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MORSE INDEX OF COMPACT MINIMAL SURFACES

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Abstract. In this paper we study the Hessian at critical points of energy function on Teichmüller space $T(R)$ and apply it to the index of minimal surfaces

1. Introduction

Let N be an n dimensional compact Riemannian manifold and M be a compact Riemannian surface of genus γ. Let ϕ be a smooth map from M into N. Then ϕ induces the map $\phi_\#$ of $\pi_1(M,*)$ into $\pi_1(N,\phi(*))$, where $*$ is a fixed point of M. Let $L_1^2(M,N)$ denote the space of maps of M into N having square integrable first derivations in the distribution sense. Shoen and Yau [2] proved that there exists an energy minimizing harmonic map among $\{\,\phi \in L_1^2(M,N) \,:\, \phi_\# = \tau^{-1} f_\# \tau\,\}$, where τ is some curve from $\phi(*)$ to $f(*)$.

Let R be a fixed compact Riemannian surface of genus γ. A pair of compact Riemannian surface M and a differential homeomorphism f of R onto M is denoted by (M,f). We define an equivalence relation for the pairs as follows. (M_2,f_2) is said to be equivalent to (M_1,f_1) if and only if there exists a biholomorphic map h of M_1 onto M_2 such that h is homotopic to $f_2 f_1^{-1}$. This space of all equivalence classes is called the Teichmüller space $T(R)$ of R.

In this paper we study the Hessian at critical points of energy function on $T(R)$ and apply it to the index of minimal surfaces.

2. Preliminaries

Let M be a compact two dimensional Riemannian manifold and consider the Euclidean space $\mathbb{R}^k$. Let $L^2(M,\mathbb{R}^k)$ denote the Hilbert

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space of square integrable maps from M into $\mathbb{R}^k$ with inner product

$$(f, g) = \int_M \langle f(x), g(x) \rangle \, dv$$

and norm

$$|f| = (f, f)^{\frac{1}{2}},$$

where $\langle \ , \ \rangle$ is the ordinary inner product in $\mathbb{R}^k$ and dv is the volume element of M. Let $L_1^2(M, \mathbb{R}^k)$ denote the Hilbert space of maps having square integrable first derivatives in the distribution sense. The inner product and norm on $L_1^2(M, \mathbb{R}^k)$ are

$$(f, g)_1 = (f, g) + \int_M \langle df(x), dg(x) \rangle \, dv$$

and

$$|f|_1 = (f, f)_1^{\frac{1}{2}},$$

where $\langle df(x), dg(x) \rangle$ is the natural inner product in $\mathrm{Hom}(M_x, \mathbb{R}^k)$. A sequence $\{ f_i \}$ in $L_1^2(M, \mathbb{R}^k)$ is said to be converge weakly to $f \in L_1^2(M, \mathbb{R}^k)$ if for every $g \in L_1^2(M, \mathbb{R}^k)$ we have

$$\lim_{i \to \infty} (f_i, g)_1 = (f, g)_1.$$

The energy of a map $f \in L_1^2(M, \mathbb{R}^k)$ is given by

$$E(f) = \int_M \langle df(x), dg(x) \rangle \, dv.$$

Let N be another compact Riemannian manifold of dimension n and by the Nash imbedding theorem we suppose that N is imbedded in $\mathbb{R}^k$. We define

$$L_1^2(M, N) = \{ f \in L_1^2(M, \mathbb{R}^k) \, : \, f(x) \in N \text{ for a.e. } x \in M \}.$$

The definition implies that $L_1^2(M, N)$ is closed subset of $L_1^2(M, \mathbb{R}^k)$. Let $\gamma_1, \ldots, \gamma_\ell$ be imbedded closed curves on M which form a generating set for $\pi_1(M,*)$. For convenience we assume that $\gamma_i(0) = *$ for $1 \le i \le \ell$, where γ_i is defined on $[-2,2]$ and that

$$\gamma_i(t) = \gamma_j(t) \quad \text{for } t \in (-1,1),\ 1 \le i,j \le \ell.$$

Let T_i be a Tubular neighborhood of γ_i in M such that

$$\Psi_i : S^1 \times [-1,1] \to T$$

is a smooth immersion. For $s \in [-1,1]$, let $\gamma_i^s : S^1 \to M$ be the curve defined by

$$\gamma_i^s(t) = \Psi_i(t,s)$$

and suppose that

$$\gamma_i^s(t) = \gamma_j^s(t)$$

for $1 \le i,j \le \ell,\ t \in (-1,1)$ and $\gamma_i^0 = \gamma_i$.

