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# ON HENSTOCK STIELTJES INTEGRAL

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Abstract. In this paper, we define the Henstock-Stieltjes integral for real-valued function and prove some of its properties.

## 1. Introduction and Prelininaries

It is well known that the Riemann integral is not adequate for advanced mathematics, since there are many functions that are not Riemann integrable, and since the integral does not possess sufficiently strong convergence theorems. To correct these deficiencies, Lebesgue developed his integral around the turn of the present century, and his integral has become the official integral in mathematical reseach. However, there are also diffiulties with the Lebesgue integral. We need more general than the Lebesgue integral. In his study of differential equations during the 1950's, J. Kurzweil introduced a generalized version of the Riemann integral. In the 1960's Henstock made the first systematic study of this new integral, but some reason it has not become well known. Its definition is Riemann-like , but its power is super-Lebesgue. Sergio S. Cao[1] generalize the definition of the Henstock integral for real-valued functions to functions taking values in Banach spaces and investigate some of its properties. Many authors have studied Henstock integral( [2], [3], [4] ). In this paper, we define the Henstock-Stieltjes integral for real-valued function and investigate some of its basic properties. We begin with some terminology notations.

Definition 1.1. Let $\delta(\cdot)$ be a positive function defined on the interval $[a, b]$. A tagged interval $(x, [c, d])$ consists of an inteval $[c, d] \subseteq [a, b]$

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and a point $x \in [c, d]$. The tagged interval $(x, [c, d])$ is subordinate to $\delta$ if

$$[c, d] \subseteq (x - \delta(x), x + \delta(x)).$$

The letter $\mathcal{P}$ will be used to denote finite collections of non-overlapping tagged intervals. Let $\mathcal{P} = \{(x_i, [c_i, d_i]) : 1 \leq i \leq n\}$ be such a collection in $[a, b]$. We adopt the following terminology.

(1) The points $\{x_i\}$ are the tags of $\mathcal{P}$ and the intervals $\{[c_i, d_i]\}$ are the intervals of $\mathcal{P}$,
(2) If $(x_i, [c_i, d_i])$ is subordinate to $\delta$ for each $i$, then $\mathcal{P}$ is subordinate to $\delta$,
(3) If $\mathcal{P}$ is subordinate to $\delta$ and $[a, b] = \cup_{i=1}^{n}[c_i, d_i]$, then $\mathcal{P}$ is a tagged partition of $[a, b]$ that is subordinate to $\delta$.

Let $\mathcal{P} = \{(x_i, [c_i, d_i] : 1 \leq i \leq n\}$ be a finite collection of non-overlapping tagged intervals in $[a, b]$, let $f : [a, b] \to R$, let $F$ be a function defined on the subintervals of $[a, b]$ and let $\alpha$ be an increasing function on $[a, b]$. We will use the following notation:

$$f_\alpha(\mathcal{P}) = \sum_{i=1}^{n} f(x_i)(\alpha(d_i) - \alpha(c_i))$$

## 2. Henstock- Stieltjes Integal

We now present the definition of the Henstock-Stieltjes integral. If $\alpha(x) = x$, the weight assigned to tagged interval $(x_i, [c_i, d_i])$ is $f(x_i)(\alpha(d_i) - \alpha(c_i)) = f(x_i)(d_i - c_i)$. Thus the Henstock-Stieltjes integral include the Henstock integral as a special case.

Definition 2.1. Let $\alpha$ be an increasing function on $[a, b]$. A function $f : [a, b] \to R$ is Henstock-Stieltjes integrable with respect to $\alpha$ on $[a, b]$ if there exists a real number $L$ with the following property : for each $\epsilon > 0$, there exists a positive function $\delta$ on $[a, b]$ such that $|f_\alpha(\mathcal{P}) - L| < \epsilon$ whenever $\mathcal{P}$ is a tagged partition of $[a, b]$ that is subordinate to $\delta$ . The function $f$ is Henstock-Stieltjes integrable on a measurable set $E \subseteq [a, b]$ with respect to $\alpha$ if $f_{\chi_E}$ is Henstock - Stieltjes integrable with respect to $\alpha$ on $[a, b]$.

