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BANACH-STEINHAUS PROPERTIES OF LOCALLY CONVEX SPACES

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Abstract. Banach-Steinhaus type results are established for sequentially continuous operators and bounded operators between locally convex spaces without barrelledness.

In the past, all of Banach-Steinhaus type results have been established only for some special classes of locally convex spaces, e.g., barrelled spaces ([1,2,3]), s-barrelled spaces ([4]), strictly s-barrelled spaces ([5]), etc. Recently, Li Ronglu and Min-Hyung Cho ([6]) have obtained a Banach-Steinhaus type result which is valid for every locally convex space as follows.

Theorem 1 ([6], Th. B). *Let (X, λ) and (Y, μ) be locally convex spaces and $T_n : X \to Y$ a $\lambda - \mu$ continuous linear operators, $n \in \mathbb{N}$. If $\lim_n T_n x = Tx$ exists in $Y, \mu)$ for each $x \in X$, then the limit operator T is $\beta(X, X')$-μ continuous and, in particular, continuous if X is barrelled.*

In this paper we would like to present Banach-Steinhaus type results for sequentially continuous operators and bounded operators and bounded operators between locally convex spaces without barrelledness requirement. By the agency of these results, we show that an important topology of uniform convergence on conditionally weak* sequentially compact sets can be incompatible for some linear dual pairs.

Let X and Y be locally convex spaces. An operator $T : X \to Y$ is said to be sequentially continuous if $\{x_n\}$ is a sequence in X such that $x_n \to x$, then $Tx_n \to Tx$; T is said to be bounded if T sends bounded sets into bounded sets. Clearly, continuous operators are sequentially continuous, and sequentially continuous operators are bounded but,

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in general, converse implications fail. Let X', X^s and X^b denote the families of continuous linear functions, sequentially continuous linear functionals and bounded linear functionals on X, respectively. In general, $X' \subsetneqq X^s \subsetneqq X^b$.

For a linear dual pair (E, F) let $\beta(E, F)$ denote the strongest (E, F)-polar topology on E which is just the topology of uniform convergence on $\sigma(F, E)$-bounded subsets of F. Thus, $x_a \xrightarrow{\beta(E,F)} x$ if and only if for every $\sigma(F, E)$-bounded subset A of F, $\lim_a f(x_a) = f(x)$ uniformly in $f \in A$.

THEOREM 2. *Let X and (Y, μ) be locally convex spaces and $T_n : X \to Y$ sequentially continuous linear operators, $n \in \mathbb{N}$. If $\lim_n T_n x = Tx$ exists in (Y, μ) for each $x \in X$, then the limit operator T is $\beta(X, X^s) - \mu$ continuous.*

Proof. Every locally convex topology has a local base of neighborhoods of 0 which are barrels so μ is a (Y, Y')-polar topology and, hence, there is a family $\mathcal{F}$ of $\sigma(Y', Y)$-bounded subsets of Y' such that μ is just the topology of uniform convergence on sets in $\mathcal{F}$.

Let $A \in \mathcal{F}$ and $x \in X$. Since $T_n x \to Tx$, $\lim_n f(T_n x) = f(Tx)$ uniformly in $f \in A$ and, hence, there is an $n_0 \in \mathbb{N}$ such that $|f(T_n x) < |f(Tx)| + 1$ for all $f \in A$ and $n \geq n_0$. Observe that A is $\sigma(Y', Y)$-bounded, there is an $M > 0$ such that $\sup_{f \in A} |f(Tx)| \leq M$ and $\sup_{f \in A, 1 \leq n \leq n_0} |f(T_n x)| \leq M$. Therefore, $\sup_{f \in A, n \in \mathbb{N}} |f(T_n x)| \leq M + 1$, i.e., $\{f \circ T_n : f \in A, n \in \mathbb{N}\}$ is a $\sigma(X^s, X)$-bounded subset of X^s.

