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A NOTE ON THE $\theta_3(0,\tau)$

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Abstract. Let $\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$, where $q = e^{2\pi i\tau}$ and $\tau \in \mathbb{C}$. Then the transformation

$$g(\tau) = \rho \frac{\{\eta(\frac{\tau+1}{2})\eta(\frac{\tau+2}{2})\}^{16}}{\eta(\tau)^{24}} \left(\bar{\rho}\eta(\frac{\tau+1}{2})^8 + \eta(\frac{\tau+2}{2})^8\right)^2$$

is holomorphic for Im $\tau > 0$, and has the property

$$g(\tau+1) = g(\tau), \quad g(-\frac{1}{\tau}) = \tau^{12}g(\tau). \text{ (Theorem)}$$

1. Preliminaries

The study of theta-series is an important part of number theory. This paper mainly deals with theta-series and Dedekind η-function. We describe some preliminaries on theta-series which are needed in section 2. Let $\tau \in \mathfrak{h}$, where $\mathfrak{h} = \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$.

The *Dedekind sum* $s(\mu,\nu)$ is defined to be

$$s(\mu,\nu) = \sum_{j=1}^{\nu-1} \frac{j}{\nu}\left(\frac{j\mu}{\nu} - \left[\frac{j\mu}{\nu}\right] - \frac{1}{2}\right).$$

(The square brackets denote the greatest integer function.[1])

The Dedekind η-function satisfies the identities

$$(*) \qquad \eta(\tau+1) = e^{\frac{2\pi i}{24}}\eta(\tau), \quad \text{and } \eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\,\eta(\tau).$$

Here we take the branch of $\sqrt{\ }$ which is positive on the positive real axis.

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Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ with $c > 0$. The Dedekind η-function satisfies the transformation formula

$$\eta(\gamma\tau) = e^{2\pi i \Phi(\gamma)/24}\sqrt{-i(c\tau+d)}\,\eta(\tau),$$

where $\sqrt{\ }$ is the branch of the square root which is positive on the positive real axis, $\Phi(\gamma)$ is given by the formula

$$\Phi(\gamma) = \frac{1}{c} + \frac{d}{c} - 12s(d,c),$$

and $s(\mu,\nu)$ is Dedekind sum defined above.

Let $\theta_3(v,\tau) = 1 + 2\sum_{n=1}^{\infty} q^{\frac{1}{2}n^2}\cos(n\pi v)$. Then $\theta_3(v,\tau)$ can be expressed by infinite products which exhibit its zero.

Proposition 1. *([4]) We have*

$$\theta_3(v,\tau) = \prod_{n=1}^{\infty}(1-q^n)\prod_{n=1}^{\infty}(1+q^{n-\frac{1}{2}}e^{2\pi v})\prod_{n=1}^{\infty}(1+q^{n-\frac{1}{2}}e^{-2\pi v}),$$

where $q = e^{2\pi i\tau}$.

Proposition 2. *([4]) Let* $\Delta(\tau)$ *be modular discriminant. We have*

$$\begin{aligned}\Delta(\tau) &= (2\pi)^{12} q \sum_{n=1}^{\infty}\prod_{n=1}^{\infty}(1-q^n)^{24} \\ &= 16\pi^{12}\theta_3(0,\tau)^8\theta_3(0,\tau+1)^8(\theta_3(0,\tau)^4 - \theta_3(0,\tau+1)^4)^2.\end{aligned}$$

Proposition 3.

$$\eta\left(\frac{\tau+1}{2}\right)^2 = e^{\frac{\pi i}{12}}\theta_3(0,\tau)\eta(\tau).$$