THEOREM 2.2. *Let $\alpha$ be an increasing function on $[a,b]$ and let $f=0$ nearly everywhere on $[a,b]$. If $\alpha \in C^1[a,b]$, then $f$ is Henstock-Stieltjes intrgrable with respect to $\alpha$ on $[a,b]$ and $\int_a^b f d\alpha = 0$.*

*Proof.* Let $\{a_n | n \in Z^+\} = \{x \in [a,b] : f(x) \neq 0\}$ and let $\epsilon > 0$. Since $\alpha \in C^1[a,b]$, there exists $M$ such that $|\alpha'(x)| \leq M$ for all $x \in [a,b]$. By the mean-value theorem, there exists $x_{i_0} \in (c_i, d_i)$ such that

$$\alpha(d_i) - \alpha(c_i) = \alpha'(x_{i_0})(d_i - c_i)$$

Define a positive function $\delta$ by

$$\delta(x) = \begin{cases} 1 & \text{if } x \in [a,b] - \{a_n | n \in Z^+\} \\ \frac{\epsilon}{|f(a_n)| M 2^{n+1}} & \text{if } x = a_n. \end{cases}$$

Suppose that $\mathcal{P} = \{(x_i, [c,d]) : 1 \leq i \leq q\}$ is a tagged partition of $[a,b]$ that is subordinate to $\delta$ and assume that each tag occurs only once. Let $\pi$ be the set of all indices $i$ such that $x_i \in \{a_n : x \in Z^+\}$ and for each $i \in \pi$, choose $n_i$ so that $x_i = a_{n_i}$. Then

$$\begin{aligned} |f_\alpha(\mathcal{P}) &= |\sum_{i\in\pi} f(x_i)[\alpha(d_i) - \alpha(c_i)]| \\ &= |\sum_{i\in\pi} f(x_i)\alpha'(x_{i_0})(d_i - c_i)| \\ &\leq \sum_{i\in\pi} 2|f(a_{n_i})| M\delta(a_{n_i}) \leq \sum_{i\in\pi} \epsilon 2^{-n_i} < \epsilon \end{aligned}$$

Hence, the function $f$ is Henstock-Stieltjes integrable with respect to $\alpha$ on $[a,b]$ and $\int_a^b f d\alpha = 0$. □

Given a set $E$ and a point $x$, let $\rho(x,E) = \inf\{|y-x| : y \in E\}$ be the distance from $x$ to $E$. Note that $\rho(x,E) > 0$ if $x \notin E$ and $E$ is closed.

THEOREM 2.3. *Let $\alpha$ be an increasing function on $[a,b]$ and let $f=0$ almost everywhere on $[a,b]$. If $\alpha \in C^1[a,b]$, then $f$ is Henstock-Stieltjes integrable with respect to $\alpha$ on $[a,b]$ and $\int_a^b f d\alpha = 0$.*

*Proof.* Since $\alpha \in C^1[a,b]$, there exists $M$ such that $|\alpha'(x)| \leq M$, for all $x \in [a,b]$. By the mean-value theorem , there exists $x_{i_0} \in (c_i, d_i)$ such that

$$\alpha(d_i) - \alpha(c_i) = \alpha'(x_{i_0})(d_i - c_i)$$

Let $E = \{x \in [a,b] : f(x) \neq 0\}$ and for each positive integer $n$, let $E_n = \{x \in E : n-1 \leq |f(x)| < n\}$. These sets are disjoint and each has measure zero. For each $n$, choose an open set $O_n$ such that $E_n \subseteq O_n$ and $\mu(O_n) < \frac{\epsilon}{n2^n M}$. Define a positive function $\delta$ on $[a,b]$ by

$$\delta(x) = \begin{cases} 1 & \text{if } x \in [a,b] - E; \\ \rho(x, O_n^c) & \text{if } x \in E_n. \end{cases}$$

