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# FUZZY $r$-SEMICONTINUOUS, $r$-SEMIOPEN AND $r$-SEMICLOSED MAPS

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Abstract. In this paper, we investigate some conditions which are equivalent to fuzzy $r$-homeomorphisms and give some characterizing theorems for fuzzy $r$-semicontinuous, $r$-semiopen and $r$-semiclosed maps.

## 1. Introduction

Chang [2] introduced fuzzy topological spaces and some authors [4, 5, 7] introduced new definitions of fuzzy topology as a generalization of Chang's fuzzy topology. We introduced fuzzy $r$-semiopen sets and fuzzy $r$-semicontinuous maps which are generalization of fuzzy semiopen sets and fuzzy semicontinuous maps in Chang's fuzzy topology, respectively [6]. In this paper, we investigate some conditions which are equivalent to fuzzy $r$-homeomorphisms and give some characterizing theorems for fuzzy $r$-semicontinuous, $r$-semiopen and $r$-semiclosed maps.

## 2. Preliminaries

In this paper, $I$ denotes the unit interval $[0, 1]$ of the real line and $I_0 = (0, 1]$. A member $\mu$ of $I^X$ is called a *fuzzy set* of $X$. For any $\mu \in I^X$, $\mu^c$ denotes the complement $1-\mu$. By $\tilde{0}$ and $\tilde{1}$ we denote constant maps on $X$ with value 0 and 1, respectively. All other notations are standard notations of fuzzy set theory.

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Definition 2.1. [3, 6] Let $(X, \mathcal{T})$ be a fuzzy topological space. For each $r \in I_0$ and for each $\mu \in I^X$, the *fuzzy $r$-closure* is defined by

$$\mathrm{cl}(\mu, r) = \bigwedge\{\rho \in I^X \mid \mu \leq \rho, \mathcal{F}_{\mathcal{T}}(\rho) \geq r\}$$

and the *fuzzy $r$-interior* is defined by

$$\mathrm{int}(\mu, r) = \bigvee\{\rho \in I^X \mid \mu \geq \rho, \mathcal{T}(\rho) \geq r\}.$$

¿From now on, for $r \in I_0$ we will call $\mu$ a *fuzzy $r$-open set* of $X$ if $\mathcal{T}(\mu) \geq r$, $\mu$ a *fuzzy $r$-closed set* of $X$ if $\mathcal{F}(\mu) \geq r$. Note that $\mu$ is fuzzy $r$-closed if and only if $\mu = \mathrm{cl}(\mu, r)$ and $\mu$ is fuzzy $r$-open if and only if $\mu = \mathrm{int}(\mu, r)$.

Definition 2.2. [6] Let $\mu$ be a fuzzy set of a fuzzy topological space $(X, \mathcal{T})$ and $r \in I_0$. Then $\mu$ is said to be

(1) *fuzzy $r$-semiopen* if there is a fuzzy $r$-open set $\rho$ in $X$ such that $\rho \leq \mu \leq \mathrm{cl}(\rho, r)$,
(2) *fuzzy $r$-semiclosed* if there is a fuzzy $r$-closed set $\rho$ in $X$ such that $\mathrm{int}(\rho, r) \leq \mu \leq \rho$.

Definition 2.3. [6] Let $(X, \mathcal{T})$ be a fuzzy topological space. For each $r \in I_0$ and for each $\mu \in I^X$, the *fuzzy $r$-semiclosure* is defined by

$$\mathrm{scl}(\mu, r) = \bigwedge\{\rho \in I^X \mid \mu \leq \rho,\ \rho \text{ is fuzzy } r\text{-semiclosed}\}$$

and the *fuzzy $r$-semiinterior* is defined by

$$\mathrm{sint}(\mu, r) = \bigvee\{\rho \in I^X \mid \mu \geq \rho,\ \rho \text{ is fuzzy } r\text{-semiopen}\}.$$

Obviously $\mathrm{scl}(\mu, r)$ is the smallest fuzzy $r$-semiclosed set which contains $\mu$ and $\mathrm{sint}(\mu, r)$ is the greatest fuzzy $r$-semiopen set which contained in $\mu$. Also, $\mathrm{scl}(\mu, r) = \mu$ for any fuzzy $r$-semiclosed set $\mu$ and $\mathrm{sint}(\mu, r) = \mu$ for any fuzzy $r$-semiopen set $\mu$. Moreover, we have

$$\mathrm{int}(\mu, r) \leq \mathrm{sint}(\mu, r) \leq \mu \leq \mathrm{scl}(\mu, r) \leq \mathrm{cl}(\mu, r).$$

It is obvious that any fuzzy $r$-open ($r$-closed) set is fuzzy $r$-semiopen ($r$-semiclosed). But the converse need not be true. The intersection (union) of any two fuzzy $r$-semiopen ($r$-semiclosed) sets need not be fuzzy $r$-semiopen ($r$-semiclosed).