Now let $\{x_k\}$ be a sequence in X such that $x_k \xrightarrow{\beta(X,X^s)} x$, and $A \in \mathcal{F}$, $\varepsilon > 0$. Since $\{f \circ T_n : f \in A, n \in \mathbb{N}\}$ is $\sigma(X^s, X)$-bounded, $\lim_k f(T_n x_k) = f(T_n x)$ uniformly in both $f \in A$ and $n \in \mathbb{N}$. Hence there is a $k_0 \in \mathbb{N}$ such that

$$|f(T_n x_k) - f(T_n x)| < \frac{\epsilon}{3}, \quad \forall f \in A,\ n \in \mathbb{N},\ k \geq k_0.$$

Fix a $k \geq k_0$. Since $\lim_n f(T_n x_k) = f(Tx_k)$ uniformly in $f \in A$ and $\lim_n f(T_n x) = f(Tx)$ uniformly in $f \in A$, there is an $n_0 \in \mathbb{N}$ such that

$$|f(T_{n_0} x_k) - f(Tx_k)| < \frac{\epsilon}{3}, \quad |f(T_{n_0} x) - f(Tx)| < \frac{\epsilon}{3}, \quad \forall f \in A.$$

Therefore,

$$|f(Tx_k) - f(Tx)| \le |f(Tx_k) - f(T_{n_0}x_k)| + |f(T_{n_0}x_k) - f(T_{n_0}x)| + |f(T_{n_0}x) - f(Tx)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}, \quad \forall f \in A.$$

This shows that $\lim_k f(Tx_k) = f(Tx)$ uniformly in $f \in A$. Since $A \in \mathcal{F}$ is arbitrary, $\lim_k Tx_k = Tx$ in (Y, μ). Thus T is $\beta(X, X^s)$-μ sequentially continuous. □ □

A linear dual pair (E, F) is said to be Banach-Mackey if $\sigma(E, F)$-bounded subsets of E are $\beta(E, F)$-bounded, i.e., $\{f(x) : x \in B,\ f \in A\}$ is bounded for every $\sigma(E, F)$-bounded subset B of E and $\sigma(F, E)$-bounded subset A of F. A locally convex space X is called a Banach-Mackey space if (X, X') is Banach-Mackey. It is easy to see that for subsets of a locally convex space X the $\sigma(X, X')$-boundedness, the $\sigma(X, X^s)$-boundedness and the $\sigma(X, X^b)$-boundedness are equivalent. Hence if X is not Banach-Mackey, i.e., (X, X') is not Banach-Mackey, both (X, X^s) and (X, X^b) are not Banach-Mackey because $X' \subset X^s \subset X^b$. This shows that (X, X^b) is not Banach-Mackey for many locally convex spaces. A recent result due to Li Ronglu and C. Swartz ([7], Th. 8) can be stated in the following way.

Proposition 3. *Let X be a locally convex space. The followings are equivalent.*

(1) *(X, X^b) is Banach-Mackey.*
(2) *If $\{f_n\} \subset X^b$ and $\lim_n f_n(x) = f(x)$ exists at each $x \in X$, then the limit functional f is in X^b.*

Now let X and Y be locally convex spaces and $T_n : X \to Y$ bounded linear operators for $n \in \mathbb{N}$. Proposition 3 shows that the limit operator T can be unbounded even if $T_n x \to Tx$ at each $x \in X$, i.e., $T(B)$ can be unbounded for some bounded $B \subset X$. Hence we would like to show that for some kind of bounded set B, $T(B)$ must be bounded. To see this we recall that in a duality pair (E, F) a subset A of F is said to be conditionally $\sigma(F, E)$-sequentially compact if every sequence $\{f_n\}$ in A has a subsequence $\{f_{n_k}\}$ such that $\lim_k f_{n_k}(x)$ exists at each $x \in E$.

Let $\eta(E, F)$ denote the topology of uniform convergence on conditionally $\sigma(F, E)$-sequentially compact subsets of F. P.Dierolf ([8]) has shown that in the case of linear dual pair (E, F) the topology $\eta(E, F)$ and the weak topology $\sigma(E, F)$ have the same subseries convergent series.

THEOREM 4. *Let X and Y be locally convex spaces and $T_n : X \to Y$ bounded linear operators, $n \in \mathbb{N}$. If* weak-$\lim_n T_n x = Tx$ *exists at each $x \in X$, then the limit operator T sends $\eta(X, X^b)$-bounded sets into bounded sets.*

Proof. Let $y' \in Y'$. Then $\lim_n y'(T_n x) = y'(Tx)$ for each $x \in X$ so $\{y' \circ T_n : n \in \mathbb{N}\}$ is conditionally $\sigma(X^b, X)$-sequentially compact. Suppose that B is a $\eta(X, X^b)$-bounded subset of X and $\{x_k\} \subset B$. Then $\frac{1}{k} x_k \xrightarrow{\eta(X,X^b)} 0$, so for every $y' \in Y'$ $\lim_k \frac{1}{k} y'(T_n x_k) = 0$ uniformly in $n \in \mathbb{N}$.

