Kangweon-Kyungki Math. Jour. **6** (1998), No. 1, pp. 9–15

CONVERGENCE OF $C-$SEMIGROUPS

Young S. Lee

Abstract. In this paper, we show convergence and approximation theorem for C-semigroups. And we study the problem of approximation of an exponentially bounded C-semigroup on a Banach space X by a sequence of exponentially bounded C-semigroup on X_n.

1. Introduction

Let X be a Banach space and let A be a linear operator from $D(A) \subset X$ into X. Given $x \in X$, the abstract Cauchy problem consists of finding a solution $u(t)$ to the following initial value problem

$$\text{(IVP)} \qquad \begin{aligned} \frac{du}{dt} &= Au, \quad t \geq 0 \\ u(0) &= x. \end{aligned}$$

From Theorem 4.1 of [2], if A is the generator of C-semigroup $\{S(t) : t \geq 0\}$, then the abstract Cauchy problem (IVP) has a unique solution for all $x \in C(D(A))$, given by $u(t) = S(t)C^{-1}x$.

In this paper, we consider, roughly speaking, the continuous dependency of an exponentially bounded C-semigroups on its generators. That is, the solution of (IVP) depends continuously on the operator A.

In section 2, we introduce the definition of C-semigroups and some known results about C-semigroups and its generators. And we show that the convergence of a sequence of generators implies the convergence of the corresponding C-semigroups. In this section, we assume

Received November 21, 1997.
1991 Mathematics Subject Classification: Primary 47H05, Secondary 46H20.
Key words and phrases: C-semigroup, convergence, approximation.
This research is supported by Seoul Women's University.

that the approximating C-semigroups are defined on the same space as the limit C-semigroup.

In section 3, we introduce the approximating sequence $\{X_n\}$ of Banach spaces to a Banach space X. Then we consider the approximation of a C-semigroup of operators defined on a Banach space X by a sequence of C-semigroups acting in other Banach spaces.

We will write $D(A)$ and $R(A)$ for the domain and range of operator A, respectively.

2. Convergence of C-semigroups.

Definition. Suppose that C is a bounded linear injective operator with dense range. The family of bounded linear operators $\{S(t) : t \geq 0\}$ is a C-semigroup if it satisfies the following conditions:

(1) $S(0) = C$.
(2) $S(t)S(s) = CS(t+s)$ for $t,\ s \geq 0$.
(3) $S(t)x$ is continuous in t for each $x \in X$.

A C-semigroup $\{S(t) : t \geq 0\}$ is said to be exponentially bounded if there exist M and ω such that $||S(t)|| \leq Me^{\omega t}$ for all $t \geq 0$.

If $C = I$, the identity operator on X, then a C-semigroup is a (C_0) semigroup in the ordinary sense (see [3] and [5]). In this case, a (C_0) semigroup is always exponentially bounded. But there exists a C-semigroup which is not exponentially bounded (see [1]). By (2), we have $S(t)C = CS(t)$ for all $t \geq 0$.

Definition. The linear operator A is called the generator of a C-semigroup $\{S(t) : t \geq 0\}$ if

$$D(A) = \{x : \lim_{h \to 0} (S(h)x - Cx)/h \text{ exists and is in } R(C)\},$$

$$Ax = C^{-1}(\lim_{h \to 0} \frac{S(h)x - Cx}{h}).$$

The following lemma is known in [1, 2].

Lemma 2.1. *Suppose that A is the generator of a C-semigroup $\{S(t) : t \geq 0\}$. Then*

(1) *$D(A)$ is dense.*
(2) *If $x \in D(A)$, then for all $t \geq 0$, $S(t)x \in D(A)$, $S(t)x$ is a differentiable function of t and $(d/dt)S(t)x = AS(t)x = S(t)Ax$.*

Lemma 2.2. *Let A be the generator of an exponentially bounded C_1-semigroup $\{S(t) : t \geq 0\}$ with $||S(t)|| \leq Me^{\omega t}$ for all $t \geq 0$. Let B be the generator of an exponentially bounded C_2-semigroup $\{T(t) : t \geq 0\}$ with $||T(t)|| \leq Me^{\omega t}$ for all $t \geq 0$. Suppose that $S(t)T(s) = T(s)S(t)$ for $t,\ s \geq 0$. Then for $x \in D(A) \cap D(B)$,*

$$||S(t)C_2x - T(t)C_1x|| \leq tM^2e^{\omega t}||Ax - Bx||.$$

Proof. Let $x \in D(A) \cap D(B)$. First, we will show that $S(t)x \in D(B)$ for $t \geq 0$ and $BS(t)x = S(t)Bx$. Since $x \in D(B)$, $\lim_{h\to 0}(T(h)x - C_2x)/h$ exists and is in $R(C_2)$. So

$$\begin{aligned}\lim_{h\to 0} \frac{T(h)S(t)x - C_2S(t)x}{h} &= S(t) \lim_{h\to 0} \frac{T(h)x - C_2x}{h} \\ &= S(t)C_2Bx = C_2S(t)Bx.\end{aligned}$$

