Kangweon-Kyungki Math. Jour. **6** (1998), No. 1, pp. 27–45

SOME PROPERTIES OF FUZZY QUASI-UNIFORM SPACES

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Abstract. We will define a fuzzy quasi-uniform space and investigate some properties of fuzzy quasi-uniform spaces. We will show that the fuzzy bitopology and the fuzzy quasi-proximity can be induced by a fuzzy quasi-uniformity.

1. Introduction

In [7,8,9], S.K. Samanta introduced the fuzziness in the concept of openness of a fuzzy subset as a generalization of Chang's fuzzy topology. Moreover, S.K. Samanta [6] introduced the concept of gradations of fuzzy proximity and fuzzy uniformity. It was shown that this fuzzy proximity and fuzzy uniformity are more general than that of Artico and Moresco [1] and that of B. Hutton [3].

On the other hand, M.H. Ghanim et al.[5] introduced fuzzy proximity spaces with somewhat different definition of S.K. Samanta [6]. In [11], we defined a fuzzy quasi-proximity space in view of the definition of M.H. Ghanim et al.[5] and investigated some properties of fuzzy quasi-proximities.

In this paper, we will define a fuzzy quasi-uniform space in view of the definition of Samanta [6] and investigate some properties of fuzzy quasi-uniform spaces. We will show that the fuzzy bitopology and the fuzzy quasi-proximity can be induced by a fuzzy quasi-uniformity.

In this paper, all the notations and the definitions are standard in fuzzy set theory.

Received December 11, 1997.
1991 Mathematics Subject Classification: 54A40.
Key words and phrases: fuzzy quasi-proximity, fuzzy quasi-uniformity.

2. Preliminaries

Definition 2.1. [9] Let X be a nonempty set. A function $\mathcal{T} : I^X \to I$ is called a *gradation of openness* on X if it satisfies the following conditions:

(O1) $\mathcal{T}(\tilde{0}) = \mathcal{T}(\tilde{1}) = 1$, where $\tilde{0}(x) = 0$ and $\tilde{1}(x) = 1$ for all $x \in X$.
(O2) $\mathcal{T}(\mu_1 \wedge \mu_2) \geq \mathcal{T}(\mu_1) \wedge \mathcal{T}(\mu_2)$.
(O3) $\mathcal{T}(\bigvee_{i\in\Gamma} \mu_i) \geq \bigwedge_{i\in\Gamma} \mathcal{T}(\mu_i)$.

The pair $(X, \mathcal{T})$ is called a *fuzzy topological space.*

Let $\mathcal{T}$ be a gradation of openness on X and $\mathcal{F} : I^X \to I$ be defined by $\mathcal{F}(\lambda) = \mathcal{T}(\tilde{1} - \lambda)$. Then $\mathcal{F}$ is called a *gradation of closedness* on X.

Let $(X, \mathcal{T})$ be a fuzzy topological space, then for each $r \in I$, $\mathcal{T}_r = \{\mu \in I^X \mid \mathcal{T}(\mu) \geq r\}$ is a Chang's fuzzy topology on X.

Let $\mathcal{T}_1, \mathcal{T}_2$ be gradations of openness on X. The space $(X, \mathcal{T}_1, \mathcal{T}_2)$ is called a *fuzzy bitopological space.*

From the definition of M.H. Ghanim [5], we can define a fuzzy quasi-proximity.

Definition 2.2. [11] A function $\delta : I^X \times I^X \to I$ is said to be a *fuzzy quasi-proximity* on X which satisfies the following conditions:

(FQP1) $\delta(\tilde{0}, \tilde{1}) = 0$ and $\delta(\tilde{1}, \tilde{0}) = 0$.
(FQP2) (1) $\delta(\mu, \rho \vee \lambda) = \delta(\mu, \lambda) \vee \delta(\rho, \lambda)$.
(2) $\delta(\mu \vee \rho, \lambda) = \delta(\mu, \lambda) \vee \delta(\rho, \lambda)$.
(FQP3) If $\delta(\mu, \rho) < r$ for $r \in (0, 1]$, then there exists $\lambda \in I^X$ such that $\delta(\mu, \lambda) < r$ and $\delta(\tilde{1} - \lambda, \rho) < r$.
(FQP4) If $\delta(\mu, \rho) \neq 1$, then $\mu \leq \tilde{1} - \rho$.

The pair (X, δ) is called a *fuzzy quasi-proximity space.*

A fuzzy quasi-proximity space (X, δ) is called a *fuzzy proximity space* if the following is satisfied:

(FP) $\delta(\lambda, \mu) = \delta(\mu, \lambda)$, for all $\lambda, \mu \in I^X$.

Let δ_1, δ_2 be quasi-proximities on X. We say that δ_2 is *finer* than δ_1 (δ_1 is *coarser* than δ_2) iff for any $\lambda, \mu \in I^X$, $\delta_2(\lambda, \mu) \leq \delta_1(\lambda, \mu)$.

Theorem 2.3. [5] *Let (X, δ) be a fuzzy (quasi-)proximity space. Then, for each $r \in (0, 1]$ the family $\delta_r = \{(\mu, \rho) \in I^X \times I^X \mid \delta(\mu, \rho) \geq r\}$ is a classical fuzzy (quasi-)proximity space on X.*

Let (X, δ) be a fuzzy quasi-proximity space. We define $\delta^{-1}(\lambda, \mu) = \delta(\mu, \lambda)$ for every $\lambda, \mu \in I^X$. Then δ^{-1} is a fuzzy quasi-proximity on X.

THEOREM 2.4. [11] *Let (X, δ) be a fuzzy quasi-proximity space. We define, for $\lambda, \mu \in I^X$,*

$$\delta^*(\lambda, \mu) = inf\{\bigvee_{j,k}(\delta(\lambda_j, \mu_k) \wedge \delta^{-1}(\lambda_j, \mu_k))\}.$$

where for every finite families $(\lambda_j), (\mu_k)$ such that $\lambda = \bigvee \lambda_j$ and $\mu = \bigvee \mu_k$. Then the structure δ^* is the coarsest fuzzy proximity on X which is finer than δ and δ^{-1}.

THEOREM 2.5. [6,11] *Let δ be a fuzzy quasi-proximity on X. For each $r \in [0, 1), \lambda \in I^X$, we define*

$$cl_\delta(\lambda, r) = \bigwedge\{\tilde{1} - \rho \mid \delta(\rho, \lambda) < 1 - r\}.$$

Then it satisfies the followings:

(i) $cl_\delta(\tilde{0}, r) = \tilde{0},\ cl_\delta(\tilde{1}, r) = \tilde{1}$.
(ii) $cl_\delta(\lambda, r) \geq \lambda$ *and* $cl_\delta(\lambda_1, r) \leq cl_\delta(\lambda_2, r)$, *if* $\lambda_1 \leq \lambda_2$.
(iii) $cl_\delta(cl_\delta(\lambda, r), r) = cl_\delta(\lambda, r)$.
(iv) $cl_\delta(\lambda \vee \mu, r) = cl_\delta(\lambda, r) \vee cl_\delta(\mu, r)$.
(v) $cl_\delta(\lambda, r) \leq cl_\delta(\lambda, r')$, *if* $r \leq r'$, *where* $r, r' \in [0, 1)$.

THEOREM 2.6. [6] *Let (X, δ) be a fuzzy quasi-proximity space. The function $\mathcal{F}_\delta : I^X \to I$ defined by*

$$\mathcal{F}_\delta(\lambda) = \bigvee\{r \in [0, 1) \mid cl_\delta(\lambda, r) = \lambda\}, \quad \lambda \in I^X.$$

Then $\mathcal{F}_\delta$ is a gradation of closedness on X.

3. Some properties of fuzzy quasi-uniformity spaces

In [3], B. Hutton expanded the concept of entourages of the uniformity as following results.

