ESTIMATIONS OF THE GENERALIZED REIDMEISTER NUMBERS II

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Abstract. This paper is a continuation of [1]. Let $\sigma(X,x_0,G)$ be the fundamental group of a transformation group $(X,G)$. Let $R(\varphi,\psi)$ be the generalized Reidemeister number for an endomorphism $(\varphi,\psi) : (X,G) \to (X,G)$. The main results in this paper concern the conditions for $R(\varphi,\psi) = |\text{Coker}(1 - (\varphi,\psi)_\sigma)|$.

1. Introduction

F. Rhodes introduced the concept of the fundamental group $\sigma(X,x_0,G)$ of a transformation group $(X,G)$, a group $G$ of homeomorphisms of a space $X$, as a generalization of the fundamental group $\pi_1(X,x_0)$ of a topological space $X$ in [6]. Recently, we gave a definition of the generalized Reidemeister number $R(\varphi,\psi)$ for an endomorphism $(\varphi,\psi) : (X,G) \to (X,G)$ and studied the algebraic computations of $R(\varphi,\psi)$ in [1] and [5].

This article deals with the problem of determining the conditions for $R(\varphi,\psi) = |\text{Coker}(1 - (\varphi,\psi)_\sigma)|$ as a continuation of [1].

Throughout this paper, the space $X$ is assumed to be a compact connected polyhedron. In this paper, we follow F. Rhodes [6] for the basic terminologies.

2. Preliminaries

Let $(X,G)$ be a transformation group and let $(\varphi,\psi) : (X,G) \to (X,G)$ be an endomorphism. Since $\varphi(gx) = (\psi g)(\varphi x)$ for every pair $(x,g)$, if $\alpha$ is a path in $X$ of order $g$ with base-point $x_0$, then $\varphi \alpha$ is a path in $X$ of order $\psi(g)$ with base-point $\varphi(x_0)$. Furthermore, if two
path $\alpha$ and $\beta$ of the same order $g$ is homotopic, $\alpha \simeq \beta$, then $\varphi \alpha \simeq \varphi \beta$. Thus $(\varphi, \psi)$ induces a homomorphism

$$(\varphi, \psi)_* : \sigma(X, x_0, G) \to \sigma(X, \varphi(x_0), G)$$

defined by $(\varphi, \psi)_*[\alpha; g] = [\varphi \alpha; \psi(g)]$.

If $\lambda$ is a path from $\varphi(x_0)$ to $x_0$, then $\lambda$ induces an isomorphism

$$\lambda_* : \sigma(X, \varphi(x_0), G) \to \sigma(X, x_0, G)$$

defined by $\lambda_*[\alpha; g] = [\lambda \rho + \alpha + g \lambda; g]$ for each $[\alpha; g] \in \sigma(X, \varphi(x_0), G)$, where $\rho(t) = 1-t$. This isomorphism $\lambda_*$ depends only on the homotopy class of $\lambda$.

Conveniently, we denote by $(\varphi, \psi)_\sigma$ the composition $\lambda_* (\varphi, \psi)_*$. 

**Definition.** ([5]) Let $(\varphi, \psi)_\sigma : \sigma(X, x_0, G) \to \sigma(X, x_0, G)$ be a homomorphism. Two elements $[\alpha; g_1], [\beta; g_2]$ in $\sigma(X, x_0, G)$ are said to be $(\varphi, \psi)_\sigma$–equivalent, $[\alpha; g_1] \sim [\beta; g_2]$, if there exists $[\gamma; g] \in \sigma(X, x_0, G)$ such that

$$(\alpha; g_1) = (\gamma; g)[\beta; g_2](\varphi, \psi)_\sigma(\gamma; g^{-1}).$$

For an endomorphism $(\varphi, \psi) : (X, G) \to (X, G)$, the Reidemeister number $R(\varphi, \psi)$ of $(\varphi, \psi)$ is defined to be the numbers of equivalence classes of $\sigma(X, x_0, G)$ under $(\varphi, \psi)_\sigma$–equivalence.

### 3. The estimates of the generalized Reidemeister number

In this section, we always assume that the group $G$ is an abelian.

Let $C(\sigma(X, x_0, G))$ be a commutator subgroup of $\sigma(X, x_0, G)$ and let

$$\tilde{\sigma}(X, x_0, G) = \sigma(X, x_0, G)/C(\sigma(X, x_0, G)).$$

Then $\theta_\sigma : \sigma(X, x_0, G) \to \tilde{\sigma}(X, x_0, G)$ is a canonical homomorphism such that $Ker \theta_\sigma = C(\sigma(X, x_0, G))$. Let $\eta_\sigma : \tilde{\sigma}(X, x_0, G) \to Coker(1 - (\varphi, \psi)_\sigma)$ be the natural projection. Then $\eta_\sigma \theta_\sigma$ is an epimorphism.

**Theorem 3.1.** ([5]) If $(\varphi, \psi) : (X, G) \to (X, G)$ is an endomorphism, then $R(\varphi, \psi) \geq |Coker(1 - (\varphi, \psi)_\sigma)|$, where $1$ and $(\varphi, \psi)_\sigma$ denote respectively the identity isomorphism and the endomorphism of $\tilde{\sigma}(X, x_0, G)$ induced by $(\varphi, \psi)$. Furthermore, if $\sigma(X, x_0, G)$ is abelian,

$$R(\varphi, \psi) = |Coker(1 - (\varphi, \psi)_\sigma)|.$$
Theorem 3.2. If the epimorphism $\eta_\sigma \theta_\sigma$ induces a one-one correspondence between the set of $(\varphi, \psi)_\sigma$–equivalent classes and $\text{Coker}(1 - (\varphi, \psi)_\sigma)$, then $[\alpha; g_1] \sim [\beta; g_2]$ implies $[\alpha; g_1][\gamma; g] \sim [\beta; g_2][\gamma; g]$ for any $[\gamma; g] \in \sigma(X, x_0, G)$.