Hence $S(t)x \in D(B)$ for $t \geq 0$, and

$$S(t)Bx = C_2^{-1}(\lim_{h\to 0} \frac{T(h)S(t)x - C_2S(t)x}{h}) = BS(t)x.$$

Consider

$$\begin{aligned}\frac{d}{dt}(T(t-s)S(s)x) &= -T(t-s)BS(s)x + T(t-s)S(s)Ax \\ &= T(t-s)S(s)(Ax - Bx).\end{aligned}$$

Integrating this equation from 0 to t, we obtain

$$\begin{aligned}\int_0^t T(t-s)S(s)(Ax - Bx)ds &= \int_0^t \frac{d}{ds}(T(t-s)S(s)x)ds \\ &= C_2S(t)x - T(t)C_1x = S(t)C_2x - T(t)C_1x.\end{aligned}$$

Therefore, we have

$$
\begin{aligned}
||S(t)C_2x - T(t)C_1x|| &\le \int_0^t ||T(t-s)S(s)(Ax - Bx)||ds \\
&\le \int_0^t ||\,T(t-s)||\,||\,S(s)||\,||\,Ax - Bx||\,ds \\
&\le \int_0^t Me^{\omega(t-s)}Me^{\omega s}||Ax - Bx||ds \\
&= tM^2e^{\omega t}||Ax - Bx||. \qquad \square
\end{aligned}
$$

THEOREM 2.3. *Let A be the generator of an exponentially bounded C-semigroup $\{S(t) : t \ge 0\}$ with $||S(t)|| \le Me^{\omega t}$ for $t \ge 0$. For each n, A_n be the generator of an exponentially bounded C-semigroup $\{S_n(t) : t \ge 0\}$ with $||S_n(t)|| \le Me^{\omega t}$ for $t \ge 0$. Suppose that $D(A) \subset D(A_n)$ for all n and $S_n(t)S(s) = S(s)S_n(t)$ for all n and $t,\ s \ge 0$. If*

$$\lim_{n\to\infty} A_n x = Ax$$

for all $x \in D(A)$, then

$$\lim_{n\to\infty} S_n(t)x = S(t)x$$

for all $x \in X$ and the convergence is uniform on bounded t-intervals.

Proof. Let $x \in D(A)$ and $0 \le t \le T$. Then, by Lemma 2.3 with $C = C_1 = C_2$, we have

$$
\begin{aligned}
||CS_n(t)x - CS(t)x|| &\le tM^2e^{\omega t}||A_nx - Ax|| \\
&\le TMe^{\omega T}||A_nx - Ax||.
\end{aligned}
$$

Thus we have

$$\lim_{n\to\infty} CS_n(t)x = CS(t)x$$

for all $x \in D(A)$ and the convergence is uniform on bounded t-intervals. Since C is injective,

$$\lim_{n\to\infty} S_n(t)x = S(t)x.$$

By Lemma 2.1, $D(A)$ is dense. So the result follows. □

By the similar argument in Lemma 2.2 and Theorem 2.3, we have the following theorem. In Theorem 2.4, we assume that A_n are bounded linear operators. This is a special case of convergence theorem which can be shown by a simple proof.

THEOREM 2.4. *Let A be the generator of an exponentially bounded C-semigroup $\{S(t) : t \geq 0\}$ with $||S(t)|| \leq Me^{\omega t}$ for $t \geq 0$. For each n, let A_n be the generator of an exponentially bounded C-semigroup $\{S_n(t) : t \geq 0\}$ with $||S_n(t)|| \leq Me^{\omega t}$ for $t \geq 0$. Suppose that each A_n is a bounded linear operator on X and $A_n S(t) = S(t)A_n$ for all n and $t \geq 0$. If*

$$\lim_{n\to\infty} A_n x = Ax$$

for all $x \in D(A)$, then

$$\lim_{n\to\infty} S_n(t)x = S(t)x$$

for all $x \in X$ and the convergence is uniform on bounded t-intervals.