Let Ω_X denote the family of all functions $U : I^X \to I^X$ with the following properties:

(1) $U(\tilde{0}) = \tilde{0}$, $\mu \leq U(\mu)$, for every $\mu \in I^X$.
(2) $U(\bigvee \mu_i) = \bigvee U(\mu_i)$, for $\mu_i \in I^X$.

For $U \in \Omega_X$, the function $U^{-1} \in \Omega_X$ is defined by

$$U^{-1}(\mu) = \bigwedge \{\rho \mid U(\tilde{1} - \rho) \leq \tilde{1} - \mu\}.$$

For $U, V \in \Omega_X$, we define, for all $\mu \in I^X$,

$$(U \sqcap V)(\mu) = \bigwedge \{U(\mu_1) \vee V(\mu_2) \mid \mu_1 \vee \mu_2 = \mu\}, \quad U \circ V(\mu) = U(V(\mu)).$$

Then $U \sqcap V, U \circ V \in \Omega_X$ from Lemma 2 of [3].

LEMMA 3.1. *For any $U, V, W, U_1, V_1 \in \Omega_X$, the following properties hold:*

(1) *If $U \leq U_1$ and $V \leq V_1$, then $U \sqcap V \leq U_1 \sqcap V_1$.*
(2) *$U \sqcap V \leq U$, $U \sqcap V \leq V$ and $U \sqcap U = U$.*
(3) *$(U^{-1})^{-1} = U$.*
(4) *$U \leq V$ iff $U^{-1} \leq V^{-1}$.*
(5) *Let a function $D : I^X \to I^X$ be defined by*

$$D(\mu) = \begin{cases} \tilde{1} & \text{if } \mu \neq \tilde{0} \\ \tilde{0} & \text{if } \mu = \tilde{0}. \end{cases}$$

Then $D = D^{-1} \in \Omega_X$ and $U \sqcap D = U$.
(6) *$U(\mu) \leq \lambda$ iff $U^{-1}(\tilde{1} - \lambda) \leq \tilde{1} - \mu$, for $\mu, \lambda \in I^X$.*
(7) *$U(\tilde{1} - U^{-1}(\lambda)) \leq \tilde{1} - \lambda$ for $\lambda \in I^X$.*
(8) *$(V \circ U)^{-1} = U^{-1} \circ V^{-1}$.*
(9) *$(U \sqcap V)^{-1} = U^{-1} \sqcap V^{-1}$.*
(10) *$(U \sqcap V) \sqcap W = U \sqcap (V \sqcap W)$.*

Proof. (1) and (2) are easily proved from the definition of $U \sqcap V$.

(3) and (4) are proved from Lemma 3.8 of [10].

(5) From the definition of D, we have $D = D^{-1} \in \Omega_X$.

From (2), we have $U \sqcap D \leq U$. Suppose that $U \sqcap D \not\geq U$. Then there exist $x \in X$, $\mu \in I^X$ such that $(U \sqcap D)(\mu)(x) < U(\mu)(x)$. From the definition of $U \sqcap D$, there exist $\mu_1, \mu_2 \in I^X$ such that $\mu_1 \vee \mu_2 = \mu$ and

$$(U \sqcap D)(\mu)(x) \leq U(\mu_1)(x) \vee D(\mu_2)(x) < U(\mu)(x).$$

If $\mu_2 \neq \tilde{0}$, then $D(\mu_2) = \tilde{1}$. If $\mu_2 = \tilde{0}$, then $U(\mu)(x) < U(\mu)(x)$. It is a contradiction. Hence $U \sqcap D \geq U$.

(6) If $U(\mu) \leq \lambda$, we have $U^{-1}(\tilde{1}-\lambda) = \bigwedge\{\rho \mid U(\tilde{1}-\rho) \leq \lambda\} \leq \tilde{1}-\mu$.
If $U^{-1}(\tilde{1}-\lambda) \leq \tilde{1}-\mu$, we have, by (3), $U(\mu) = (U^{-1})^{-1}(\mu) = \bigwedge\{\rho \mid U^{-1}(\tilde{1}-\rho) \leq \tilde{1}-\mu\} \leq \lambda$.

(7) Since $U^{-1}(\tilde{1}-(\tilde{1}-\lambda)) \leq \tilde{1}-(\tilde{1}-U^{-1}(\lambda))$, by (6), we have $U(\tilde{1}-U^{-1}(\lambda)) \leq \tilde{1}-\lambda$.

(8) For $\mu \in I^X$, we have

$$\begin{aligned} U^{-1}(V^{-1}(\mu)) &= \bigwedge\{\rho \mid U(\tilde{1}-\rho) \leq \tilde{1}-V^{-1}(\mu)\} \\ &= \bigwedge\{\rho \mid V(U(\tilde{1}-\rho)) \leq \tilde{1}-\mu\} \quad \text{(by (6))} \\ &= (V \circ U)^{-1}(\mu). \end{aligned}$$

(9) By (2) and (4), we have $(U \sqcap V)^{-1} \leq U^{-1} \sqcap V^{-1}$.
On the other hand, by (2),(3) and (4), $(U^{-1} \sqcap V^{-1})^{-1} \leq U \sqcap V$ which implies $U^{-1} \sqcap V^{-1} \leq (U \sqcap V)^{-1}$.

(10) Suppose that $(U \sqcap V) \sqcap W \not\leq U \sqcap (V \sqcap W)$. There exist $\mu \in I^X,\ x \in X,\ c \in (0,1)$ such that

$$((U \sqcap V) \sqcap W)(\mu)(x) > c > (U \sqcap (V \sqcap W))(\mu)(x).$$

By the definition of $U \sqcap (V \sqcap W)$, there exist $\mu_1, \mu_2 \in I^X, c_1 \in (0,1)$ with $\mu = \mu_1 \vee \mu_2$ such that

$$U(\mu_1)(x) \vee (V \sqcap W)(\mu_2)(x) < c_1 < c.$$

Again, by the definition of $V \sqcap W$, there exist $\mu_3, \mu_4 \in I^X$ with $\mu_2 = \mu_3 \vee \mu_4$ such that

$$U(\mu_1)(x) \vee V(\mu_3)(x) \vee W(\mu_4)(x) \leq c_1 < c.$$

On the other hand, since $\mu = \mu_1 \vee (\mu_3 \vee \mu_4) = (\mu_1 \vee \mu_3) \vee \mu_4$, we have

$$\begin{aligned} U(\mu_1) \vee V(\mu_3) \vee W(\mu_4) &\geq (U \sqcap V)(\mu_1 \vee \mu_3) \vee W(\mu_4) \\ &\geq ((U \sqcap V) \sqcap W)(\mu_1 \vee \mu_3 \vee \mu_4) \\ &= ((U \sqcap V) \sqcap W)(\mu). \end{aligned}$$

It is a contradiction. Hence $(U \sqcap V) \sqcap W \leq U \sqcap (V \sqcap W)$.

Similarly, we have $(U \sqcap V) \sqcap W \geq U \sqcap (V \sqcap W)$. □

In [3], *a classical fuzzy uniformity* on X is a subset $\mathcal{U}$ of Ω_X such that it satisfies the following conditions:

(U1) If $U_1, U_2 \in \mathcal{U}$, then $U_1 \sqcap U_2 \in \mathcal{U}$.
(U2) For $U \in \mathcal{U}$, there exists $U_1 \in \mathcal{U}$ such that $U_1 \circ U_1 \leq U$.
(U3) If $U_1 \geq U$ and $U \in \mathcal{U}$, then $U_1 \in \mathcal{U}$.
(U4) $\mathcal{U} \neq \emptyset$.
(U5) If $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$.

