Kangweon-Kyungki Math. Jour. **6** (1998), No. 1, pp. 1–10

ON THE MCSHANE-STIELTJES REPRESENTABLE OPERATORS AND NEARLY MCSHANE-STIELTJES REPRESENTABLE OPERATORS

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Abstract. We introduce the notions of McShane-Stieltjes representable operators nearly McShane-Stieltjes representable operators and then investigate some properties of theses operators

1. Introduction

Some generalizations of the Riemann integral have been studied for real-valued functions. One of these generalizations leads to an integral, called the McShane integral, that is equivalent to the Lebesgue integral of real-valued functions and leads to Riemann-Stieltjes integral. In the late 1960's, E. J. McShane broadened the class of tagged partition by not insisting that the tag of an integral belong to the interval, this produced the extend Lebesgue integral as a limit of suitable Riemann sum. Gordon [6] introduced the

Received November 14, 1997.

1991 Mathematics Subject Classification: 28B5.

Key words and phrases: McShane-Stieltjes representable operators, nearly Mc-Shane-Stieltjes representable operator.

McShane integral of Banach-valued functions and study its basic properties. Ju Han Yoon, Gwang Sik Eun and Young Chan Lee [5] introduce the McShane-Stieltjes integral for real-valued functions. Year by year the representation of linear operators on the Banach space has been studied the Bochner representable, Pettis representable by many authors. Also in 1989, Kaufman, Petrakis, Riddle and Uhl introduced nearly representable operators which is a generalization of representable operators. More recently, C.K. Park[2] has been studied the McShane representable operators and nearly McShane representable operators. In this paper we introduce the notions of McShane-Stieltjes representable operators and nearly McShane-Stieltjes representable operators and then investigate some properties of theses operators

2. Preliminaries

Throughout this papers X is a Banach space and unless otherwise stated we always assume that α is an increasing function on [0, 1].

Definition 2.1. Let δ be a positive function defined on the interval [0, 1]. A tagged interval ([a, b], x) consists of an interval [a, b] $\subset$ [0, 1] and a point x in [0,1]. The tagged interval ([a, b], x) is subordinate to δ if [a, b] $\subset$ $(x - \delta(x), x + \delta(x))$.

Note that this x may not be a point in [a, b]. Capital letter P will be used to denote finite collections of non-overlapping tagged intervals.

Let $P = \{([a_i, b_i], x_i) : 1 \leq i \leq n\}$ be such collection in

[0, 1]. We adopt the following terminology :

(1) The points $\{x_i : 1 \leq i \leq n\}$ are tags of P.
(2) The intervals $\{[a_i, b_i] : 1 \leq i \leq n\}$ are called intervals of P.
(3) If $([a_i, b_i], x_i)$ is subordinate to δ for each i, then we write P is sub δ.
(4) If $[0, 1] = \cup_{i=1}^{n}[a_i, b_i]$, then $P = \{([a_i, b_i], x_i) : 1 \leq i \leq n\}$ is called a tagged partition (or McShane partition) of [0,1].
(5) If P is a tagged partition of [0, 1] and if P is sub δ, then we write P is sub δ on [0, 1].

Definition 2.2. The function $f : [0, 1] \to X$ is the McShane integrable on [0, 1] if there exists a vector z in X with the following property : for each $\epsilon > 0$ there exists a positive function δ on [0, 1] such that $\|f(P) - z\| < \epsilon$ whenever P is sub δ on [0, 1]. The function f is the McShane integrable on the set $E \subset [0, 1]$ if the function $f\chi_E$ is the McShane integrable on [0, 1].

For the real-valued functions the McShane integral and the Lebesgue integral are equivalent, and the McShane integrable function need not be a measurable. R. A. Gordon[6] proved that every Bochner integrable function and a measurable Pettis integrable function are McShane integrable, respectively. Also every McShane integrable function is Pettis integrable.