3. Variation of the Space

Let X and X_n be Banach spaces. Suppose that for each n, there exists a bounded linear operator $P_n : X \to X_n$ such that

(1) $||P_n|| \leq M_1$, where M_1 is independent of n.
(2) $\lim_{n\to\infty} ||P_n x|| = ||x||$ for each $x \in X$.
(3) there exists a constant M_2 such that for each $x_n \in X_n$ there exists $x \in X$ such that

$$x_n = P_n x \quad \text{and} \quad ||x|| \leq M_2||x_n||.$$

A sequence $\{x_n\}$, $x_n \in X_n$, is said to converge to $x \in X$, denoted by $x_n \to x$, if

$$\lim_{n\to\infty} ||x_n - P_n x|| = 0.$$

Note that the limit of such convergent sequence is unique.

Let $\{A_n : X_n \to X_n\}$ be a sequence of operators. We say that the sequence $\{A_n\}$ is said to converge strongly to an operator $A : X \to X$, denoted by $A_n \to_s A$, if

$$A_n P_n x \to Ax, \quad \text{that is,} \quad \lim_{n\to\infty} ||P_n Ax - A_n P_n x|| = 0.$$

For more information and examples about the approximating sequences of Banach spaces, see [4, 6]. The result that we present in this section is rather special, because we make a stronger assumption on the generators. Since our goal in this section is to show that this method is valid for C-semigroups, we will make a strong assumption on the generators in order to avoid some of technicalities.

THEOREM 3.1. *Let A be the generator of an exponentially bounded C-semigroup $\{S(t) : t \geq 0\}$ in X with $||S(t)|| \leq Me^{\omega t}$ for all $t \geq 0$. Let A_n be the generator of an exponentially bounded C_n-semigroup in X_n with $||S_n(t)|| \leq Me^{\omega t}$ for all $t \geq 0$. Suppose that $C_n \to_s C$,* $\lim_{n\to\infty} ||A_n P_n - P_n A|| = 0$ *and $P_n(D(A)) \subset D(A_n)$ for each n. Then*

$$S_n(t) \to_s S(t).$$

Proof. Let $x \in D(A)$ and $0 \leq t \leq T$. Then

$$\frac{d}{ds}(S_n(t-s)P_n S(s)x) = S_n(t-s)(P_n A - A_n P_n)S(s)x.$$

Integrating this equation from 0 to t, we obtain

$$C_n P_n S(t)x - S_n(t)P_n Cx = \int_0^t S_n(t-s)(P_n A - A_n P_n)S(s)x ds.$$

So we have

$$\begin{aligned} &||C_n P_n S(t)x - S_n(t)P_n Cx|| \\ &\qquad \leq \int_0^t ||\, S_n(t-s)||\, ||\, (P_n A - A_n P_n)||\, ||\, S(s)||\, ||\, x||\, ds \\ &\qquad \leq tM^2 e^{\omega t} ||\, P_n A - A_n P_n||\, ||\, x||\, . \end{aligned}$$

Hence $\lim_{n\to\infty} ||C_n P_n S(t)x - S_n(t)P_n Cx|| = 0$.

Since $C_n \to_s C$, we have

$$\lim_{n\to\infty} ||P_n CS(t)x - S_n(t)P_n Cx|| = 0.$$

Thus the result holds for $C(D(A))$. Since C has the dense range and $D(A)$ is dense, the result follows. $\square$

If $X_n = X$ for all n with $P_n = I$, the identity operator on X, then we have the following result.

COROLLARY 3.2. *Let A be the generator of an exponentially bounded C-semigroup $\{S(t) : t \geq 0\}$ with $||S(t)|| \leq Me^{\omega t}$ for all $t \geq 0$. Let A_n be the generator of an exponentially bounded C_n-semigroup with $||S_n(t)|| \leq Me^{\omega t}$ for all $t \geq 0$. Suppose that $\lim_{n\to\infty} C_n x = Cx$ for all x, $\lim_{n\to\infty} ||A_n - A|| = 0$ and $D(A) \subset D(A_n)$ for each n. Then*

$$\lim_{n\to\infty} S_n(t)x = S(t)x$$

for all $x \in X$ and the convergence is uniform on bounded t-intervals.

References

1. E.B. Davies and M.M.H. Pang, *The Cauchy problem and a generalization of the Hille-Yosida theorem*, Proc. London Math. Soc. **55** (1987), 181 - 248.
2. R. deLaubenfels, *C-semigroups and the Cauchy problem*, J. Funct. Anal. **111** (1993), 44 - 61.
3. J.A. Goldstein, *Semigroup of linear operators and applications*, Oxford, New York, 1985.
4. T. Kato, *Perturbation theory for linear operators*, Springer Verlag, New York, 1966.
5. A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Springer Verlag, New York, 1983.
6. H.F. Trotter, *Approximation of semi-groups of operators*, Pacific J. Math. **8** (1958), 887 - 919.

Department of Mathematics
Seoul Women's University
Seoul 139 - 774, Korea
E-mail: younglee @ swift2.swu.ac.kr