Suppose that $\mathcal{P}$ is a tagged partition that is subordinate to $\delta$. For each $n$, let $\mathcal{P}_n$ be the subset of $\mathcal{P}$ that has tags in $E_n$. If $I$ is an interval in $\mathcal{P}_n$, then $I \subseteq O_n$. We thus have

$$\begin{aligned} |f_\alpha(\mathcal{P})| = \sum_{n=1}^{\infty} |f_\alpha(\mathcal{P}_n)| &= \sum_{n=1}^{\infty} \Big| \sum_{x_{n_i} \in E_n} f(x_{n_i})[\alpha(d_{n_i}) - \alpha(c_{n_i})]\Big| \\ &= \sum_{n=1}^{\infty} \Big| \sum_{x_{n_i} \in E_n} f(x_{n_i})\alpha'(x_{n_{i_0}})(d_{n_i} - c_{n_i})\Big| \\ &< \sum_{n=1}^{\infty} nM\mu(O_n) < \sum_{n=1}^{\infty} \epsilon 2^{-n} = \epsilon \end{aligned}$$

Hence, the function $f$ is Henstock-Stieltjes integrable with respect to $\alpha$ on $[a,b]$ and $\int_a^b f d\alpha = 0$. □

We next verify the basic properties of the Henstock-Stieltjes integral. Just as in the Henstock integral, there is a Cauchy criterion for function to be Henstock-Stieltjes integrable. This is the content of the next Theorem.

THEOREM 2.4. *Let $\alpha$ be an increasing function on $[a,b]$. A fuction $f : [a,b] \to R$ is Henstock-Stieltjes integrable with respect to $\alpha$ on $[a,b]$ if and only if for each $\epsilon > 0$ there exists a positive function $\delta$ on $[a,b]$ such that $|f_\alpha(\mathcal{P}_1) - f_\alpha(\mathcal{P}_2)| < \epsilon$ whenever $\mathcal{P}_1$ and $\mathcal{P}_2$ are tagged partitions of $[a,b]$ that are subordinate to $\delta$.*

*Proof.* Suppose first that $f$ is Henstock-Stietjes integrable with respect to $\alpha$ and $\epsilon > 0$. There exists a positive function $\delta$ on $[a, b]$ such that

$$|f_\alpha(\mathcal{P}_1) - \int_a^b f d\alpha| < \frac{\epsilon}{2}, \quad |f_\alpha(\mathcal{P}_2) - \int_a^b f d\alpha| < \frac{\epsilon}{2}$$

whenever $\mathcal{P}_1$ and $\mathcal{P}_2$ are tagged partitions of $[a, b]$ that are subordinate to $\delta$. Then

$$\begin{aligned} |f_\alpha(\mathcal{P}_1) - f_\alpha(\mathcal{P}_2)| &\leq |f_\alpha(\mathcal{P}_1) - \int_a^b f d\alpha| + |f_\alpha(\mathcal{P}_2) - \int_a^b f d\alpha| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Hence, the Cauchy criterion is satisfied.

Conversely, suppose that the Cauchy criterion is satisfied. For each positive integer $n$, choose a positive function $\delta_n$ on $[a, b]$ such that

$$|f_\alpha(\mathcal{P}_1) - f_\alpha(\mathcal{P}_2)| < \frac{1}{n}$$

whenever $\mathcal{P}_1$ and $\mathcal{P}_2$ are tagged partitions of $[a, b]$ that are subordinate to $\delta_n$. We may assume that the sequence $\{\delta_n\}$ is nonincreasing. For each $n$, let $\mathcal{P}_n$ be a tagged partition of $[a, b]$ that is subordinate to $\delta_n$. The sequence $\{f_\alpha(\mathcal{P}_n)\}$ is a Cauchy sequence since

$$m > n \geq N \quad \text{implies} \quad |f_\alpha(\mathcal{P}_m) - f_\alpha(\mathcal{P}_n)| < \frac{1}{N}$$

Let $L$ be the limit of this sequence and let $\epsilon > 0$. Choose a positive integer $N$ such that

$$\frac{1}{N} < \frac{\epsilon}{2} \quad \text{and} \quad |f_\alpha(\mathcal{P}_n) - L| < \frac{\epsilon}{2} \quad \text{for all } n \leq N$$

Let $\mathcal{P}$ be a tagged partition of $[a, b]$ that is subordinate to $\delta_N$ on $[a, b]$ and compute

$$|f_\alpha(\mathcal{P}) - L| \leq |f_\alpha(\mathcal{P}) - f_\alpha(\mathcal{P}_N)| + |f_\alpha(\mathcal{P}_N) - L| < \frac{1}{N} + \frac{\epsilon}{2} < \epsilon$$

