \sum <w, Q_1c^j_2y^i_2u > c^j_2y^i_2u$$

$$= \sum <w, c^j_2y^i_2e(u + y_2) > c^j_2y^i_2u$$

shows $w = c^j_2y^i_1e$ or $w = c^j_2y^i_2e$. Since $Q_{0*}c^j_2y^i_2e = c^j_2y^i_2e$, the case $w = c^j_2y^i_2e$ gives a relation such that $2x(j, i + 1) + \cdots = 0$, which is contained in (3.14). Hence we need only the case $w = c^j_2y^i_2e$,

$$Q_{n*}w = \sum <c^j_2y^i_2e, Q_n, c^j_2y^i_2e > c^j_2y^i_2u$$

$$= \sum <c^j_2y^i_2e, f_n, u^iy^j_22^n-1-te > c^{i-1}_2y^{e-1}_2u$$

$$= f_n, 2t+1, c^j_2y^i_2(2^n-1-2t-1)u.$$
Therefore we get
\[ \sum_n v_n \left( \sum_t f_{n,2t+1} X(j-t, i-2^n + 2t + 2) \right) = 0. \]

**Theorem 3.10.** There is a $BP_\ast$-module isomorphism
\[ BP_\ast(BD) = BP_\ast \left\{ X(j,i), X'(0,i') \mid j, i \geq 0, i \geq 2 \right\} / R \]
where $R = ((3.10), (3.11), (3.14), (3.15)) \mod (2, v_1, \ldots)^2$.

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