It follows from [1] that for a.e. $s \in [-1,1]$, $f|_{\gamma_i^s}$ is continuous and achieve its restricted value on γ_i^s in the L_1^2 sense. Fix such an $s_0 \in [-1,1]$ and let $* = \gamma_i^{s_0}(0)$. Define a map

$$f_\# : \pi_1(M,*) \to \pi_1(N, f(*))$$

by $f_\#(\gamma_i^{s_0}) = f(\gamma_i^{s_0})$ for $1 \le i \le \ell$ on the generators and extend $f_\#$ to be a group homomorphism. Let $\phi \in C^\infty(M,N)$ be a given smooth maps from M to N and let

$$\mathcal{F} = \{ f \in L_1^2(M,N) : f_\# = \tau^{-1}\phi_\#\tau \text{ for some curve from } f(*) \text{ to } \phi(*)\}.$$

Let $I = \inf\{\, E(f) \,:\, f \in \mathcal{F}\,\}$. Then there exists $f \in \mathcal{F}$ such that $E(f) = I$.

Theorem 1. [2]. *There exists a smooth harmonic map $f : M \to N$ with $E(f) = I$ and so that $f_\# : \pi_1(M,*) \to \pi_1(N, f(*))$ and $\phi_\# : \pi_1(M,*) \to \pi_1(N,\phi(*))$ are related by $\tau^{-1} f_\# \tau = \phi_\#$ for some curve τ from $\phi(*)$ to $f(*)$.*

3. Main Theorems

Let R be a fixed compact Riemannian surface of genus γ. A pair of compact Riemannian surface M of genus γ and differential homeomorphism f of R onto M is denoted by (M, f). We define an equivalence relation for the pairs as follows. (M_2, f_2) is said to be equivalent to (M_1, f_1) if and only if there exists a biholomorphic map h of M_1 onto M_2 such that h is homotopic to $f_2 f_1^{-1}$. This space of all equivalence classes is called the Teichmüller space $T(R)$ of R. Let $[(M, f)]$ denote the point of $T(R)$. Let ϕ be a smooth map of R into an n dimensional Riemannian manifold N. Let $\tilde{f}_1$ be the energy minimizing harmonic map of M_1 into N for ϕf_1^{-1}. When (M_2, f_2) is equivalent to (M_1, f_1) by a biholomorphic map h, if $\tilde{f}_2$ is an energy minimizing harmonic map of M_2 into N for ϕf_2^{-1}, $\tilde{f}_2 h$ becomes an energy minimizing harmonic map of M_1 into N for ϕf_1^{-1}. Thus the energy of $\tilde{f}_1$ and $\tilde{f}_2$ are same which defines the energy function E_ϕ on $T(R)$ by giving the energy of an correspondent energy minimizing harmonic map at $[(M, f)]$. Let ζ be a parameter of a neighborhood of a point $[(M, f)] \in T(R)$. Then there exists the Riemannian metric g_ζ on R whose scalar curvature is -1 which gives the complex structure corresponding to ζ. Let $[g_\zeta]$ denote the point of $T(R)$ for ζ. Furthermore, for the almost complex structure J_ζ corresponding to g_ζ, we may consider that $[J_\zeta]$ also denotes the same point of $T(R)$. We denote by (R, g_ζ) the compact Riemannian surface compatible with g_ζ.