Hence, the function $f$ is Henstock-Stieltjes integrable with respect to $\alpha$ on $[a, b]$. □

Theorem 2.5. *Let $\alpha$ be an increasing function on $[a, b]$. Let $f : [a, b] \to R$. If $f$ is Henstock-Stieltjes integrable with respect to $\alpha$ on $[a, b]$, then $f$ is Henstock-Stieltjes integrable with respect to $\alpha$ on every subinterval of $[a, b]$.*

*Proof.* Let $[c, d] \subseteq [a, b]$ and let $\epsilon > 0$. Choose a positive function $\delta$ on $[a, b]$ such that $|f_\alpha(\mathcal{P}_1) - f_\alpha(\mathcal{P}_2)| < \epsilon$ whenever $\mathcal{P}_1$ and $\mathcal{P}_2$ are tagged partitions of $[a, b]$ that are subordinate to $\delta$. Fix tagged partition $\mathcal{P}_a$ of $[a, c]$ and $\mathcal{P}_b$ of $[d, b]$ that are subordinate to $\delta$. Let $\mathcal{P}'_1$ and $\mathcal{P}'_2$ be tagged partitions of $[c, d]$ that are subordinate to $\delta$ and define $\mathcal{P}_1 = \mathcal{P}_a \cup \mathcal{P}'_1 \cup \mathcal{P}_b$ and $\mathcal{P}_2 = \mathcal{P}_a \cup \mathcal{P}'_2 \cup \mathcal{P}_b$. Then $\mathcal{P}_1$ and $\mathcal{P}_2$ are tagged partitions of $[a, b]$ that are subordinate to $\delta$ and

$$|f_\alpha(\mathcal{P}'_1) - f_\alpha(\mathcal{P}'_2)| = |f_\alpha(\mathcal{P}_1) - f_\alpha(\mathcal{P}_2)| < \epsilon.$$

Hence, the function $f$ is Henstock-Stieltjes integrable with respect to $\alpha$ on every subinterval of $[a, b]$. □

Theorem 2.6. *Let $\alpha$ be an increasing function on $[a, b]$. Let $f : [a, b] \to R$ and let $c \in (a, b)$. If $f$ is Henstock-Stieltjes integrable with respect to $\alpha$ on each of the intervals $[a, c]$ and $[c, b]$, then $f$ is Henstock- Stieltjes integrable with respect to $\alpha$ on $[a, b]$ and $\int_a^b f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha$.*

*Proof.* Let $\epsilon > 0$ . By hypothesis, there exists a positive function $\delta_1$ on $[a, c]$ such that $|f_\alpha(\mathcal{P}) - \int_a^c f d\alpha| < \frac{\epsilon}{2}$ whenever $\mathcal{P}$ is tagged partition of $[a, c]$ that is subordinate to $\delta_1$ and a positive function $\delta_2$ on $[c, b]$ such that $|f_\alpha(\mathcal{P}) - \int_a^b f d\alpha| < \frac{\epsilon}{2}$ whenever $\mathcal{P}$ is tagged partition of $[c, d]$ that is subordinate to $\delta_2$. Defint $\delta$ on $[a, b]$ by

$$\delta(x) = \begin{cases} \min\{\delta_1(x), c - x\}, & \text{if } a \le x < c; \\ \min\{\delta_1(c), \delta_2(c)\}, & \text{if } x = c; \\ \min\{\delta_2(x), x - c\}, & \text{if } c < x \le b. \end{cases}$$

Let $\mathcal{P}$ be a tagged partition of $[a, b]$ that is subordinate to $\delta$ and suppose that each tag occurs only once. Note that $\mathcal{P}$ must be of the form

