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SEQUENTIAL COMPACTNESS AND SEMICOMPACTNESS

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Abstract. In this paper, we introduce two notions of compactness defined by sequential convergence and compare them.

1. Preliminaries

Let X be a set and I be the closed unit interval. Then a function F from X into I is called a *fuzzy set* in X. For any fuzzy set F, $\{x \in X \mid F(x) > 0\}$ is called the support of F and denoted by supp$F$, i.e., supp$F = \{x \in X \mid F(x) > 0\}$. And for any $\alpha \in (0, 1]$, a fuzzy set x_α in X is called a *fuzzy point* if its support is a singleton $\{x\}$ and its value is α on its support. That is,

$$x_\alpha(y) = \begin{cases} \alpha & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

Definition 1.1. Let X be a nonempty set and I be the closed unit interval. A family δ of functions from X into I is called a *fuzzy topology* on X if

(1) $\emptyset,\ X \in \delta$
(2) for all $U_i \in \delta$, $\cup U_i \in \delta$
(3) if $U_1,\ U_2 \in \delta$, then $U_1 \cap U_2 \in \delta$.

The pair (X, δ) is called a *fuzzy topological space*. A member of δ is called an *open set*. And a fuzzy set F in X is said to be *closed* if $F^c = X - F$ is open in X, i.e., $F^c \in \delta$.

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Definition 1.2. Let $\{P_n\}$ be a sequence of fuzzy points and P a fuzzy point in a fuzzy topological space (X, δ). We say that $\{P_n\}$ *converges* to P, or P is a *limit* of the sequence $\{P_n\}$ and write $P_n \to P$ if for every Q-neighborhood A of P there is a natural number m such that P_nQA for all $n \geq m$.

Remarks. Note that, given any fuzzy point P in X, every sequence $\{P_n\}$ of fuzzy points such that $P_nQ(1-P)$ for all $n \geq m$ converges to P.

Definition 1.3. Let $\{P_n\}$ be a sequence of fuzzy points and P a fuzzy point in a fuzzy topological space $(X, \Im)$. Then P is said to be a *limit value* of the sequence $\{P_n\}$ if there is a subsequence of $\{P_n\}$ converging to P.

One has that every limit of a sequence is one of its limit values.

Definition 1.4. Let $\{P_n\}$ be a sequence of fuzzy points and P a fuzzy point in a fuzzy topological space (X, δ). Then P is said to be a *cluster* for the sequence $\{P_n\}$ if for every Q-neighborhood A of P and for every natural number m there is a natural number $n \geq m$ such that P_nQA.

Remarks. It is easy to see that every limit value of a sequence is a cluster of the sequence.

Definition 1.5. A fuzzy topological space (X, δ) is said to be C_1 if every fuzzy point P in X has a *countable fundamental Q-neighborhood system* (briefly C.F.Q.N.S.).

Lemma 1.6. *If (X, δ) is C_1 fuzzy topological space, then for every fuzzy point in X there exists a C.F.Q.N.S. $\{A_i\}$ such that $A_1 \supset A_2 \supset \cdots \supset A_i \supset \cdots$.*

Proof. By assumption, there exists a C.F.Q.N.S. $B = \{B_i\}$ of P. Define $A_1 = B_1,\ A_2 = B_1 \cap B_2, \ldots, A_n = \cap_{i=1}^{n} B_i, \ldots$. Clearly, $A_1 \supset A_2 \supset \cdots \supset A_i \supset \cdots$. In order to prove that these Q-neighborhood of P form an F.Q.N.S. of P, let A be a Q-neighborhood of P. There exists $B_i \in B$ such that $B_i \subset A$. Since PQB_i for every $i = 1, 2, \ldots, n$, $PQ(\cap_{i=1}^{n} B_i) = A_n \subset A$. □ □

THEOREM 1.7. *Let* $(X, \Im)$ *be a* C_1 *fuzzy topological space,* $\{P_n\}$ *be a sequence of fuzzy points and* P *a fuzzy point in* X*. If* P *is a cluster for the sequence* $\{P_n\}$*, then* P *is one of its limit value.*

Proof. By Lemma 1.6, there exists a C.F.Q.N.S. $\{A_i\}$ such that $A_1 \supset A_2 \supset \cdots \supset A_i \supset \cdots$. Since P is a cluster for $\{P_n\}$, for every $n \in \mathbb{N}$ there is $k(n) \in \mathbb{N}$ such that $P_{k(n)} Q A_n$. We define, in this way, a sequence of natural numbers $\{k(n)\}$ which can be taken to be strictly increasing. To show that the subsequence $\{P_{k(n)}\}$ converges to P, let A be Q-neighborhood of P. There exists $n_0 \in \mathbb{N}$ such that $PQA_n \subset A$; but $A_n \subset A_{n_0}$ for each $n \geq n_0$, and this implies $P_{k(n)} Q A_n \subset A_{n_0} \subset A$ for each $n \geq n_0$. □ □

2. semicompact and sequential compact

DEFINITION 2.1. A fuzzy topological space (X, δ) is said to be *semicompact* if every sequence of fuzzy points in X has a cluster.