Let g_t be a one parameter family in g_ζ and $S(g_t) = S_t$ the harmonic map for ϕ with respect to g_t. Then the energy function along t is defined by the energy $E(g_t, S_t)$ of S_t with respect to g_t:

$$(1) \qquad E_\phi([g_t]) = E(g_t, S_t) = \int_R g_t^{ij} < \frac{\partial S_t}{\partial x^i}, \frac{\partial S_t}{\partial x^j} > \sqrt{g_t}\, dx^1 dx^2,$$

where (x^1, x^2) is a local coordinate system of R, $\frac{\partial S_t}{\partial x^i} = S_{t_x}(\frac{\partial}{\partial x^i})$ and $\sqrt{g_t}$ means $\sqrt{\det(g_{t_{ij}})}$. The harmonic map equation is given by

$$(2) \qquad \frac{1}{\sqrt{g_t}} \frac{\partial}{\partial x^i}(g_t^{ij} \sqrt{g_t} \frac{\partial S_t^\alpha}{\partial x^i}) + \Gamma_{\gamma\beta}^\alpha(S_t) \frac{\partial S_t^\gamma}{\partial x^i} \frac{\partial S_t^\beta}{\partial x^j} g_t^{ij} = 0,$$

where $(x^1, \cdots, x^n)$ is the local coordinate system of N, $\Gamma_{\gamma\beta}^\alpha$ means the Christoffel symbols with respect to this local coordinate system.

Let $h_{t_{ij}}$ be the first variation of g_t. Then $h_{t_{ij}}$ is trace free for $g_{t_{ij}}$, because g_t has a constant scalar curvature -1 and we may consider that $h_{ij}(=h_{0ij})$ is divergence for g_{ij} which implies that $h_{zz}dz^2$ ($h_{zz} = \frac{1}{4}(h_{11}-h_{22}-2ih_{12})$ is a holomorphic quadratic differential for (R,g). The first and second differential of g_t^{ij} are given by

$$(g_t^{ij})' = -g_t^{i\ell}g_t^{jm}(h_{t\ell m})',$$

$$(g_t^{ij})''_{t=0} = h^{i\ell}g^{jm}h_{\ell m} + g^{i\ell}h^{jm}h_{\ell m} - g^{i\ell}g^{jm}(g_{t\ell m})''_{t=0}.$$

Since

$$(\sqrt{g_t})' = \frac{1}{2}g_t^{ij}(g_{tij})'\sqrt{g_t},$$

$$(\sqrt{g_t})''_{t=0} = -\frac{1}{2}h^{ij}\sqrt{g} + \frac{1}{2}g^{ij}(g_{tij})''_{t=0}\sqrt{g} + \frac{1}{2}g^{ij}(g_{tij})'_{t=o}(\sqrt{g_t})'_{t=0},$$

and $h_{t_{ij}}$ is trace free for $g_{t_{ij}}$, we get

$$-h^{ij}h_{ij} + g^{ij}(g_{t_{ij}})'' = 0. \tag{3}$$

Let $V(h)$ be the variation vector field $(S_t)'_{t=0}$. Then the first variation of $E_\phi([g_t])$ is given by

$$\begin{aligned} DE_\phi([g_t])h_t = &\int_R (g_t^{ij})'\langle\frac{\partial S_t}{\partial x^i},\frac{\partial S_t}{\partial x^j}\rangle\sqrt{g_t}\,dx^1dx^2 \\ &- 2\int_R \langle g_t^{ij}\sqrt{g_t}\frac{\partial S_t}{\partial x^i},(\frac{\partial S_t}{\partial x^i})'\rangle\,dx^1dx^2. \end{aligned}$$

Since S_t is a harmonic map with respect to g_t,

$$DE_\phi([g_t])h_t = \int_R (g_t^{ij})'\langle\frac{\partial S_t}{\partial x^i},\frac{\partial S_t}{\partial x^j}\rangle\sqrt{g_t}\,dx^1dx^2.$$