## 3. Fuzzy $r$-semicontinuous maps

DEFINITION 3.1. [6] Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map from a fuzzy topological space $X$ to another fuzzy topological space $Y$ and $r \in I_0$. Then $f$ is called

(1) a *fuzzy $r$-continuous* map if $f^{-1}(\mu)$ is a fuzzy $r$-open set of $X$ for each fuzzy $r$-open set $\mu$ of $Y$, or equivalently, $f^{-1}(\mu)$ is a fuzzy $r$-closed set of $X$ for each fuzzy $r$-closed set $\mu$ of $Y$,
(2) a *fuzzy $r$-open* map if $f(\mu)$ is a fuzzy $r$-open set of $Y$ for each fuzzy $r$-open set $\mu$ of $X$,
(3) a *fuzzy $r$-closed* map if $f(\mu)$ is a fuzzy $r$-closed set of $Y$ for each fuzzy $r$-closed set $\mu$ of $X$,
(4) a *fuzzy $r$-homeomorphism* if $f$ is bijective, fuzzy $r$-continuous and fuzzy $r$-open.

DEFINITION 3.2. [6] Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map from a fuzzy topological space $X$ to another fuzzy topological space $Y$ and $r \in I_0$. Then $f$ is called

(1) a *fuzzy $r$-semicontinuous* map if $f^{-1}(\mu)$ is a fuzzy $r$-semiopen set of $X$ for each fuzzy $r$-open set $\mu$ of $Y$, or equivalently, $f^{-1}(\mu)$ is a fuzzy $r$-semiclosed set of $X$ for each fuzzy $r$-closed set $\mu$ of $Y$,
(2) a *fuzzy $r$-semiopen* map if $f(\mu)$ is a fuzzy $r$-semiopen set of $Y$ for each fuzzy $r$-open set $\mu$ of $X$,
(3) a *fuzzy $r$-semiclosed* map if $f(\mu)$ is a fuzzy $r$-semiclosed set of $Y$ for each fuzzy $r$-closed set $\mu$ of $X$.

It is clear that every fuzzy $r$-continuous($r$-open, $r$-closed) map is a fuzzy $r$-semicontinuous($r$-semiopen, $r$-semiclosed) map for each $r \in I_0$. However the converse need not be true.

THEOREM 3.3. [6] *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map and $r \in I_0$. Then $f$ is fuzzy $r$-continuous if and only if $f(\mathrm{cl}(\rho, r)) \le \mathrm{cl}(f(\rho), r)$ for each fuzzy set $\rho$ of $X$.*

THEOREM 3.4. *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map and $r \in I_0$. Then $f$ is fuzzy $r$-closed if and only if $f(\mathrm{cl}(\rho, r)) \ge \mathrm{cl}(f(\rho), r)$ for each fuzzy set $\rho$ of $X$.*

*Proof.* Let $f$ be a fuzzy $r$-closed map and $\rho$ any fuzzy set of $X$. Note $\mathrm{cl}(\rho, r)$ is a fuzzy $r$-closed set of $X$. Since $f$ is fuzzy $r$-closed, $f(\mathrm{cl}(\rho, r))$ is a fuzzy $r$-closed set of $Y$. Thus

$$f(\mathrm{cl}(\rho, r)) = \mathrm{cl}(f(\mathrm{cl}(\rho, r)), r) \geq \mathrm{cl}(f(\rho), r).$$

Conversely, let $\rho$ be fuzzy $r$-closed in $X$. Then $\mathrm{cl}(\rho, r) = \rho$. By hypothesis,

$$\mathrm{cl}(f(\rho), r) \leq f(\mathrm{cl}(\rho, r)) = f(\rho) \leq \mathrm{cl}(f(\rho), r).$$

Thus $\mathrm{cl}(f(\rho), r) = f(\rho)$ and hence $f(\rho)$ is fuzzy $r$-closed in $Y$. Therefore $f$ is fuzzy $r$-closed. □

¿From Theorem 3.3 and Theorem 3.4 we have the following result.