Here, each subset can be identified with its characteristic function. Accordingly, a classical uniformity $\mathcal{U}$ can be interpreted as a function from Ω_X into $\{0, 1\}$.

S.K. Samanta [6] introduced the concept of a gradation of uniformity to allow grades of uniformity to be any value in the unit interval $I = [0, 1]$ instead of $\{0, 1\}$.

From the definition of S.K. Samanta [6], we can define a fuzzy quasi-uniformity.

Definition 3.2. A function $\mathcal{U} : \Omega_X \to I$ is said to be a *fuzzy quasi-uniformity* on X if it satisfies the following conditions:

(FQU1) For $U_1, U_2 \in \Omega_X$, we have $\mathcal{U}(U_1 \sqcap U_2) \geq \mathcal{U}(U_1) \wedge \mathcal{U}(U_2)$.
(FQU2) For $U \in \Omega_X$, there exists $U_1 \in \Omega_X$ with $U_1 \circ U_1 \leq U$ such that $\mathcal{U}(U_1) \geq \mathcal{U}(U)$.
(FQU3) If $U_1 \geq U$, then $\mathcal{U}(U_1) \geq \mathcal{U}(U)$.
(FQU4) There exists $U \in \Omega_X$ such that $\mathcal{U}(U) = 1$.

The pair $(X, \mathcal{U})$ is said to be a *fuzzy quasi-uniform space*.

A fuzzy quasi-uniform space $(X, \mathcal{U})$ is called a *fuzzy uniform space* if the following is satisfied:

(FU) for $U \in \Omega_X$, there exists $U_1 \in \Omega_X$ with $U_1 \leq U^{-1}$ such that $\mathcal{U}(U_1) \geq \mathcal{U}(U)$.

Let $\mathcal{U}_1, \mathcal{U}_2$ be fuzzy quasi-uniformities on X. We say that $\mathcal{U}_1$ is *finer* than $\mathcal{U}_2$ (or $\mathcal{U}_2$ is *coarser* than $\mathcal{U}_1$), denoted by $\mathcal{U}_2 \leq \mathcal{U}_1$, iff for any $U \in \Omega_X$, $\mathcal{U}_2(U) \leq \mathcal{U}_1(U)$.

Remark 1. (1) Let $(X, \mathcal{U})$ be a fuzzy quasi-uniform space. By (FQU1), (FQU3) and Lemma 3.1 (2), we have $\mathcal{U}(U_1 \sqcap U_2) = \mathcal{U}(U_1) \wedge \mathcal{U}(U_2)$.

(2) If $(X,\mathcal{U})$ is a fuzzy uniform space, then, by (FU), (FQU3) and Lemma 3.1 (3), we have $\mathcal{U}(U)=\mathcal{U}(U^{-1})$.

(3) Let $(X,\mathcal{U})$ be a fuzzy quasi-uniform space. Since $U \leq D$ for all $U \in \Omega_X$ by Lemma 3.1 (5), we have $\mathcal{U}(D) = 1$ by (FQU3) and (FQU4).

Theorem 3.3. [5] *Let $(X,\mathcal{U})$ be a fuzzy (quasi-)uniform space. Then, for each $r \in (0,1]$, the family $\mathcal{U}_r = \{U \in \Omega_X \mid \mathcal{U}(U) \geq r\}$ is a classical fuzzy (quasi-)uniform space on X.*

Theorem 3.4. *Let $(X,\mathcal{U})$ be a fuzzy quasi-uniform space. We define for $U \in \Omega_X$, $\mathcal{U}^{-1}(U)=\mathcal{U}(U^{-1})$. Then the structure $\mathcal{U}^{-1}$ is a fuzzy quasi-uniformity on X.*

Proof. (FQU1) For $U_1, U_2 \in \Omega_X$, we have

$$\begin{aligned}\mathcal{U}^{-1}(U_1 \sqcap U_2) &= \mathcal{U}((U_1 \sqcap U_2)^{-1})\\ &= \mathcal{U}(U_1^{-1} \sqcap U_2^{-1}) \qquad \text{(by Lemma 3.1(9))}\\ &\geq \mathcal{U}(U_1^{-1}) \wedge \mathcal{U}(U_2^{-1})\\ &= \mathcal{U}^{-1}(U_1) \wedge \mathcal{U}^{-1}(U_2).\end{aligned}$$

(FQU2) For $U \in \Omega_X$, there exists $U_1 \in \Omega_X$ with $U_1 \circ U_1 \leq U^{-1}$ such that $\mathcal{U}(U_1) \geq \mathcal{U}(U^{-1})$. By (3),(4) and (8) of Lemma 3.1, since $U_1 \circ U_1 \leq U^{-1}$ iff $U_1^{-1} \circ U_1^{-1} \leq U$, there exists $U_1^{-1} \in \Omega_X$ with $U_1^{-1} \circ U_1^{-1} \leq U$ such that $\mathcal{U}^{-1}(U_1^{-1}) = \mathcal{U}(U_1) \geq \mathcal{U}^{-1}(U)$.

(FQU3) If $U_1 \geq U$, then, by (4) of Lemma 3.1, $U_1^{-1} \geq U^{-1}$. Hence $\mathcal{U}^{-1}(U_1) \geq \mathcal{U}^{-1}(U)$.

(FQU4) There exists $U \in \Omega_X$ such that $\mathcal{U}(U) = \mathcal{U}^{-1}(U^{-1}) = 1$. □

We will define the coarsest fuzzy uniformity on X which is finer than $\mathcal{U}$ and $\mathcal{U}^{-1}$.

Theorem 3.5. *Let $\mathcal{U}$ and $\mathcal{U}^{-1}$ be fuzzy quasi-uniformities on X. We define, for $U \in \Omega_X$,*

$$\mathcal{U}^*(U) = sup\{\mathcal{U}(U_1) \wedge \mathcal{U}^{-1}(U_2) \mid U_1 \sqcap U_2 \leq U\}.$$

Then the structure $\mathcal{U}^*$ is the coarsest fuzzy uniformity on X which is finer than $\mathcal{U}$ and $\mathcal{U}^{-1}$.

Proof. First, we will show that the structure $\mathcal{U}^*$ is a fuzzy uniformity on X.

(FQU1) For any $U, V \in \Omega_X$, we will show that

$$\mathcal{U}^*(U \sqcap V) \geq \mathcal{U}^*(U) \wedge \mathcal{U}^*(V).$$

If $\mathcal{U}^*(U) = 0$ or $\mathcal{U}^*(V) = 0$, it is trivial.

If $\mathcal{U}^*(U) \neq 0$ and $\mathcal{U}^*(V) \neq 0$, for ϵ with $\mathcal{U}^*(U) \wedge \mathcal{U}^*(V) > \epsilon > 0$, there exist families $(U_i)_{i=1,2}, (V_j)_{j=1,2}$ such that

$$\mathcal{U}(U_1) \wedge \mathcal{U}^{-1}(U_2) \geq \mathcal{U}^*(U) - \epsilon, \;\; U_1 \sqcap U_2 \leq U,$$
$$\mathcal{U}(V_1) \wedge \mathcal{U}^{-1}(V_2) \geq \mathcal{U}^*(V) - \epsilon, \;\; V_1 \sqcap V_2 \leq V.$$

Since $(U_1 \sqcap U_2) \sqcap (V_1 \sqcap V_2) = (U_1 \sqcap V_1) \sqcap (U_2 \sqcap V_2) \leq U \sqcap V$ by Lemma 3.1 (10),

$$\mathcal{U}^*(U \sqcap V) \geq \mathcal{U}(U_1 \sqcap V_1) \wedge \mathcal{U}^{-1}(U_2 \sqcap V_2) \geq (\mathcal{U}^*(U) \wedge \mathcal{U}^*(V)) - \epsilon.$$

Since ϵ is arbitrary, this gives the desired result.

