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# PETTIS INTEGRABILITY

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Abstract. In this paper, we have some characterizations of Pettis integrability of bounded weakly measurable function $f : \Omega \longrightarrow X^*$ determined by separable subspace of $X^*$.

## 1. Introduction

The theory of integration of functions with values in a Banach space has long been a fruitful area of study. Since the invention of the Pettis integral over forty years ago, the problem of recognizing the Pettis integrability of a function has been much studied.

In this paper we are going to study Pettis integrability of bounded weakly measurable function $f : \Omega \longrightarrow X^*$ determined by separable subspace of $X^*$.

We will show that if $f : \Omega \longrightarrow X^*$ is a bounded weakly measurable function determined by a separable subspace of $X^*$ that has the WRNP, then $\{f(\cdot)x : \|x\| \le 1\}$ is weakly precompact in $L_\infty(\mu)$.

## 2. Notation and Preliminaries

Let $(\Omega, \Sigma, \mu)$ be a finite measure space and $X$ be a Banach space with dual $X^*$. If $f : \Omega \to X^*$ is bounded weakly measurable, then it can easily be shown that for every $E \in \Sigma$, there exists ${x_E}^* \in X^*$ such that for every $x \in X$,

$$ {x_E}^*(x) = \int_E \hat{x} \cdot f d\mu $$

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and for every $E \in \Sigma$, there exists $x_E{}^{***} \in X^{***}$ such that for every $x^{**} \in X^{**}$,

$$x_E{}^{***}(x^{**}) = \int_E x^{**} \cdot f d\mu.$$

The element $x_E{}^*$ is called the weak* integral of $f$ over $E$, denoted by $w^* - \int_E f d\mu$, and $x_E{}^{***}$ is called the Dunford integral of $f$ over $E$, denoted by $D - \int_E f d\mu$.

In the case that $D - \int_E f d\mu \in X^*$ for each $E \in \Sigma$, then $f$ is called Pettis integrable and we write $P - \int_E f d\mu$ instead of $D - \int_E f d\mu$ to denote the Pettis integral of $f$ over $E$.

A subset $K$ of $X$ is called a weak Radon-Nikodym set if for every finite measure space $(\Omega, \Sigma, \mu)$ and every bounded linear operator $S : L_1(\mu) \longrightarrow X$ for which $S(\chi_E/\mu(E))$ belongs to $K$ for each $E \in \Sigma$ with $\mu(E) \neq 0$, the operator $S$ is represented by a Pettis integrable function with values in $K$. A Banach space $X$ is said to have the weak Radon-Nikodym property(WRNP) if its closed unit ball, $B_X$, is a weak Radon-Nikodym set.

The following theorem proved in Riddle, Saab and Uhl[4].

THEOREM 1. *Each of the following statements about an operator $T : X \longrightarrow Y$ implies all the others:*

(a) *The set $T(B_X)$ is weakly precompact.*
(b) *The operator $T$ factors through a Banach space that contains no copy of $l_1$.*
(c) *The set $T^*(B_{Y^*})$ is a weak Radon-Nikodym set.*
(d) *The adjoint operator $T^*$ factors through a Banach space with the weak Radon-Nikodym property.*

If $F$ is a finite set in Banach space $X$ and $\epsilon > 0$, let

$$K(F, \epsilon) = \{x^* \in X^* : \|x^*\| \le 1 \text{ and } |x^*(x)| \le \epsilon \text{ for every } X \text{ in } F\}.$$

In [3], Huff shows that if $f : \Omega \longrightarrow X$ is a weakly measurable function and the operator $T : X^* \longrightarrow L_1(\mu)$ defined by $T(X^*) = x^* f$, then the following statements are equivalent:

(a) $f$ is Pettis integrable.

(b) $T$ is weakly compact operator and $\{0\} = \cap\{T(K(F,\epsilon)) : F \subset X$, $F$ is finite and $\epsilon > 0\}$.

(c) $T$ is weak$^*$ - to - weak continuous.

## 3. Main results

We define a bounded weakly measurable function $f : \Omega \longrightarrow X$ to be determined by a subspace $G$ of Banach space $X$ if for each $x^* \in X^*$, $x^*$ restricted to $G$ equals zero the $x^*f$ equals zero almost everywhere.

PROPOSITION 2. *Let $f : \Omega \longrightarrow X$ be a bounded weakly measurable function determined by a separable subspace of $X$. Then $f$ is Pettis integrable.*

*Proof.* Define $T : X^* \longrightarrow L_1(\mu)$ by $T(X^*) = x^*f$ . Then $T$ is weakly compact, by the boundedness of $f$. By corollary 4. of [3], if $h \in \cap_{(F,\epsilon)} T(K(F,\epsilon))$, then $h = 0$ almost everywhere. □ □

Let $(\Omega, \Sigma, \mu)$ be a measure space, let $E$ be a measurable set and let $f : \Omega \longrightarrow X^*$ be a bounded function we define the $w^*$-core of $f$ over $E$, denoted by $Cor_f{}^*(E)$, to be that subset of $X^*$ given by

$$Cor_f{}^*(E) = \cap w^* - \overline{Co}\{f(E \backslash A) : \mu(A) = 0, A \in \Sigma\}.$$

In [1] Andrews show that for each $E \in \Sigma$,

$$Cor_f{}^*(E) = w^* - \overline{Co}\{\frac{w^* - \int_B f d\mu}{\mu(B)} : B \subset E, B \in \Sigma, \mu(B) > 0\}.$$

THEOREM 3. *Let $f : \Omega \to X^*$ be a bounded weakly measurable function determined by a separable subspace of $X^*$. If $Cor_f{}^*(\Omega)$ is a weak Radon-Nikodym set, then $f$ is weak$^*$ equivalent to a Pettis integrable function that takes its value in $Cor_f{}^*(\Omega)$.*

*Proof.* By Proposition 2, $f$ is Pettis integrable. Define a measure $\nu : \Sigma \to X^{**}$ by $\nu(E) = w^* - \int_E f d\mu$ for all $E$ in $\Sigma$. Then

$$\{\frac{\nu(E)}{\mu(E)} | E \in \Sigma, \mu(E) > 0\} \subset Cor_f{}^*(\Omega).$$

Since $Cor_f{}^*(\Omega)$ is a weak Radon-Nikodym set, the measure $\nu$ has a Pettis integrable derivative $g$ whose range lies in $Cor_f{}^*(\Omega)$.

Hence, $P - \int_E g d\mu = \nu(E) = P - \int_E f d\mu$. □ □

THEOREM 4. *Let $f : \Omega \longrightarrow X^*$ be a bounded weakly measurable function determined by a separable subspace of $X^*$. If $X^*$ has the WRNP, then the set $\{f(\cdot)x : \|x\| \leq 1\}$ is weakly precompact in $L_\infty(\mu)$.*

*Proof.* By Proposition 2, $f$ is Pettis integrable. Define an operator $T : X \longrightarrow L_\infty(\mu)$ by $T(x) = f(\cdot)x$. Then the adjoint operator $T^*$ is weak$^*$ - to - weak$^*$ continuous and maps the unit ball of $L_\infty(\mu)^*$ onto a weak$^*$ compact convex subset of $k(B_{X^*})$, which certainly is a weak Radon-Nikodym set by Theorem 1 of [5].

Hence, by Theorem 1, the set $\{f(\cdot)x : \|x\| \leq 1\}$ is weakly precompact in $L_\infty(\mu)$. □ □

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