THEOREM 3.5. *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a bijection and $r \in I_0$. Then the following statements are equivalent :*

(1) *$f$ is a fuzzy $r$-homeomorphism.*
(2) *$f$ is fuzzy $r$-continuous and fuzzy $r$-closed.*
(3) *$f(\mathrm{cl}(\rho, r)) = \mathrm{cl}(f(\rho), r)$ for each fuzzy set $\rho$ of $X$.*

The notion of fuzzy $r$-semicontinuity can be restated in terms of fuzzy $r$-closure and fuzzy $r$-interior.

THEOREM 3.6. *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map and $r \in I_0$. Then the following statements are equivalent :*

(1) *$f$ is a fuzzy $r$-semicontinuous map.*
(2) *$\mathrm{int}(\mathrm{cl}(f^{-1}(\mu), r), r) \leq f^{-1}(\mathrm{cl}(\mu, r))$ for each fuzzy set $\mu$ of $Y$.*
(3) *$f(\mathrm{int}(\mathrm{cl}(\rho, r), r)) \leq \mathrm{cl}(f(\rho), r)$ for each fuzzy set $\rho$ of $X$.*

*Proof.* (1) ⇒ (2) Let $f$ be a fuzzy $r$-semicontinuous map and $\mu$ any fuzzy set of $Y$. Then $\mathrm{cl}(\mu, r)$ is a fuzzy $r$-closed set of $Y$. Since $f$ is fuzzy $r$-semicontinuous, $f^{-1}(\mathrm{cl}(\mu, r))$ is a fuzzy $r$-semiclosed set of $X$. Thus

$$f^{-1}(\mathrm{cl}(\mu, r)) \geq \mathrm{int}(\mathrm{cl}(f^{-1}(\mathrm{cl}(\mu, r)), r), r) \geq \mathrm{int}(\mathrm{cl}(f^{-1}(\mu), r), r).$$

(2) ⇒ (3) Let $\rho$ be a fuzzy set of $X$. Then $f(\rho)$ is a fuzzy set of $Y$. By (2),

$$f^{-1}(\mathrm{cl}(f(\rho), r)) \geq \mathrm{int}(\mathrm{cl}(f^{-1}f(\rho), r), r) \geq \mathrm{int}(\mathrm{cl}(\rho, r), r).$$

Hence

$$\mathrm{cl}(f(\rho), r) \geq ff^{-1}(\mathrm{cl}(f(\rho), r)) \geq f(\mathrm{int}(\mathrm{cl}(\rho, r), r)).$$

(3) ⇒ (1) Let $\mu$ be a fuzzy $r$-closed set of $Y$. Then $f^{-1}(\mu)$ is a fuzzy set of $X$. By (3),

$$f(\mathrm{int}(\mathrm{cl}(f^{-1}(\mu), r), r)) \leq \mathrm{cl}(ff^{-1}(\mu), r) \leq \mathrm{cl}(\mu, r) = \mu.$$

So

$$\mathrm{int}(\mathrm{cl}(f^{-1}(\mu), r), r) \leq f^{-1}f(\mathrm{int}(\mathrm{cl}(f^{-1}(\mu), r), r)) \leq f^{-1}(\mu).$$

Thus $f^{-1}(\mu)$ is a fuzzy $r$-semiclosed set of $X$ and hence $f$ is a fuzzy $r$-semicontinuous map. □

We already knew the following theorem.

THEOREM 3.7. [6] *Let* $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ *be a map and* $r \in I_0$. *Then the following statements are equivalent :*

(1) $f$ *is a fuzzy* $r$*-semicontinuous map.*
(2) $f(\mathrm{scl}(\rho, r)) \leq \mathrm{cl}(f(\rho), r)$ *for each fuzzy set* $\rho$ *of* $X$.
(3) $\mathrm{scl}(f^{-1}(\mu), r) \leq f^{-1}(\mathrm{cl}(\mu, r))$ *for each fuzzy set* $\mu$ *of* $Y$.
(4) $f^{-1}(\mathrm{int}(\mu, r)) \leq \mathrm{sint}(f^{-1}(\mu), r)$ *for each fuzzy set* $\mu$ *of* $Y$.