(FQU2) Let $U \in \Omega_X$ be given. If $\mathcal{U}^*(U) = 0$, then there exists the identity function $E \in \Omega_X$ with $E \circ E \leq U$ such that $\mathcal{U}^*(E) \geq 0$.

If $\mathcal{U}^*(U) \neq 0$, for ϵ with $\mathcal{U}^*(U) > \epsilon > 0$, there exist $U_1, U_2 \in \Omega_X$ such that

$$\mathcal{U}(U_1) \wedge \mathcal{U}^{-1}(U_2) \geq \mathcal{U}^*(U) - \epsilon, \;\; U_1 \sqcap U_2 \leq U.$$

Since $\mathcal{U}$ and $\mathcal{U}^{-1}$ are fuzzy quasi-uniformities, by (FQU2), there exist $V_1, V_2 \in \Omega_X$ such that

$$V_1 \circ V_1 \leq U_1, \;\; \mathcal{U}(V_1) \geq \mathcal{U}(U_1), \quad V_2 \circ V_2 \leq U_2, \;\; \mathcal{U}^{-1}(V_2) \geq \mathcal{U}^{-1}(U_2).$$

By Lemma 3.1 (2), we have $(V_1 \sqcap V_2) \circ (V_1 \sqcap V_2) \leq V_i \circ V_i$ for $i = 1, 2$. Let $V = V_1 \sqcap V_2$ be given. Then we have

$$V \circ V \leq (V_1 \circ V_1) \sqcap (V_2 \circ V_2) \leq U_1 \sqcap U_2 \leq U,$$

$$\mathcal{U}^*(V) \geq \mathcal{U}(V_1) \wedge \mathcal{U}^{-1}(V_2) \geq \mathcal{U}(U_1) \wedge \mathcal{U}^{-1}(U_2) \geq \mathcal{U}^*(U) - \epsilon.$$

(FQU3) By the definition of $\mathcal{U}^*$, it is trivial.

(FQU4) There exists $U \in \Omega_X$ such that $\mathcal{U}(U) = \mathcal{U}^{-1}(U^{-1}) = 1$. By the definition of $\mathcal{U}^*$, we have $\mathcal{U}^*(U \sqcap U^{-1}) = 1$.

(FU) For $U \in \Omega_X$, there exists $U^{-1} \in \Omega_X$ with $U^{-1} \leq U^{-1}$ such that

$$\begin{aligned}\mathcal{U}^*(U^{-1}) &= sup\{\mathcal{U}(U_1) \wedge \mathcal{U}^{-1}(U_2) \mid U_1 \sqcap U_2 \leq U^{-1}\} \\ &= sup\{\mathcal{U}(U_1) \wedge \mathcal{U}^{-1}(U_2) \mid U_1^{-1} \sqcap U_2^{-1} \leq U\} \\ &= sup\{\mathcal{U}^{-1}(U_1^{-1}) \wedge \mathcal{U}(U_2^{-1}) \mid U_1^{-1} \sqcap U_2^{-1} \leq U\} \\ &= \mathcal{U}^*(U).\end{aligned}$$

Second, we will prove that the structure $\mathcal{U}^*$ is finer than $\mathcal{U}$ and $\mathcal{U}^{-1}$. For $U \in \Omega_X$,

$$\begin{aligned}\mathcal{U}^*(U) &= sup\{\mathcal{U}(U_1) \wedge \mathcal{U}^{-1}(U_2) \mid U_1 \sqcap U_2 \leq U\} \\ &\geq \mathcal{U}(U) \wedge \mathcal{U}^{-1}(D) && (\text{ by Lemma 3.1(5)}) \\ &= \mathcal{U}(U). && (\text{ by Remark 1.(3)})\end{aligned}$$

Similarly, we have $\mathcal{U}^* \geq \mathcal{U}^{-1}$.

Finally, if $\mathcal{V} \geq \mathcal{U}$ and $\mathcal{V} \geq \mathcal{U}^{-1}$, we have, for $U \in \Omega_X$,

$$\begin{aligned}\mathcal{U}^*(U) &= sup\{\mathcal{U}(U_1) \wedge \mathcal{U}^{-1}(U_2) \mid U_1 \sqcap U_2 \leq U\} \\ &\leq sup\{\mathcal{V}(U_1) \wedge \mathcal{V}(U_2) \mid U_1 \sqcap U_2 \leq U\} \\ &\leq sup\{\mathcal{V}(U_1 \sqcap U_2) \mid U_1 \sqcap U_2 \leq U\} \\ &= \mathcal{V}(U).\end{aligned}$$

□

If $\mathcal{U}$ is a classical fuzzy uniformity on X, we define $i_{\mathcal{U}}(\mu) = sup\{\rho \mid U(\rho) \leq \mu \text{ for some } U \in \mathcal{U}\}$. Then $i_{\mathcal{U}}$ is an interior operator on I^X. We will expand it from the following theorem.

Theorem 3.6. *Let* $(X,\mathcal{U})$ *be a fuzzy quasi-uniform space. Define, for each* $r \in [0,1)$, $\lambda \in I^X$,

$$i_{\mathcal{U}}(\lambda, r) = \bigvee\{\mu \in I^X \mid U(\mu) \leq \lambda \text{ for some } U \text{ with } \mathcal{U}(U) > r\}.$$

Then it satisfies the followings:

(i) $i_{\mathcal{U}}(\tilde{0}, r) = \tilde{0}, \quad i_{\mathcal{U}}(\tilde{1}, r) = \tilde{1}$.

(ii) $i_{\mathcal{U}}(\lambda, r) \le \lambda$ *and if* $\lambda_1 \le \lambda_2$, *then* $i_{\mathcal{U}}(\lambda_1, r) \le i_{\mathcal{U}}(\lambda_2, r)$.

(iii) $i_{\mathcal{U}}(i_{\mathcal{U}}(\lambda, r), r) = i_{\mathcal{U}}(\lambda, r)$.

(iv) $i_{\mathcal{U}}(\lambda \wedge \mu, r) = i_{\mathcal{U}}(\lambda, r) \wedge i_{\mathcal{U}}(\mu, r)$.

(v) $i_{\mathcal{U}}(\lambda, r) \le i_{\mathcal{U}}(\lambda, r')$, *if* $r \ge r'$.

Proof. (i) Suppose that there exists $x \in X$ such that $i_{\mathcal{U}}(\tilde{0}, r)(x) > 0$. Then there exist $U \in \Omega_X, \mu \in I^X$ such that $\mathcal{U}(U) > r$ and $U(\mu) \le \tilde{0}$ with

$$i_{\mathcal{U}}(\tilde{0}, r)(x) \ge \mu(x) > 0.$$

Since $0 < \mu(x) \le U(\mu(x))$, it is a contradiction. Hence $i_{\mathcal{U}}(\tilde{0}, r) = \tilde{0}$.

For all $U \in \Omega_X$, since $U(\tilde{1}) \le \tilde{1}$, we have $i_{\mathcal{U}}(\tilde{1}, r) = \tilde{1}$ by (FQU4).

(ii) Suppose that $i_{\mathcal{U}}(\lambda, r) \not\le \lambda$. Then there exists $x \in X$ such that $i_{\mathcal{U}}(\lambda, r)(x) > \lambda(x)$. By the definition of $i_{\mathcal{U}}(\lambda, r)$, there exist $U \in \Omega_X, \mu \in I^X$ such that $\mathcal{U}(U) > r$ and $U(\mu) \le \lambda$ with

$$i_{\mathcal{U}}(\lambda, r)(x) \ge \mu(x) > \lambda(x).$$

This yields a contradiction since $\mu \le U(\mu) \le \lambda$.