$\mathcal{P}_a \cup (c, [u, v]) \cup \mathcal{P}_b$ where the tags of $\mathcal{P}_a$ are less than $c$ and the tags of $\mathcal{P}_b$ are greater than $c$. Let $\mathcal{P}_1 = \mathcal{P}_a \cup (c, [u, c])$ and let $\mathcal{P}_2 = \mathcal{P}_b \cup (c, [c, v])$. Then $\mathcal{P}_1$ is a tagged partition of $[a, c]$ that is subordinate to $\delta_1$ and $\mathcal{P}_2$ is a tagged partition of $[c, b]$ that is subordinate to $\delta_2$. Since $f_\alpha(\mathcal{P}) = f_\alpha(\mathcal{P}_1) + f_\alpha(\mathcal{P}_2)$, we obtain

$$\begin{aligned} |f_\alpha(\mathcal{P}) - &\int_a^c f d\alpha - \int_c^b f d\alpha| \\ &\leq |f_\alpha(\mathcal{P}_1) - \int_a^c f d\alpha| + |f_\alpha(\mathcal{P}_2) - \int_c^b f d\alpha| < \epsilon. \end{aligned}$$

Hence the function $f$ is Henstock-Stieltjes integrable with respect to $\alpha$ on $[a, b]$ and $\int_a^c f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha$. □

The next theorem shows the linearity of the Henstock-Stieltjes integrals.

THEOREM 2.7. *Let $\alpha$ be an increasing function on $[a, b]$. Let $f$ and $g$ be Henstock-Stieltjes integrable with respect to $\alpha$ on $[a, b]$. Then*

(1) *$kf$ is Henstock-Stieltjes integrable with respect to $\alpha$ on $[a, b]$ and $\int_a^b kf d\alpha = k \int_a^b f d\alpha$ for each $k \in R$;*
(2) *$f + g$ is Henstock-Stieltjes integrable with respect to $\alpha$ on $[a, b]$ and $\int_a^b (f + g) d\alpha = \int_a^b f d\alpha + \int_a^b g d\alpha$.*

*Proof.* (1) Let $f$ be Henstock-Stieltjes integrable with respect to $\alpha$ on $[a, b]$. Case 1. $k = 0$. Of course, (1) is obvious. Case 2. $k \neq 0$. There exists a positive function $\delta$ on $[a, b]$ such that $|f_\alpha(\mathcal{P}) - \int_a^b f d\alpha| < \frac{\epsilon}{|k|}$ whenever $\mathcal{P}$ is a tagged partition of $[a, b]$ that is subordinate to $\delta$. Then

$$|(kf)_\alpha(\mathcal{P}) - k \int_a^b f d\alpha| = |k| \cdot |f_\alpha(\mathcal{P}) - \int_a^b f d\alpha| < \epsilon$$

Hence $kf$ is Henstock-Stieltjes integrable with respect to $\alpha$ on $[a, b]$ and $\int_a^b kf d\alpha = k \int_a^b f d\alpha$ for each $k \in R$.

(2) Let f and g be Henstock-Stieltjes integrable with respect to $\alpha$ on $[a, b]$. There exists a positive function $\delta_1$ on $[a, b]$ such that $|f_\alpha(\mathcal{P}_1) -$

$\int_a^b f d\alpha| < \frac{\epsilon}{2}$ whenever $\mathcal{P}_1$ is tagged partitions of $[a, b]$ that is subordinate to $\delta_1$ and a positive function $\delta_2$ such that $|g_\alpha(\mathcal{P}_2) - \int_a^b g d\alpha| < \frac{\epsilon}{2}$ whenever $\mathcal{P}_2$ is tagged partitions of $[a, b]$ that is subordinate to $\delta_2$. Define $\delta$ on $[a, b]$ by

$$\delta(x) = \min\{\delta_1(x), \delta_2(x)\}$$

Let $\mathcal{P}$ be a tagged partition of $[a, b]$ that is subordinate to $\delta$. Then

$$\begin{aligned} &|(f+g)_\alpha(\mathcal{P}) - (\int_a^b f d\alpha + \int_a^b g d\alpha)| \\ &\leq |f_\alpha(\mathcal{P}) - \int_a^b f d\alpha| + |g_\alpha(\mathcal{P}) - \int_a^b g d\alpha| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Hence $f + g$ is Henstock-Stieltjes integrable with respect to $\alpha$ on $[a, b]$ and $\int_a^b (f + g) d\alpha = \int_a^b f d\alpha + \int_a^b g d\alpha$. □

We will use the following notation : $f^+(x) = \max\{f(x), 0\}$, $f^-(x) = \max\{-f(x), 0\}$.