Definition 2.3. A function $f : [0, 1] \to X$ is McShane-Stieltjes integrable with respect to α with McShane-Stieltjes integral $x \in X$ if for each $\varepsilon > 0$ there exists a positive function δ on [0, 1] such that $\|x - \sum_{i=1}^{n} f(t_i)[\alpha(b_i) - \alpha(a_i)]\| < \epsilon$

whenever McShane partition $\{([a_i, b_i], t_i) : 1 \leq i \leq n\}$ of [0, 1] is subordinate to δ. In this case, we write $x = \int_0^1 f d\alpha$. A function $f : [0,1] \to X$ is McShane-Stieltjes integrable with respect to α on a set $E \subset [0,1]$ if $f\chi_E$ is McShane-Stieltjes integrable with respect to α on [0, 1].

3. The McShane-Stieltjes representable operators

Definition 3.1. A bounded linear operator $T : L_1[0,1] \to X$ is the Bochner(resp. Pettis, McShane) representable if there exists a Bochner integrable and essentially bounded(resp. Pettis integrable and scalarly essentially bounded, scalarly essentially bounded McShane integrable) function $g : [0,1] \to X$ such that for every $f \in L_1[0,1], T(f) = \int_0^1 fgd\mu$ (resp. $(P) - \int_0^1 fgd\mu, (M) - \int_0^1 fgd\mu$).

Definition 3.2. A bounded linear operator $T : L_1[0,1] \to X$ is McShane-Stieltjes representable with respect to α if there exists a scalarly essentially bounded McShane-Stieltjes integrable function $g : [0,1] \to X$ with respect to α such that for every $f \in L_1[0,1], T(f) = \int_0^1 fgd\alpha$.

Theorem 3.3. *If $f : [0,1] \to X$ is a McShane-Stieltjes integrable function with respect to α and if $T : X \to Y$ is a bounded linear operator, then the composition $T \circ f : [0,1] \to Y$ is a McShane-Stieltjes integrable function with respect to α and $T(\int_0^1 f d\alpha) = \int_0^1 T \circ f d\alpha$.*

Proof. If $T = 0$, then it is clear. Suppose that $T \neq 0$. Let $\int_0^1 f d\alpha = x$. Then for given $\epsilon > 0$ there exists a positive function δ on [0, 1] such that $\|x - \sum_{i=1}^n f(t_i)[\alpha(b_i) - \alpha(a_i)]\| < \frac{\epsilon}{\|T\|}$ whenever McShane partition $\{([a_i, b_i], t_i) : 1 \leq i \leq n\}$ of [0, 1] is subordinate to δ. Hence $\|Tx - \sum_{i=1}^n (T \circ f)(t_i)[\alpha(b_i) - \alpha(a_i)]\| \leq \|T\| \|x - \sum_{i=1}^n f(t_i)[\alpha(b_i) - \alpha(a_i)\| < \epsilon$ whenever McShane partition $\{([a_i, b_i], t_i) : 1 \leq i \leq n\}$ of [0, 1] is subordinate to δ. Therefore $T \circ f : [0, 1] \to Y$ is a McShane-Stieltjes integrable function with respect to α and $\int_0^1 T \circ f d\alpha = Tx = T(\int_0^1 f d\alpha)$. □

Theorem 3.4. *If $T : L_1[0, 1] \to X$ is a McShane-Stieltjes representable with respect to α and $S : X \to Y$ is any bounded linear operator, then the composition $S \circ T : L_1[0, 1] \to Y$ is McShane-Stieltjes representable with respect to α.*

Proof. Suppose that $T : L_1[0, 1] \to X$ is McShane-Stieltjes representable with respect to α and $S : X \to Y$ is any bounded linear operator. Then $S \circ T : L_1[0, 1] \to Y$ is clearly a bounded linear operator and there exists a scalarly essentially bounded McShane-Stieltjes integrable function $g : [0, 1] \to X$ with respect to α such that $T(f) = \int_0^1 fg d\alpha$ for all $f \in L_1[0, 1]$. By theorem 3.3, $S \circ g : [0, 1] \to Y$ is also a scalarly essentially bounded McShane-Stieltjes integrable function with respect to α. For each $f \in L_1[0, 1]$, $(S \circ T)(f) = S(\int_0^1 fg d\alpha) = \int_0^1 S \circ (fg) d\alpha = \int_0^1 f(S \circ g) d\alpha$. Therefore $S \circ T : L_1[0, 1] \to Y$ is McShane-Stieltjes representable with respect to α. □