Proof. Note that the $\eta_\sigma \theta_\sigma$ images of all elements of a $(\varphi, \psi)_\sigma$–equivalent class are the same element of $\text{Coker}(1 - (\varphi, \psi)_\sigma)$, that is, if $[\alpha; g_1] \sim [\beta; g_2]$, then $\eta_\sigma \theta_\sigma([\alpha; g_1]) = \eta_\sigma \theta_\sigma([\beta; g_2])$ (See proof of Theorem 3.5 in [5]). Since $\eta_\sigma \theta_\sigma$ is a homomorphism,

$$\eta_\sigma \theta_\sigma([\alpha; g_1][\gamma; g]) = \eta_\sigma \theta_\sigma([\alpha; g_1]) + \eta_\sigma \theta_\sigma([\gamma; g]) = \eta_\sigma \theta_\sigma([\beta; g_2]) + \eta_\sigma \theta_\sigma([\gamma; g]) = \eta_\sigma \theta_\sigma([\beta; g_2][\gamma; g]).$$

Hence from the assumption of Theorem, we obtain

$$[\alpha; g_1][\gamma; g] \sim [\beta; g_2][\gamma; g].$$

Corollary 3.3. If $(\varphi, \psi) : (X, G) \to (X, G)$ is an endomorphism, then the following statements are equivalent:

(1) The epimorphism $\eta_\sigma \theta_\sigma$ induces a one-one correspondence between the set of $(\varphi, \psi)_\sigma$–equivalent classes and $\text{Coker}(1 - (\varphi, \psi)_\sigma)$.

(2) For any $[\gamma; g] \in \sigma(X, x_0, G)$, $[\alpha; g_1] \sim [\beta; g_2]$ implies $[\alpha; g_1][\gamma; g] \sim [\beta; g_2][\gamma; g]$.

(3) For any $[\alpha; g_1], [\beta; g_2], [\gamma; g_3] \in \sigma(X, x_0, G)$,

$$[\alpha; g_1][\beta; g_2][\gamma; g_3] \sim [\beta; g_2][\alpha; g_1][\gamma; g_3].$$

Proof. For (2) $\Rightarrow$ (3) $\Rightarrow$ (1), we refer to [1, Lemma 2.4 and Theorem 3.2]. Hence it is clear from Theorem 3.2.

Corollary 3.4. If one of the three statements in Corollary 3.3 holds, then

$$R(\varphi, \psi) = |\text{Coker}(1 - (\varphi, \psi)_\sigma)|.$$
Lemma 3.5. ([1]) Let \((\varphi, \psi)_\sigma : \sigma(X, x_0, G) \rightarrow \sigma(X, x_0, G)\) be a homomorphism. Then, for any \([\alpha; g_1], [\beta; g_2] \in \sigma(X, x_0, G)\),

1. \([\alpha; g_1][\beta; g_2] \sim [\beta; g_2](\varphi, \psi)_\sigma([\alpha; g_1])\).
2. \([\alpha; g_1] \sim (\varphi, \psi)_\sigma([\alpha; g_1])\).

Theorem 3.6. Let \(Z(\sigma(X, x_0, G))\) be a center of \(\sigma(X, x_0, G)\). If \((\varphi, \psi)_\sigma\) image of \(\sigma(X, x_0, G)\) is contained in \(Z(\sigma(X, x_0, G))\), that is,

\((\varphi, \psi)_\sigma(\sigma(X, x_0, G)) \subseteq Z(\sigma(X, x_0, G))\),

then

\[R(\varphi, \psi) = |\text{Coker}(1 - (\varphi, \psi)_\sigma)|.\]

Proof. It is sufficient to prove that the third statement of Corollary 3.3 holds. For any \([\alpha; g_1], [\beta; g_2], [\gamma; g_3] \in \sigma(X, x_0, G)\), from (2) of Lemma 3.5 and hypothesis of Theorem,

\[\begin{align*}
[\alpha; g_1][\beta; g_2][\gamma; g_3] & \sim (\varphi, \psi)_\sigma([\alpha; g_1][\beta; g_2][\gamma; g_3]) \\
& = (\varphi, \psi)_\sigma([\alpha; g_1])(\varphi, \psi)_\sigma([\beta; g_2])(\varphi, \psi)_\sigma([\gamma; g_3]) \\
& = (\varphi, \psi)_\sigma([\beta; g_2])(\varphi, \psi)_\sigma([\alpha; g_1])(\varphi, \psi)_\sigma([\gamma; g_3]) \\
& = (\varphi, \psi)_\sigma([\beta; g_2][\alpha; g_1])(\varphi, \psi)_\sigma([\gamma; g_3]) \\
& \sim [\beta; g_2][\alpha; g_1][\gamma; g_3].
\end{align*}\]

Corollary 3.7. ([1]) Let \((\varphi, \psi)_\sigma : \sigma(X, x_0, G) \rightarrow \sigma(X, x_0, G)\) be a homomorphism. If \(\sigma(X, x_0, G)\) is abelian, then

\[R(\varphi, \psi) = |\text{Coker}(1 - (\varphi, \psi)_\sigma)|.\]

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