2. Main theorem

THEOREM. *Let* $\rho = e^{\frac{2\pi i}{3}}$ *and* $-\overline{\rho} = e^{\frac{2\pi i}{6}}$. *Then the transformation*

$$g(\tau) = \rho \frac{\{\eta(\frac{\tau+1}{2})\eta(\frac{\tau+2}{2})\}^{16}}{\eta(\tau)^{24}} \left(\overline{\rho}\eta(\frac{\tau+1}{2})^8 + \eta(\frac{\tau+2}{2})^8\right)^2$$

is holomorphic for Im $\tau > 0$, *and has the property*

$$g(\tau+1) = g(\tau), \quad g(-\frac{1}{\tau}) = \tau^{12} g(\tau).$$

Proof. By Proposition 1, we obtain

$$(**) \qquad \theta_3(0,\tau) = e^{-\frac{\pi i}{12}} \frac{\eta(\frac{\tau+1}{2})^2}{\eta(\tau)}, \quad \theta_3(0,\tau+1) = e^{-\frac{\pi i}{12}} \frac{\eta(\frac{\tau+2}{2})^2}{\eta(\tau+1)}.$$

Now we compute

$$\begin{aligned}
g(\tau) &= \rho \frac{\{\eta(\frac{\tau+1}{2})\eta(\frac{\tau+2}{2})\}^{16}}{\eta(\tau)^{24}} \left(\overline{\rho}\eta(\frac{\tau+1}{2})^8 + \eta(\frac{\tau+2}{2})^8\right)^2 \\
&= -\frac{\eta(\frac{\tau+1}{2})^{16}\eta(\frac{\tau+2}{2})^{16}}{\eta(\tau)^{24}e^{\frac{4\pi}{3}}} \left(\eta(\frac{\tau+1}{2})^8 e^{\frac{\pi i}{3}} - \eta(\frac{\tau+2}{2})^8\right)^2 \\
&= -\frac{\eta(\frac{\tau+1}{2})^{16}\eta(\frac{\tau+2}{2})^{16}}{\eta(\tau)^{16}e^{\frac{16\pi i}{12}}\eta(\tau)^{16}} \left(\eta(\frac{\tau+1}{2})^8 e^{\frac{4\pi i}{12}}\eta(\tau)^4 - \eta(\frac{\tau+2}{2})^8\eta(\tau)^4\right)^2 \\
&= -\left(\frac{\eta(\frac{\tau+1}{2})\eta(\frac{\tau+2}{2})}{\eta(\tau)\eta(\tau+1)}\right)^{16} \left(\eta(\frac{\tau+1}{2})^8\eta(\tau+1)^4 - \eta(\frac{\tau+2}{2})^8\eta(\tau)^4\right)^2 \\
&\qquad \text{(by } (*)) \\
&= -e^{-\frac{24\pi i}{12}} \frac{\eta(\frac{\tau+1}{2})^{16}\eta(\frac{\tau+2}{2})^{16}}{\eta(\tau)^8\eta(\tau+1)^8} \left(\frac{\eta(\frac{\tau+1}{2})^8\eta(\tau+1)^4 - \eta(\frac{\tau+2}{2})^8\eta(\tau)^4}{\eta(\tau)^4\eta(\tau+1)^4}\right)^2 \\
&= -e^{-\frac{8\pi i}{12}} \frac{\eta(\frac{\tau+1}{2})^{16}}{\eta(\tau)^8} e^{-\frac{8\pi i}{12}} \frac{\eta(\frac{\tau+2}{2})^{16}}{\eta(\tau+1)^8} \left(e^{-\frac{4\pi i}{12}} \frac{\eta(\frac{\tau+1}{2})^8}{\eta(\tau)^4} - e^{-\frac{4\pi i}{12}} \frac{\eta(\frac{\tau+2}{2})^8}{\eta(\tau+1)^4}\right)^2 \\
&= -\theta_3(0,\tau)^8\theta_3(0,\tau+1)^8\{\theta_3(0,\tau)^4 - \theta_3(0,\tau+1)^4\}^2. \qquad \text{(by}(**))
\end{aligned}$$

Using proposition 2, $\Delta(\tau) = 16\pi^{12}\theta_3(0,\tau)^8\theta_3(0,\tau+1)^8(\theta_3(0,\tau)^4 - \theta_3(0,\tau+1)^4)^2$. Thus $\Delta(\tau) = -16\pi^{12}g(\tau)$ is a cusp form([3]), then $g(\tau)$ is holomorphic for Im $\tau > 0$.

$g(\tau)$ under the transformations $\tau \to -\frac{1}{\tau}$ and $\tau \to \tau+1$ follows from the property of $\Delta(\tau)$. □

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