If $\lambda_1 \le \lambda_2$, by the definition of $i_{\mathcal{U}}$, we have $i_{\mathcal{U}}(\lambda_1, r) \le i_{\mathcal{U}}(\lambda_2, r)$.

(iii) By (ii), we have $i_{\mathcal{U}}(i_{\mathcal{U}}(\lambda, r), r) \le i_{\mathcal{U}}(\lambda, r)$.

Suppose that $i_{\mathcal{U}}(i_{\mathcal{U}}(\lambda, r), r) \not\ge i_{\mathcal{U}}(\lambda, r)$. Then there exists $x \in X$ such that $i_{\mathcal{U}}(i_{\mathcal{U}}(\lambda, r), r)(x) < i_{\mathcal{U}}(\lambda, r)(x)$. By the definition of $i_{\mathcal{U}}(\lambda, r)$, there exist $\rho \in I^X$, $U \in \Omega_X$ such that $\mathcal{U}(U) > r$, $U(\rho) \le \lambda$ and

$$i_{\mathcal{U}}(i_{\mathcal{U}}(\lambda, r), r)(x) < \rho(x) \le i_{\mathcal{U}}(\lambda, r)(x).$$

On the other hand, for $U \in \Omega_X$, by (FQU2), there exists $U_1 \in \Omega_X$ with $U_1 \circ U_1 \le U$ such that $\mathcal{U}(U_1) \ge \mathcal{U}(U) > r$ and $U_1(U_1(\rho)) \le \lambda$. By the definition of $i_{\mathcal{U}}(\lambda, r)$, we have $U_1(\rho) \le i_{\mathcal{U}}(\lambda, r)$. By the definition of $i_{\mathcal{U}}(i_{\mathcal{U}}(\lambda, r), r)$, it follows that $i_{\mathcal{U}}(i_{\mathcal{U}}(\lambda, r), r) \ge \rho$. It is a contradiction.

(iv) By (ii), we have $i_{\mathcal{U}}(\lambda \wedge \mu, r) \le i_{\mathcal{U}}(\lambda, r) \wedge i_{\mathcal{U}}(\mu, r)$.

We must show that $i_{\mathcal{U}}(\lambda \wedge \mu, r) \ge i_{\mathcal{U}}(\lambda, r) \wedge i_{\mathcal{U}}(\mu, r)$. Suppose that there exists $x \in X$ such that $i_{\mathcal{U}}(\lambda \wedge \mu, r)(x) < i_{\mathcal{U}}(\lambda, r)(x) \wedge i_{\mathcal{U}}(\mu, r)(x)$.

Then there exist $\rho_1, \rho_2 \in I^X$ with

$$i_{\mathcal{U}}(\lambda \wedge \mu, r)(x) < \rho_1(x) \wedge \rho_2(x) \le i_{\mathcal{U}}(\lambda, r)(x) \wedge i_{\mathcal{U}}(\mu, r)(x)$$

such that $\mathcal{U}(U_1) > r,\ \ U_1(\rho_1) \le \lambda$ and $\mathcal{U}(U_2) > r,\ \ U_2(\rho_2) \le \mu$.

It follows that $\mathcal{U}(U_1 \sqcap U_2) > r$ and

$$\begin{aligned}(U_1 \sqcap U_2)(\rho_1 \wedge \rho_2) &\le (U_1 \sqcap U_2)(\rho_1) \wedge (U_1 \sqcap U_2)(\rho_2) \\ &\le U_1(\rho_1) \wedge U_2(\rho_2) \quad \text{(by Lemma 3.1 (2))} \\ &\le \lambda \wedge \mu.\end{aligned}$$

Hence $i_{\mathcal{U}}(\lambda \wedge \mu, r) \ge \rho_1 \wedge \rho_2$. It is a contradiction.

(v) It is easily proved from the definition of $i_{\mathcal{U}}$. □

Let $(X,\mathcal{U})$ be fuzzy a quasi-uniform space. Similarly, we can define $i_{\mathcal{U}^{-1}}(\lambda, r)$ and $i_{\mathcal{U}^*}(\lambda, r)$.

THEOREM 3.7. *Let $(X,\mathcal{U})$ be a fuzzy quasi-uniform space. The function $\mathcal{T}_{\mathcal{U}} : I^X \to I$ is defined by*

$$\mathcal{T}_{\mathcal{U}}(\lambda) = \bigvee\{r \in [0,1) \mid i_{\mathcal{U}}(\lambda, r) = \lambda\}, \quad \text{for } \lambda \in I^X.$$

Then $\mathcal{T}_{\mathcal{U}}$ is a gradation of openness on X.

Proof. (O1) It is easily proved from Theorem 3.6 (i).

(O2) For $\lambda_1, \lambda_2 \in I^X$, suppose that there exists $c \in (0,1)$ such that

$$\mathcal{T}_{\mathcal{U}}(\lambda_1 \wedge \lambda_2) \le c < \mathcal{T}_{\mathcal{U}}(\lambda_1) \wedge \mathcal{T}_{\mathcal{U}}(\lambda_2).$$

Then there exist $r_1, r_2 \in (0,1)$ such that $r_1, r_2 > c,\ \ i_{\mathcal{U}}(\lambda_1, r_1) = \lambda_1,\ \ i_{\mathcal{U}}(\lambda_2, r_2) = \lambda_2$.

If $r = r_1 \wedge r_2$, by Theorem 3.6 (v) and (iv), we have

$$\lambda_1 \wedge \lambda_2 \le i_{\mathcal{U}}(\lambda_1, r) \wedge i_{\mathcal{U}}(\lambda_2, r) = i_{\mathcal{U}}(\lambda_1 \wedge \lambda_2, r).$$

By Theorem 3.6 (ii) , it follows that $\lambda_1 \wedge \lambda_2 = i_{\mathcal{U}}(\lambda_1 \wedge \lambda_2, r)$. Therefore $\mathcal{T}_{\mathcal{U}}(\lambda_1 \wedge \lambda_2) \ge r$. It is a contradiction. Hence $\mathcal{T}_{\mathcal{U}}(\lambda_1 \wedge \lambda_2) \ge \mathcal{T}_{\mathcal{U}}(\lambda_1) \wedge \mathcal{T}_{\mathcal{U}}(\lambda_2)$.

(O3) Suppose that there exist $c \in (0,1),\ \lambda_i \in I^X$ such that

$$\mathcal{T}_{\mathcal{U}}(\bigvee_{i \in \Gamma} \lambda_i) \le c < \bigwedge_{i \in \Gamma} \mathcal{T}_{\mathcal{U}}(\lambda_i).$$

Then for all $i \in \Gamma$, there exist $r_i > c$ such that $i_{\mathcal{U}}(\lambda_i, r_i) = \lambda_i$.
If $r = \bigwedge_{i\in\Gamma} r_i$, we have

$$\begin{aligned}\bigvee_{i\in\Gamma} \lambda_i &= \bigvee_{i\in\Gamma} i_{\mathcal{U}}(\lambda_i, r_i) \\ &\leq \bigvee_{i\in\Gamma} i_{\mathcal{U}}(\lambda_i, r) \qquad \text{(by Theorem 3.6 (v))} \\ &\leq i_{\mathcal{U}}(\bigvee_{i\in\Gamma} \lambda_i, r). \qquad \text{(by Theorem 3.6 (ii))}\end{aligned}$$

By Theorem 3.6 (ii) , it follows that $\bigvee_{i\in\Gamma} \lambda_i = i_{\mathcal{U}}(\bigvee_{i\in\Gamma} \lambda_i, r)$. Therefore $\mathcal{T}_{\mathcal{U}}(\bigvee_{i\in\Gamma} \lambda_i) \geq r \geq c$. It is a contradiction. □

Similarly, since $(X,\mathcal{U}^{-1})$ be a fuzzy quasi-uniform space, $\mathcal{T}_{\mathcal{U}^{-1}}$ is a gradation of openness on X. The space $(X, \mathcal{T}_{\mathcal{U}}, \mathcal{T}_{\mathcal{U}^{-1}})$ is called a fuzzy bitopological space induced by $(X,\mathcal{U})$.