Now fix an $y' \in Y'$ and $\epsilon > 0$. There is a $k_0 \in \mathbb{N}$ such that $\left|\frac{1}{k} y'(T_n x_k)\right| < \frac{\epsilon}{2}$ for all $n \in \mathbb{N}$ and all $k \geq k_0$. Fix a $k \geq k_0$. Since $\lim_n y'(T_n x_k) = y'(Tx_k)$ there is an $n_0 \in \mathbb{N}$ such that $|y'(T_{n_0} x_k) - y'(Tx_k)| < \frac{\epsilon}{2}$. Therefore,

$$|\frac{1}{k} y'(Tx_k)| \leq |\frac{1}{k} y'(Tx_k) - \frac{1}{k} y'(T_{n_0} x_k)| + |\frac{1}{k} y'(T_{n_0} x_k)| < \frac{\epsilon}{2k} + \frac{\epsilon}{2} \leq \epsilon.$$

This shows that $\{y'(Tx) : x \in B\}$ is bounded. Since $y' \in Y'$ is arbitrary, $\{Tx : x \in B\}$ is bounded in Y by the classical Mackey theorem. □ □

Let us denote by $\theta(X, X^b)$ the topology of uniform convergence on $\sigma(X^b, X)$-Cauchy sequences in X^b. A subset B of X is said to be $\theta(X, X^b)$-bounded if for every sequence $\{x_k\}$ in B and every $\sigma(X^b, X)$-Cauchy sequence $\{f_n\}$ in X^b, $\lim_k \frac{1}{k} f_n(x_k) = 0$ uniformly in $n \in \mathbb{N}$. Then the proof of Theorem 4 gives the following.

THEOREM 5. *Let X and Y be locally convex spaces and $T_n : X \to Y$ bounded linear operators, $n \in \mathbb{N}$. If* weak-$\lim_n T_n x = Tx$ *exists at each $x \in X$, then the limit operator T sends $\theta(X, X^b)$-bounded sets to bounded sets.*

Now we have a useful proposition as follows.

Theorem 6. *For a locally convex space X the following conditions are equivalent.*

(1) *For every locally convex space Y and for every sequence $\{T_n\}$ of bounded linear operators from X into Y such that* weak-$\lim_n T_n x = Tx$ *exists at each $x \in X$, the limit operator T is also bounded.*
(2) $(X^b, \sigma(X^b, X))$ *is sequentially complete.*

Proof. (1)$\Longrightarrow$(2). Let $\{f_n\}$ be a $\sigma(X^b, X)$-Cauchy sequence in X^b. Then $\lim_n f_n(x) = f(x)$ exists at each $x \in X$ and $f \in X^b$ by (1).

(2)$\Longrightarrow$(1). Let Y be a locally convex space and $\{T_n\}$ a sequence of bounded linear operators form X into Y such that weak-$\lim_n T_n x = T_x$ exists at each $x \in X$. Suppose that B is a bounded subset of X and $y' \in Y'$. Then $\lim_n y'(T_n x) = y'(Tx)$ at each $x \in X$. Since $y' \circ T_n \in X^b$ for all $n \in \mathbb{N}$, $y' \circ T \in X^b$ by (2). Therefore $\{y'(Tx) : x \in B\}$ is bounded and hence $\{Tx : x \in B\}$ is bounded in Y by the classical Mackey theorem. □ □

A locally convex space X is said to be semibornological if $X' = X^b$. Let φ be a family of number sequences such that each $\{t_j\} \in \varphi$ has only finite many of nonzero t_j. With the norm $\|\{t_j\}\|_\infty = \sup_j |t_j|$, $X = (\varphi, \|\cdot\|_\infty)$ is a noncomplete normed space. It is easy to see that $X' = X^b = 1^1$, i.e., X is semibornological, and $(X^b, \sigma(X^b, X)) = (X', \sigma(X', X)) = (1^1, \sigma(1^1, \varphi))$ is not sequentially complete. By Theorem 6, there exists a locally convex space Y and a sequence $\{T_n\}$ of bounded linear operators such that weak-$\lim_n T_n x = Tx$ exists at each $x \in X$ but the limit operator T is not bounded., i.e., $T(B_1)$ is unbounded in Y for the unit ball B_1 of X. Now by Theorem 4, B_1 can not be $\eta(X, X^b)$-bounded, i.e., B_1 is not $\eta(X, X')$-bounded because $X' = X^b = 1^1$. Thus, we have the following interesting fact.

Corollary 7. *For a linear dual pair (X, X') the polar topology $\eta(X, X')$ need not be compatible, e.g., $\eta(\varphi, 1^1)$ is not compatible.*

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