THEOREM 3.5. *Let bounded linear operators* $T : L_1[0,1] \to X$ *and* $G : L_1[0,1] \to X$ *be McShane-Stieltjes representable with respect to* α. *Then* $k_1T + k_2G : L_1[0,1] \to X$ *is McShane-Stieltjes representable with respect to all* k_1, k_2 *in R.*

Proof. We show that, for k in R, kT and $T + G$ are McShane-Stieltjes representable with respect to α, respectively. Suppose that a bounded linear operator $T : L_1[0,1] \to X$ is McShane-Stieltjes representable with respect to α, then there exists a scalarly essentially bounded McShane-Stieltjes integrable function $g : [0,1] \to X$ with respect to α such that $T(f) = \int_0^1 fgd\alpha$ for all $f \in L_1[0,1]$. Since $T : L_1[0,1] \to X$ is a bounded linear operator, $kT : L_1[0,1] \to X$ is a bounded linear operator for all k in R. And $kg : [0,1] \to X$ is McShane-Stieltjes integrable function with respect to α for k in R. Hence

$$(kT)(f) = kT(f) = k\int_0^1 fgd\alpha = \int_0^1 kfgd\alpha = \int_0^1 f(kg)d\alpha.$$

Thus $kT : L_1[0,1] \to X$ is McShane-Stieltjes representable with respect to α. To show that $T+G$ is McShane-Stieltjes representable with respect to α. Suppose that bounded linear operators $T : L_1[0,1] \to X$ and $G : L_1[0,1] \to X$ are McShane-Stieltjes representable with respect to α. Then there exists scalarly essentially bounded McShane-Stieltjes integrable functions $g : [0,1] \to X$, $h : [0,1] \to X$ with respect to α such that $T(f) = \int_0^1 fgd\alpha$, $G(f) = \int_0^1 fhd\alpha$ for all $f \in L_1[0,1]$, respectively. Since $T : L_1[0,1] \to X$ and $G : L_1[0,1] \to X$ are bounded linear operators, so

$T + G : L_1[0,1] \to X$ is also a bounded linear operator. And $g + h : [0,1] \to X$ is a scalarly essentially bounded McShane-Stieltjes integrable function with respect to α. Hence

$$(T+G)(f) = T(f)+G(f) = \int_0^1 fg d\alpha + \int_0^1 fh d\alpha = \int_0^1 f(g+h) d\alpha.$$

Thus $T + G : L_1[0,1] \to X$ is McShane-Stieltjes representable with respect to α. Therefore $k_1T+k_2G : L_1[0,1] \to X$ is McShane-Stieltjes representable with respect to α. □

Gordon showed that if $f : [0,1] \to X$ is Bochner integrable then f is McShane integrable. Thus if $T : L_1[0,1] \to X$ is Bochner representable then it is McShane representable. Also, Mendoza showed that if $f : [0,1] \to X$ is McShane integrable then f is Pettis integrable. So if $T : L_1[0,1] \to X$ is McShane representable then it is Pettis representable. Since McShane integrable function is McShane-Stieltjes integrable, if $T : L_1[0,1] \to X$ is McShane representable then it is McShane-Stieltjes representable with respect to α.

4. Nearly McShane-Stieltjes representable operators

Definition 4.1. A bounded linear operator $T : X \to Y$ is nearly Bochner (resp. McShane, Pettis) representable if the composition $T \circ D : L_1[0,1] \to Y$ is the Bochner (resp. McShane, Pettis) representable for every Dunford-Pettis operator $D : L_1[0,1] \to X$.

Definition 4.2. A bounded linear operator $T : X \to Y$ is nearly McShane-Stieltjes representable with respect to α if the composition $T \circ D : L_1[0,1] \to Y$ is McShane-Stieltjes representable with respect to α for every Dunford-Pettis operator $D : L_1[0,1] \to X$.