THEOREM 3.8. *Let $\mathcal{U}$ be a fuzzy quasi-uniformity on X. For each $r \in [0,1), \lambda \in I^X$, we define*

$$cl_{\mathcal{U}}(\lambda, r) = \bigwedge\{U^{-1}(\lambda) \mid \mathcal{U}(U) > r\}.$$

Then it satisfies the followings:

(i) $cl_{\mathcal{U}}(\tilde{0}, r) = \tilde{0},\ cl_{\mathcal{U}}(\tilde{1}, r) = \tilde{1}$.
(ii) $cl_{\mathcal{U}}(\lambda, r) \geq \lambda$ *and* $cl_{\mathcal{U}}(\lambda_1, r) \leq cl_{\mathcal{U}}(\lambda_2, r)$, *if* $\lambda_1 \leq \lambda_2$.
(iii) $cl_{\mathcal{U}}(cl_{\mathcal{U}}(\lambda, r), r) = cl_{\mathcal{U}}(\lambda, r)$.
(iv) $cl_{\mathcal{U}}(\lambda \vee \mu, r) = cl_{\mathcal{U}}(\lambda, r) \vee cl_{\mathcal{U}}(\mu, r)$.
(v) $cl_{\mathcal{U}}(\lambda, r) \leq cl_{\mathcal{U}}(\lambda, r')$, *if* $r \leq r'$, *where* $r, r' \in [0,1)$.

Proof. (i), (ii) and (v) are easily proved from the definition of $cl_{\mathcal{U}}$.
(iii) By (ii), we have $cl_{\mathcal{U}}(cl_{\mathcal{U}}(\lambda, r), r) \geq cl_{\mathcal{U}}(\lambda, r)$.
Suppose $cl_{\mathcal{U}}(cl_{\mathcal{U}}(\lambda, r), r) \not\leq cl_{\mathcal{U}}(\lambda, r)$. By the definition of $cl_{\mathcal{U}}(\lambda, r)$, Then there exist $U \in \Omega_X,\ x \in X$ such that $\mathcal{U}(U) > r$ and

$$cl_{\mathcal{U}}(cl_{\mathcal{U}}(\lambda, r), r)(x) > U^{-1}(\lambda)(x) \geq cl_{\mathcal{U}}(\lambda, r).$$

On the other hand, by (FQU2),for $U \in \Omega_X$, there exists $U_1 \in \Omega_X$ such that

$$U_1 \circ U_1 \leq U,\ \mathcal{U}(U_1) \geq \mathcal{U}(U) > r.$$

It follows that

$$cl_{\mathcal{U}}(cl_{\mathcal{U}}(\lambda, r), r) \leq U_1^{-1}(cl_{\mathcal{U}}(\lambda, r)) \leq U_1^{-1}(U_1^{-1}(\lambda)) \leq U^{-1}(\lambda).$$

It is a contradiction.

(iv) By (ii), we have $cl_{\mathcal{U}}(\lambda \vee \mu, r) \geq cl_{\mathcal{U}}(\lambda, r) \vee cl_{\mathcal{U}}(\mu, r)$.

Suppose that $cl_{\mathcal{U}}(\lambda \vee \mu, r) \not\leq cl_{\mathcal{U}}(\lambda, r) \vee cl_{\mathcal{U}}(\mu, r)$. Then there exist $x \in X,\ c \in I$ such that

$$cl_{\mathcal{U}}(\lambda \vee \mu, r)(x) > c > cl_{\mathcal{U}}(\lambda, r)(x) \vee cl_{\mathcal{U}}(\mu, r)(x).$$

By the definitions of $cl_{\mathcal{U}}(\lambda, r)$ and $cl_{\mathcal{U}}(\mu, r)$, there exist $U_1, U_2 \in \Omega_X$ such that

$$\mathcal{U}(U_1) > r,\ U_1^{-1}(\lambda)(x) < c \text{ and } \mathcal{U}(U_2) > r,\ U_2^{-1}(\mu)(x) < c.$$

From the definition of $U_1^{-1} \sqcap U_2^{-1}$, since $(U_1 \sqcap U_2)^{-1}(\lambda \vee \mu)(x) < c$ and $\mathcal{U}(U_1 \sqcap U_2) > r$, we have

$$cl_{\mathcal{U}}(\lambda \vee \mu, r)(x) \leq (U_1 \sqcap U_2)^{-1}(\lambda \vee \mu)(x) < c.$$

It is a contradiction. □

Lemma 3.9. *Let $(X, \mathcal{U})$ be a fuzzy quasi-uniform space. For each $\lambda \in I^X,\ r \in [0, 1)$, if we define*

$$\beta = \bigwedge\{\tilde{1} - \rho \mid U(\rho) \leq \tilde{1} - \lambda \text{ for some } U \text{ with } \mathcal{U}(U) > r\},$$

$$\gamma = \bigwedge\{U^{-1}(\lambda) \mid \mathcal{U}(U) > r\},$$

then $\beta = \gamma$.

Proof. Suppose that $\beta \not\leq \gamma$. Then there exist $x \in X, U \in \Omega_X$ such that

$$\mathcal{U}(U) > r, \quad \beta(x) > U^{-1}(\lambda)(x) \geq \gamma(x).$$

On the other hand, by Lemma 3.1(7), since $U(\tilde{1} - U^{-1}(\lambda)) \leq \tilde{1} - \lambda$, we have $\beta \leq U^{-1}(\lambda)$. It is a contradiction.

Suppose that $\beta \not\geq \gamma$. Then there exist $x \in X,\ \rho_1 \in I^X,\ U \in \Omega_X$ such that

$$\mathcal{U}(U) > r,\ U(\rho_1) \leq \tilde{1} - \lambda, \quad \beta(x) \leq \tilde{1} - \rho_1(x) < \gamma(x).$$

On the other hand, by Lemma 3.1(6), since $U(\rho_1) \leq \tilde{1} - \lambda$ iff $U^{-1}(\lambda) \leq \tilde{1} - \rho_1$, we have $\gamma \leq \tilde{1} - \rho_1$. It is a contradiction. □

In general topology, we have $\overline{A} = (int(A^c))^c$. In a sense, we will expand it from the following lemma.

Lemma 3.10. *Let $(X,\mathcal{U})$ be a fuzzy quasi-uniform space. For each $\lambda \in I^X,\ r \in [0,1)$, we have*

$$cl_{\mathcal{U}}(\lambda, r) = \tilde{1} - i_{\mathcal{U}}(\tilde{1} - \lambda, r).$$

Proof. Using Lemma 3.9, for each $\lambda \in I^X,\ r \in [0,1)$, we have

$$\begin{aligned} cl_{\mathcal{U}}(\lambda, r) &= \bigwedge\{U^{-1}(\lambda) \mid \mathcal{U}(U) > r\} \\ &= \bigwedge\{\tilde{1} - \rho \mid U(\rho) \leq \tilde{1} - \lambda \text{ for some } U \text{ with } \mathcal{U}(U) > r\} \\ &= \tilde{1} - \bigvee\{\rho \mid U(\rho) \leq \tilde{1} - \lambda \quad \text{for some } U \text{ with } \quad \mathcal{U}(U) > r\} \\ &= \tilde{1} - i_{\mathcal{U}}(\tilde{1} - \lambda, r). \end{aligned}$$

□

Theorem 3.11. *Let $(X,\mathcal{U})$ be a fuzzy quasi-uniform space. The function $\mathcal{F}_{\mathcal{U}} : I^X \to I$ defined by*

$$\mathcal{F}_{\mathcal{U}}(\lambda) = \bigvee\{r \in [0,1) \mid cl_{\mathcal{U}}(\lambda, r) = \lambda\}, \quad \lambda \in I^X.$$

Then $\mathcal{F}_{\mathcal{U}}$ is a gradation of closeness on X.