DEFINITION 2.2. A fuzzy topological space (X, δ) is said to be *sequentially compact* if every sequence of fuzzy points in X has a limit value.

PROPOSITION 2.3. *Every fuzzy sequentially compact space is semicompact.*

PROPOSITION 2.4. *If every* C_1 *fuzzy topological space is semicompact, then it is also sequentially compact.*

In the following theorems we give some characterizations of sequentially compact space.

THEOREM 2.5. *Let* $f : X \to Y$ *be any function and* P *be any fuzzy point in* X*. Then*

(1) *For* $A \in I^X$ *and* PQA*, we have* $f(P)Qf(A)$.
(2) *For* $B \in I^Y$ *and* $f(P)QB$*, we have* $PQf^{-1}(B)$.

THEOREM 2.6. *Let X and Y be fuzzy topological spaces. Let $\{P_n\}$ be a sequence of fuzzy points in X and P fuzzy point in X. If $f : X \to Y$ is Q-continuous, then $f(P)$ is a limit of $f(P_n)$ whenever P is a limit of sequence $\{P_n\}$.*

Proof. Since f is Q-continuous, $f^{-1}(A)$ is a Q-neighborhood of P for every Q-neighborhood A of $f(P)$. Since P is a limit of sequence $\{P_n\}$, for every Q-neighborhood $f^{-1}(A)$ of P, there is a natural number m such that $P_n Q f^{-1}(A)$ for all $n \geq m$. Then $f(P_n)Qf(f^{-1}(A)) = A$, since f is onto. Hence $f(P)$ is a limit of $f(P_n)$. □ □

THEOREM 2.7. *Let X and Y be fuzzy topological spaces and $f : X \to Y$ be a Q-continuous function from X to Y which is onto. If X is sequentially compact, then Y is also sequentially compact.*

Proof. Let P be a fuzzy point in X and $\{P_n\}$ a sequence of fuzzy points. For every Q-neighborhood A of $f(P)$, since f is Q-continuous, $f^{-1}(A)$ is a Q-neighborhood of P. Since X is sequentially compact there exists a subsequence $\{P_{n(k)}\}$ of $\{P_n\}$. For every Q-neighborhood $f^{-1}(A)$ of P, there is a natural number m such that $P_{n(k)} Q f^{-1}(A)$ for all $n \geq m$. Then $f(P_{n(k)})Qf(f^{-1}(A)) = A$, since f is onto. Hence for every Q-neighborhood A of $f(P)$, there is a natural number m such that $f(P_{n(k)})QA$ for all $n \geq m$. Hence there is a subsequence $f(P_{n(k)})$ of $f(P_n)$ that converges to $f(P)$. □ □

THEOREM 2.8. *Let $(X_i)_{i \in J}$ be a family of fuzzy topological spaces and P be fuzzy point in X and let $X = \prod_{i \in J} X_i$ with product fuzzy topology $\Im$. For each $i \in J$, let π_i denote the canonical projection of X onto X_i and $\{P_n\}$ sequence of fuzzy points in X. Then P is a limit of $\{P_n\}$ if and only if $\pi_i(P)$ is a limit of $\pi_i(P_n)$.*

Proof. Let $\pi_i : X \to X_i$ be the canonical projection mapping. Since π_i is Q-continuous and onto, X_i is sequentially compact when X is sequentially compact by Theorem 2.6.

Conversely, suppose that X_i is sequentially compact, let $\{P_n\}$ be a sequence of fuzzy points in X, and A be Q-neighborhood of P. Then there exists $B \in \Im$ such that $PQB \subset A$. By the definition of the defined base for the product space $\prod_i X_i$, $B = \pi_{j_1}^{-1}(E_{j_1}) \cap \cdots \cap \pi_{j_m}^{-1}(E_{j_m})$ where

E_{j_k} is an open subset of the coordinate space X_{j_k}. Recall that PQB; hence $\pi_{j_1}(P)Q\pi_{j_1}(B), \ldots, \pi_{j_m}(P)Q\pi_{j_m}(B) = E_{j_m}$. By hypothesis, $\pi_{j_i}(P)$ is a limit of $\pi_{j_i}(P_n)$. □ □

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