THEOREM 2.8. *Let $\alpha$ be an increasing function on $[a, b]$. Let $f$ and $g$ are Henstock-Stieltjes integrable with respect to $\alpha$ on $[a, b]$. If $\alpha \in C^1[a, b]$, Then*

(1) *if $f \leq g$ almost everywhere on $[a, b]$, then*

$$\int_a^b f d\alpha \leq \int_a^b g d\alpha,$$

(2) *if $f = g$ almost everywhere on $[a, b]$, then*

$$\int_a^b f d\alpha = \int_a^b g d\alpha.$$

*Proof.* (1) Suppose that $f \geq 0$ almost everywhere on $[a, b]$. Since $f^- = 0$ almost everywhere on $[a, b]$, it is Henstock-Stieltjes integrable with respect to $\alpha$ on $[a, b]$ by Theorem 2.3 and $\int_a^b f^- d\alpha = 0$. By Theorem 2.7(2), the function $f^+ = f + f^-$ is Henstock-Stieltjes integrable with respect to $\alpha$ on the $[a, b]$. Since $f^+ \geq 0$ on $[a, b]$, it is $\int_a^b f^+ d\alpha \geq 0$. Consequently, by Theorem 2.7(1) and (2),

$$\int_a^b f d\alpha = \int_a^b (f^+ - f^-) d\alpha = \int_a^b f^+ d\alpha - \int_a^b f^- d\alpha = \int_a^b f^+ d\alpha \geq 0$$

The general result now follows since $g - f \geq 0$ almost everywhere on $[a, b]$ implies

$$\int_a^b g d\alpha - \int_a^b f d\alpha = \int_a^b (g - f) d\alpha \geq 0.$$

(2) If $f = g$ almost everywhere on $[a, b]$, then $f - g = 0$ almost everywhere on $[a, b]$. By Theorem 2.3, $f - g$ is Henstock-Stieltjes integrable with respect to $\alpha$ on $[a, b]$ and $\int_a^b (f - g) d\alpha = 0$. By Theorem 2.7, the function $g = f + (g - f)$ is Henstock-Stieltjes integrable on $[a, b]$ with respect to $\alpha$ and

$$\int_a^b g d\alpha = \int_a^b f d\alpha + \int_a^b (g - f) d\alpha = \int_a^b f d\alpha. \qquad \square$$

COROLLARY 2.9. *Let $\alpha$ be an increasing function on $[a, b]$. Let $f$ and $g$ are Henstock-Stieltjes integrable with respect to $\alpha$ on [a,b]. If $f(x) \leq g(x)$ for all $x \in [a, b]$, then $\int_a^b f d\alpha \leq \int_a^b g d\alpha$.*

THEOREM 2.10. *If $f$ is continuous on $[a, b]$ and $\alpha$ is increasing on $[a, b]$, there exists $c \in [a, b]$ such that $\int_a^b f d\alpha = f(c)[\alpha(b) - \alpha(a)]$.*

*Proof.* If $\alpha(b) = \alpha(a)$, then any value of $c$ in $[a, b]$ gives the desired conclusion. Suppose that $\alpha(b) > \alpha(a)$. Then $f$ attains its maximum $M$ and minimum $m$ on $[a, b]$. we have

$$m[\alpha(b) - \alpha(a)] \leq \int_a^b f d\alpha \leq M[\alpha(b) - \alpha(a)]$$

Therefore

$$m \le \frac{\int_a^b f d\alpha}{\alpha(b) - \alpha(a)} \le M.$$

By the intermediate-value theorem, there exists $c$ in $[a, b]$ such that

$$f(c) = \frac{\int_a^b f d\alpha}{\alpha(b) - \alpha(a)}. \qquad \square$$

## References

[1] S.S. Cao, *The Henstock Integral for Banach-Valued functions*, SEA Bull. Math. **16(1)** (1992).
[2] R. Henstock, *The general of integration*, Oxford Math. Monographs, Clarendon Press, Oxford, 1991.
[3] Lee, Peng Yee, *Lanzhou lectures on Henstock integration*, World Scientific Pub. Co., Singapore, 1989.
[4] R.A. Gordon, *The Denjoy extension of the Bochner, Pettis, and Dunford integrals*, Studies Math. Vol.92, 1989.

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