Proof. For each $\lambda \in I^X$, we have

$$\begin{aligned} \mathcal{F}_{\mathcal{U}}(\lambda) &= \bigvee\{r \in [0,1) \mid cl_{\mathcal{U}}(\lambda, r) = \lambda\} \\ &= \bigvee\{r \in [0,1) \mid i_{\mathcal{U}}(\tilde{1} - \lambda, r) = \tilde{1} - \lambda\} \quad \text{(by Lemma 3.10)} \\ &= \mathcal{T}_{\mathcal{U}}(\tilde{1} - \lambda). \end{aligned}$$

Hence $\mathcal{F}_{\mathcal{U}}$ is a gradation of closeness on X. □

If $\mathcal{U}$ is a classical fuzzy uniformity on X, we define $\mu\bar{\delta}\rho$ iff $U(\mu) \leq \tilde{1} - \rho$ for some $U \in \mathcal{U}$. Then δ is a classical fuzzy proximity on X.

We will expand it from the following theorem. Thus we obtain a fuzzy quasi-proximity space from a fuzzy quasi-uniform space.

THEOREM 3.12. *Let $(X,\mathcal{U})$ be a (quasi-)fuzzy uniform space. Define, for all $\mu, \rho \in I^X$,*

$$\delta_{\mathcal{U}}(\mu,\rho) = \begin{cases} 1 - \bigvee\{\mathcal{U}(U) \mid U(\mu) \leq \tilde{1} - \rho\} & \text{if } \Theta_{\mu,\rho} \neq \emptyset \\ 1 & \text{if } \Theta_{\mu,\rho} = \emptyset \end{cases}$$

where $\Theta_{\mu,\rho} = \{U \in \Omega_X \mid U(\mu) \leq \tilde{1} - \rho\}$. Then $(X, \delta_{\mathcal{U}})$ is a (quasi-)fuzzy proximity space.

Proof. (FQP1) Since $U(\tilde{0}) = \tilde{0}$, by (FQU4), we have $\delta_{\mathcal{U}}(\tilde{0}, \tilde{1}) = 0$.
Similarly, since $U(\tilde{1}) = \tilde{1}$, by (FQU4), we have $\delta_{\mathcal{U}}(\tilde{1}, \tilde{0}) = 0$.

(FQP2) (1) First, we will show that $\Theta_{\mu,\rho\vee\lambda} = \emptyset$ iff $\Theta_{\mu,\rho} = \emptyset$ or $\Theta_{\mu,\lambda} = \emptyset$.

$$\begin{aligned} \Theta_{\mu,\rho\vee\lambda} = \emptyset &\text{ iff for each } U \in \Omega_X,\ U(\mu) \not\leq \tilde{1} - (\rho \vee \lambda) \\ &\text{ iff for each } U \in \Omega_X,\ \exists x \in X \text{ such that} \\ &\quad U(\mu)(x) + (\rho \vee \lambda)(x) > 1 \\ &\text{ iff for each } U \in \Omega_X,\ \exists x \in X \text{ such that} \\ &\quad U(\mu)(x) + \rho(x) > 1 \text{ or } U(\mu)(x) + \lambda(x) > 1 \\ &\text{ iff } \Theta_{\mu,\rho} = \emptyset \text{ or } \Theta_{\mu,\lambda} = \emptyset. \end{aligned}$$

Thus $\delta_{\mathcal{U}}(\mu,\rho) \vee \delta_{\mathcal{U}}(\mu,\lambda) = \delta_{\mathcal{U}}(\mu, \rho \vee \lambda)$ if $\Theta_{\mu,\rho\vee\lambda} = \emptyset$.

Now, we will show that if $\Theta_{\mu\vee\rho,\lambda} \neq \emptyset$,

$$\delta_{\mathcal{U}}(\mu,\rho) \vee \delta_{\mathcal{U}}(\mu,\lambda) = \delta_{\mathcal{U}}(\mu, \rho \vee \lambda).$$

If $\rho_1 \leq \rho_2$, by the definition of $\delta_{\mathcal{U}}$, $\delta_{\mathcal{U}}(\mu,\rho_1) \leq \delta_{\mathcal{U}}(\mu,\rho_2)$. Therefore $\delta_{\mathcal{U}}(\mu,\rho) \vee \delta_{\mathcal{U}}(\mu,\lambda) \leq \delta_{\mathcal{U}}(\mu, \rho \vee \lambda)$.

Suppose that there exists $r \in (0,1)$ such that

$$\delta_{\mathcal{U}}(\mu,\rho) \vee \delta_{\mathcal{U}}(\mu,\lambda) < r < \delta_{\mathcal{U}}(\mu, \rho \vee \lambda).$$

Since $\delta_{\mathcal{U}}(\mu,\rho) < r$ and $\delta_{\mathcal{U}}(\mu,\lambda) < r$, by the definition of $\delta_{\mathcal{U}}$, there exist $U_1, U_2 \in \Omega_X$ such that

$$\mathcal{U}(U_1) > 1 - r,\ \ U_1(\mu) \leq \tilde{1} - \rho, \quad \mathcal{U}(U_2) > 1 - r,\ \ U_2(\mu) \leq \tilde{1} - \lambda.$$

It follows that $\mathcal{U}(U_1 \sqcap U_2) > 1 - r$ and

$$\begin{aligned}(U_1 \sqcap U_2)(\mu) &\leq U_1(\mu) \wedge U_2(\mu) \quad \text{(by Lemma 3.1(2))}\\ &\leq (\tilde{1} - \rho) \wedge (\tilde{1} - \lambda)\\ &= \tilde{1} - (\rho \vee \lambda).\end{aligned}$$

Hence we have $\delta_{\mathcal{U}}(\mu, \rho \vee \lambda) < r$. It is contradiction.

Therefore $\delta_{\mathcal{U}}(\mu, \rho) \vee \delta_{\mathcal{U}}(\mu, \lambda) \geq \delta_{\mathcal{U}}(\mu, \rho \vee \lambda)$.

(2) We will show that $\Theta_{\mu\vee\rho,\lambda} = \emptyset$ iff $\Theta_{\mu,\lambda} = \emptyset$ or $\Theta_{\rho,\lambda} = \emptyset$.

$$\begin{aligned}\Theta_{\mu\vee\rho,\lambda} = \emptyset &\text{ iff for each } U \in \Omega_X,\ U(\mu \vee \rho) \not\leq \tilde{1} - \lambda\\ &\text{ iff for each } U \in \Omega_X,\ \exists x \in X \text{ such that}\\ &\quad U(\mu \vee \rho)(x) + \lambda(x) > 1\\ &\text{ iff for each } U \in \Omega_X,\ \exists x \in X \text{ such that}\\ &\quad U(\mu)(x) + \lambda(x) > 1 \text{ or } U(\rho)(x) + \lambda(x) > 1\\ &\text{ iff } \Theta_{\mu,\lambda} = \emptyset \text{ or } \Theta_{\rho,\lambda} = \emptyset.\end{aligned}$$

Thus $\delta_{\mathcal{U}}(\mu, \lambda) \vee \delta_{\mathcal{U}}(\rho, \lambda) = \delta_{\mathcal{U}}(\mu \vee \rho, \lambda)$ if $\Theta_{\mu\vee\rho,\lambda} = \emptyset$.