Thus the second variation $E_\phi([g_t])$ at $t=0$ becomes

$$\begin{aligned} D^2E_\phi([g])(h,h) = &\int_R [(g_t^{ij})''_{t=0}\langle\frac{\partial S}{\partial x^i},\frac{\partial S}{\partial x^j}\rangle\sqrt{g} \\ &+ 2(g_t^{ij})'_{t=0}\langle(\frac{\partial S_t}{\partial x^i})'_{t=0},\frac{\partial S}{\partial x^j}\rangle\sqrt{g}]\,dx^1dx^2, \end{aligned}$$

which implies

$$\begin{aligned} D^2E_\phi([g])(h,h) = &\int_R [(h^{i\ell}g^{jm}h_{\ell m} + g^{i\ell}h^{jm}h_{\ell m} \\ &- g^{i\ell}g^{jm}(g_{t\ell m})''_{t=0})\langle\frac{\partial S}{\partial x^i},\frac{\partial S}{\partial x^j}\rangle\sqrt{g} - 2h^{ij}\langle\frac{\partial V(h)}{\partial x^i},\frac{\partial S}{\partial x^j}\rangle\sqrt{g}]\,dx^1dx^2. \end{aligned}$$

THEOREM 2.

$$DE_\phi([g])h = -\frac{1}{2}\int_R h^{ij}\langle\frac{\partial S}{\partial x^i},\frac{\partial S}{\partial x^j}\rangle\, d\mu_g$$

$$D^2E_\phi([g])(h,h) = \int_R [(h^{i\ell}g^{jm}h_{\ell m} + g^{i\ell}h^{jm}h_{\ell m} - g^{i\ell}g^{jm}(g_{t\ell m})''_{t=0})\langle\frac{\partial S}{\partial x^i},\frac{\partial S}{\partial x^j}\rangle\sqrt{g} - 2h^{ij}\langle\frac{\partial V(h)}{\partial x^i},\frac{\partial S}{\partial x^j}\rangle\sqrt{g}]\, dx^1dx^2.$$

THEOREM 3. [2] *N has negative sectional curvatures or N is a flat torus. Let ζ_0 be a critical point of E_ϕ. Then the correspondent harmonic map of (R, g_{ζ_0}) into N is weakly conformal, that is, a branched minimal immersion.*

Proof. Let $z = x^1 + ix^2$ be a complex coordinate system. Let $\lambda|dz|^2$ express the metric g_{ζ_0} on the coordinate neighborhood. We set

$$\begin{aligned}\xi(z) &= \langle\frac{\partial S}{\partial x^1},\frac{\partial S}{\partial x^1}\rangle - \langle\frac{\partial S}{\partial x^1},\frac{\partial S}{\partial x^2}\rangle - 2i\langle\frac{\partial S}{\partial x^1},\frac{\partial S}{\partial x^2}\rangle\\ &= 4\langle\frac{\partial S}{\partial z},\frac{\partial S}{\partial z}\rangle.\end{aligned}$$

Then $\xi(z)dz^2$ is a holomorphic quadratic differential from the harmonicity of S. Note that $\xi(z)dz^2 = 0$ if and only if S is weakly conformal. Since

$$\begin{aligned}DE_\phi([g])h &= -\mathrm{Re}\int_R \frac{1}{\lambda}h_{zz}\overline{\xi(z)}\, dx^1dx^2\\ &= -\langle h_{zz}dz^2, \xi(z)dz^2\rangle w_p,\end{aligned}$$

where $\langle\ ,\ \rangle w_p$ is the Weil-Peterson metric on $T(R)$. $[g_{\zeta_0}]$ is critical if and only if $\langle h_{zz}dz^2, \xi(z)dz^2\rangle w_p = 0$ hold for all h, which implies $\xi(z)dz^2 = 0$. □

From Theorem 2 and Theorem 3, we get the following.