Theorem 4.3. *If $T : X \to Y$ is nearly McShane-Stieltjes representable with respect to α and $U : Y \to Z$ (or $V : Z \to X$) is any bounded linear operator, then $U \circ T$ (or $T \circ V$) is also nearly McShane-Stieltjes representable with respect to α.*

Proof. Assume that $T : X \to Y$ is nearly McShane-Stieltjes representable with respect to α and $U : Y \to Z$ is any bounded linear operator. Let $D : L_1[0,1] \to X$ be a Dunford-Pettis operator. Then $T \circ D : L_1[0,1] \to X$ is McShane-Stieltjes representable with respect to α. By theorem 3.4., $U \circ T \circ D : L_1[0,1] \to Z$ is also the McShane-Stieltjes representable with respect to α. This implies $U \circ T : X \to Z$ is nearly McShane-Stieltjes representable with respect to α by Definition 4.2. Suppose that $T : X \to Y$ is nearly McShane-Stieltjes representable with respect to α and $V : Z \to X$ is any bounded linear operator. Let $D : L_1[0,1] \to Z$ be a Dunford-Pettis operator. Then $V \circ D : L_1[0,1] \to X$ is the Dunford-Pettis operator. Sine $T : X \to Y$ is nearly McShane-Stieltjes representable with respect to α, $T \circ V \circ D : L_1[0,1] \to Y$ is McShane-Stieltjes representable with respect to α. Therefore $T \circ V : Z \to Y$ is nearly McShane-Stieltjes representable with respect to α. $\square$

Note that a bounded linear operator $T : L_1[0,1] \to L_1[0,1]$ is said to be positive if $T(f) \geq 0$ whenever $f \in L_1[0,1]$ and $f \geq 0$. This gives a lattice ordering of the class $L(L_1[0,1], L_1[0,1])$ of all bounded linear operators from $L_1[0,1]$ to $L_1[0,1]$. Define $T^+(f) = \sup\{T(g) : 0 \leq g \leq f\}$ for $f \in L_1[0,1]$ and $f \geq 0$. Bourgain showed that if $T : L_1[0,1] \to L_1[0,1]$ is Dunford-Pettis operator, then the positive part T^+ of T is also a Dunford-Pettis operator.

THEOREM 4.4. *A bounded linear operator $T : L_1[0,1] \to X$ is nearly McShane-Stieltjes representable with respect to α if and only if $T \circ D : L_1[0,1] \to X$ is McShane-Stieltjes representable with respect to α for all positive Dunford-Pettis operators $D : L_1[0,1] \to L_1[0,1]$.*

Proof. Suppose that a bounded linear operator $T : L_1[0,1] \to X$ is nearly McShane-Stieltjes representable with respect to α. Then the composition $T \circ D : L_1[0,1] \to Y$ is McShane-Stieltjes representable with respect to α for every Dunford-Pettis operator $D : L_1[0,1] \to X$. Since the positive Dunford-Pettis operators is a Dunford-pettis operators by Bourgain theorem. Thus if $D : L_1[0,1] \to L_1[0,1]$ is a positive Dunford-Pettis operator, then $T \circ D : L_1[0,1] \to X$ is McShane-Stieltjes representable with respect to α.

Conversely, assume that $T \circ D : L_1[0,1] \to X$ is McShane-Stieltjes representable with respect to α for all positive Dunford-Pettis operators $D : L_1[0,1] \to L_1[0,1]$. Let $S : L_1[0,1] \to L_1[0,1]$ be any Dunford-Pettis operator. Then the positive part S^+ of S and the negative part S^- of S are both Dunford-Pettis operators by Bourgain theorem. Hence $T \circ S^+$ and $T \circ S^-$ are both McShane-Stieltjes representable with respect to α by assumption and $T \circ S =$

$T \circ (S^+ - S^-) = T \circ S^+ - T \circ S^-$ is McShane-Stieltjes representable with respect to α. Therefore $T : L_1[0,1] \to X$ is nearly McShane-Stieltjes representable with respect to α.$\square$

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