Now, we will show that if $\Theta_{\mu\vee\rho,\lambda} \neq \emptyset$,

$$\delta_{\mathcal{U}}(\mu, \lambda) \vee \delta_{\mathcal{U}}(\rho, \lambda) = \delta_{\mathcal{U}}(\mu \vee \rho, \lambda).$$

Since $\mathcal{U}(\mu) \leq \mathcal{U}(\mu \vee \rho)$, by the definition of $\delta_{\mathcal{U}}$ we have

$$\delta_{\mathcal{U}}(\mu, \lambda) \leq \delta_{\mathcal{U}}(\mu \vee \rho, \lambda).$$

Similarly, we have

$$\delta_{\mathcal{U}}(\rho, \lambda) \leq \delta_{\mathcal{U}}(\mu \vee \rho, \lambda).$$

Hence

$$\delta_{\mathcal{U}}(\mu, \lambda) \vee \delta_{\mathcal{U}}(\rho, \lambda) \leq \delta_{\mathcal{U}}(\mu \vee \rho, \lambda).$$

Suppose that there exists $r \in (0, 1)$ such that

$$\delta_{\mathcal{U}}(\mu, \lambda) \vee \delta_{\mathcal{U}}(\rho, \lambda) < r < \delta_{\mathcal{U}}(\mu \vee \rho, \lambda).$$

By the definition of $\delta_{\mathcal{U}}$, there exist $U_1, U_2 \in \Omega_X$ such that

$$\mathcal{U}(U_1) > 1 - r,\ \ U_1(\mu) \leq \tilde{1} - \lambda, \quad \mathcal{U}(U_2) > 1 - r,\ \ U_2(\rho) \leq \tilde{1} - \lambda.$$

It follows that $\mathcal{U}(U_1 \sqcap U_2) > 1 - r$ and

$$(U_1 \sqcap U_2)(\mu \vee \rho) \leq U_1(\mu) \vee U_2(\rho) \quad \text{(by the definition of } U_1 \sqcap U_2\text{)}$$
$$\leq \tilde{1} - \lambda.$$

Hence we have $\delta_{\mathcal{U}}(\mu \vee \rho, \lambda) < r$. It is contradiction.

Therefore $\delta_{\mathcal{U}}(\mu, \lambda) \vee \delta_{\mathcal{U}}(\rho, \lambda) \geq \delta_{\mathcal{U}}(\mu \vee \rho, \lambda)$.

(FQP3) If $\delta_{\mathcal{U}}(\mu, \rho) < r$ for $r \in (0, 1]$, then for some $U \in \Omega_X$, $\mathcal{U}(U) > 1 - r$ and $U(\mu) \leq \tilde{1} - \rho$. By (FQU2), there exists V such that $V \circ V \leq U$ and $\mathcal{U}(V) \geq \mathcal{U}(U) > 1 - r$.

Since $V(\mu) \leq V(\mu)$ and $V \circ V(\mu) \leq \tilde{1} - \rho$, there exists $\tilde{1} - V(\mu) \in I^X$ such that $\delta_{\mathcal{U}}(\mu, \tilde{1} - V(\mu)) < r$ and $\delta_{\mathcal{U}}(V(\mu), \rho) < r$.

(FQP4) Let $\mu \not\leq \tilde{1} - \rho$ be given. Since $\mu \leq U(\mu)$, for all $U \in \Omega_X$, we have $U(\mu) \not\leq \tilde{1} - \rho$. By the definition of $\delta_{\mathcal{U}}$, we have $\delta_{\mathcal{U}}(\mu, \rho) = 1$.

(FP) Since $\mathcal{U}(U) = \mathcal{U}(U^{-1})$ and $U(\mu) \leq \tilde{1} - \rho$ iff $U^{-1}(\rho) \leq \tilde{1} - \mu$, we have $\delta_{\mathcal{U}}(\mu, \rho) = \delta_{\mathcal{U}}(\rho, \mu)$. □

THEOREM 3.13. *Let* $(X, \mathcal{U})$ *be a fuzzy quasi-uniform space. Then:*

(1) $\mathcal{F}_{\mathcal{U}} = \mathcal{F}_{\delta_{\mathcal{U}}}$, $\mathcal{F}_{\mathcal{U}^{-1}} = \mathcal{F}_{\delta_{\mathcal{U}^{-1}}}$, $\mathcal{F}_{\mathcal{U}^*} = \mathcal{F}_{\delta_{\mathcal{U}^*}}$.

(2) $(\delta_{\mathcal{U}})^{-1} = \delta_{\mathcal{U}^{-1}}$.

Proof. (1) From Theorem 2.6 and Theorem 3.11, it suffices to show that $cl_{\mathcal{U}}(\lambda, r) = cl_{\delta_{\mathcal{U}}}(\lambda, r)$, for all $\lambda \in I^X, r \in [0, 1)$. By Theorem 3.8 , Lemma 3.9 and Theorem 2.5, we have

$$cl_{\mathcal{U}}(\lambda, r) = \bigwedge\{U^{-1}(\lambda) \mid \mathcal{U}(U) > r\}$$
$$= \bigwedge\{\tilde{1} - \rho \mid U(\rho) \leq \tilde{1} - \lambda \text{ for some } U \text{ with } \mathcal{U}(U) > r\}$$
$$cl_{\delta_{\mathcal{U}}}(\lambda, r) = \bigwedge\{\tilde{1} - \rho \mid \delta_{\mathcal{U}}(\rho, \lambda) < 1 - r\}.$$

It is proved that $cl_{\mathcal{U}}(\rho, r) = cl_{\delta_{\mathcal{U}}}(\rho, r)$ from the following:

$$\delta_{\mathcal{U}}(\rho, \lambda) < 1 - r \iff 1 - \bigvee\{\mathcal{U}(U) \mid U(\rho) \leq \tilde{1} - \lambda\} < 1 - r$$
$$\iff U(\rho) \leq \tilde{1} - \lambda \text{ for some } U \text{ with } \mathcal{U}(U) > r.$$

Hence $\mathcal{F}_{\mathcal{U}} = \mathcal{F}_{\delta_{\mathcal{U}}}$.

Similarly, we have $\mathcal{F}_{\mathcal{U}^{-1}} = \mathcal{F}_{\delta_{\mathcal{U}^{-1}}}, \ \mathcal{F}_{\mathcal{U}^*} = \mathcal{F}_{\delta_{\mathcal{U}^*}}$.

(2) From the definition of $(\delta_{\mathcal{U}})^{-1}$, we must show that

$$\delta_{\mathcal{U}}(\rho,\mu) = \delta_{\mathcal{U}^{-1}}(\mu,\rho).$$

First, if $\Theta_{\rho,\mu} = \emptyset$, we will show that $\Theta_{\rho,\mu} = \emptyset$ iff $\Theta_{\mu,\rho} = \emptyset$.

$$\begin{aligned}\Theta_{\rho,\mu} = \emptyset \text{ iff for } & U \in \Omega_X,\ U(\rho) \not\leq \tilde{1} - \mu \\ \text{iff for } & U^{-1} \in \Omega_X,\ U^{-1}(\mu) \not\leq \tilde{1} - \rho \quad \text{(by Lemma 3.1 (6))} \\ \text{iff } & \Theta_{\mu,\rho} = \emptyset\ .\end{aligned}$$

Second, if $\Theta_{\rho,\mu} \neq \emptyset$, we have

$$\begin{aligned}\delta_{\mathcal{U}}(\rho,\mu) &= 1 - \bigvee\{\mathcal{U}(U) \mid U(\rho) \leq \tilde{1} - \mu\} \\ &= 1 - \bigvee\{\mathcal{U}^{-1}(U^{-1}) \mid U^{-1}(\mu) \leq \tilde{1} - \rho\} \text{ (by Lemma 3.1 (3,6))} \\ &= \delta_{\mathcal{U}^{-1}}(\mu,\rho).\end{aligned}$$

□

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