THEOREM 4. *Let $[g_t]$ be a critical point of E_ϕ. Then*

$$D^2E_\phi([g])(h,h) = \int_R [\mu h^{ij}h_{ij} - 2h^{ij}\langle\frac{\partial V(h)}{\partial x^i},\frac{\partial S}{\partial x^j}\rangle\sqrt{g}]\, dx^1dx^2,$$

where $\mu((dx^1)^2 + (dx^2)^2)$ *is the induced metric on each coordinate neighborhood.*

Let $[g_t]$ be a curve in $T(R)$. Let us suppose that $[g]$ is a critical point of E_ϕ whose second variation along $[g_t]$ is negative. As

$$A(S(g_t)) \leq E_\phi([g_t]) = E(g_t, S(g_t)) \leq E_\phi([g]) = E(g, S(g)) = A(S(g)),$$

where A means the area, we obtain

$$D^2 A(W, W) \leq D^2 E_\phi(g)(h, h), \tag{4}$$

where W is the normal component of the variation vector field of $S(g_t)$. We can see that the second variation of area is negative. Next let s be a section of the normal bundle. Let S_t be a variation in the direction s in N. Then induced tensor field $\tilde{g}_{t_{ij}}$ is given by

$$\langle \frac{\partial S_t}{\partial x^i}, \frac{\partial S_t}{\partial x^j} \rangle$$

whose the first variation $\tilde{h}_{ij}$ is $-2\sigma^s_{ij}$, where σ^s_{ij} is the second fundamental form in the direction s. Note that $\tilde{g}_{t_{ij}}$ may be degenerate on branched points. Thus we do not know whether $\tilde{g}_{t_{ij}}$ defines an almost complex structure with smooth dependent t on R. First of all, s is zero on a neighborhood of branched points. Since $S_t = S(g)$ on the neighborhoods, we have the almost complex structures J_t associated with $\tilde{g}_{t_{ij}}$ which has smooth dependence on t. That is, the almost complex structures do not exchange on the neighborhoods.

$[J_t]$ is a smooth curve in $T(R)$ such that $[J_0] = [g]$. We have to consider the tangent vector of $[J_t]$. Let $[g_t]$ be a point in $T(R)$ corresponding to $[J_t]$. Then there exists a non-negative function ρ_t such that $\tilde{g}_t = \rho_t g_t$. Note that ρ_0 is zero for only branched points. Differentiating this, we get $\tilde{h} = (\rho_t)' g + \rho_0 h$. By the minimality, $\mathrm{tr}_{\tilde{g}} \tilde{h} = 0$ holds except branched points. Hence $\mathrm{tr}_g \tilde{h} = 0$. Of course $\mathrm{tr}_g h = 0$ holds. These imply $\tilde{h} = \rho_0 h$. The holomorphic quadratic differential h as the tangent vector of $[g_t]$ is given by

$$P\left(\frac{-2\sigma^s_{zz} dz^2}{\rho_0}\right), \tag{5}$$

where P is the orthogonal projection of the space of quadratic differentials to the space Q of holomorphic quadratic differentials on (R, g). Since $E_\phi([g_t]) \leq E(g_t, S_t) = E(\tilde{g}_t, S_t) = A(S_t)$, it follows

$$D^2 E_\phi([g])(h, h) \leq D^2 A(s, s). \tag{6}$$

Using these results, we can get the following.

THEOREM 5. *For a critical point $[g]$ of E_ϕ, we have*

$$\text{index } E_\phi = \text{index } A.$$

Proof. Let T be the maximal subspace of the tangent space where $D^2 E$ is negative definite. Then $\dim T = \text{index } E_\phi$. When we transfer an element of T to the normal component of the variation vector field of $S(g_t)$, this is an injective linear map. Because if there exists an element X of the kernel, then by (4),

$$0 = D^2 A(0, 0) \leq D^2 E_\phi([g])(X, X) < 0,$$

which is a contradiction. Therefore we obtain index E_ϕ = index A. Let V be the maximal subspace of the space of sections of the normal bundle such that $D^2 A$ in the direction of sections is negative definite. Then $\dim V$ = index A is finite. So there exists a cut-off function φ such that φ are zero for some neighborhood of branched points $D^2 A$ is negative definite on the subspace φV. Let s be an element of φV. We transfer s to h given in (5). This map is linear and injective because

$$D^2 E_\phi([g]) \leq D^2 A(s, s) < 0.$$

Therefore we obtain index $E_\phi \geq$ index A. □

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