If the map $f$ is a bijection, we have:

THEOREM 3.8. *Let* $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ *be a bijection and* $r \in I_0$. *Then* $f$ *is a fuzzy* $r$*-semicontinuous map if and only if* $\mathrm{int}(f(\rho), r) \leq f(\mathrm{sint}(\rho, r))$ *for each fuzzy set* $\rho$ *of* $X$.

*Proof.* Let $f$ be a fuzzy $r$-semicontinuous map and $\rho$ any fuzzy set of $X$. Since $\text{int}(f(\rho), r)$ is fuzzy $r$-open in $Y$, $f^{-1}(\text{int}(f(\rho), r))$ is fuzzy $r$-semiopen in $X$. Since $f$ is one-to-one, we have

$$f^{-1}(\text{int}(f(\rho), r)) \leq \text{sint}(f^{-1}f(\rho), r) = \text{sint}(\rho, r).$$

Since $f$ is onto,

$$\text{int}(f(\rho), r) = ff^{-1}(\text{int}(f(\rho), r)) \leq f(\text{sint}(\rho, r)).$$

Conversely, let $\mu$ be fuzzy $r$-open of $Y$. Then $\text{int}(\mu, r) = \mu$. Since $f$ is onto,

$$f(\text{sint}(f^{-1}(\mu), r)) \geq \text{int}(ff^{-1}(\mu), r) = \text{int}(\mu, r) = \mu.$$

Since $f$ is one-to-one, we have

$$f^{-1}(\mu) \leq f^{-1}f(\text{sint}(f^{-1}(\mu), r)) = \text{sint}(f^{-1}(\mu), r) \leq f^{-1}(\mu).$$

Thus $f^{-1}(\mu) = \text{sint}(f^{-1}(\mu), r)$ and hence $f^{-1}(\mu)$ is fuzzy $r$-semiopen in $X$. Therefore $f$ is fuzzy $r$-semicontinuous. □

THEOREM 3.9. *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map and $r \in I_0$. Then the following statements are equivalent :*

(1) *$f$ is a fuzzy $r$-semiopen map.*
(2) *$f(\text{int}(\rho, r)) \leq \text{sint}(f(\rho), r)$ for each fuzzy set $\rho$ of $X$.*
(3) *$\text{int}(f^{-1}(\mu), r) \leq f^{-1}(\text{sint}(\mu, r))$ for each fuzzy set $\mu$ of $Y$.*

*Proof.* (1) ⇒ (2) Let $\rho$ be any fuzzy set of $X$. Clearly $\text{int}(\rho, r)$ is fuzzy $r$-open in $X$. Since $f$ is fuzzy $r$-semiopen, $f(\text{int}(\rho, r))$ is fuzzy $r$-semiopen in $Y$. Thus

$$f(\text{int}(\rho, r)) = \text{sint}(f(\text{int}(\rho, r)), r) \leq \text{sint}(f(\rho), r).$$

(2) ⇒ (3) Let $\mu$ be any fuzzy set of $Y$. Then $f^{-1}(\mu)$ is a fuzzy set of $X$. By (2),

$$f(\text{int}(f^{-1}(\mu), r)) \leq \text{sint}(ff^{-1}(\mu), r) \leq \text{sint}(\mu, r).$$

Thus we have

$$\mathrm{int}(f^{-1}(\mu), r) \leq f^{-1}f(\mathrm{int}(f^{-1}(\mu), r)) \leq f^{-1}(\mathrm{sint}(\mu, r)).$$

(3) $\Rightarrow$ (1) Let $\rho$ be any fuzzy $r$-open set of $X$. Then $\mathrm{int}(\rho, r) = \rho$ and $f(\rho)$ is a fuzzy set of $Y$. By (3),

$$\rho = \mathrm{int}(\rho, r) \leq \mathrm{int}(f^{-1}f(\rho), r) \leq f^{-1}(\mathrm{sint}(f(\rho), r)).$$

Hence we have

$$f(\rho) \leq ff^{-1}(\mathrm{sint}(f(\rho), r)) \leq \mathrm{sint}(f(\rho), r) \leq f(\rho).$$

Thus $f(\rho) = \mathrm{sint}(f(\rho), r)$ and hence $f(\rho)$ is fuzzy $r$-semiopen in $Y$. Therefore $f$ is fuzzy $r$-semiopen. □

THEOREM 3.10. *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map and $r \in I_0$. Then the following statements are equivalent :*

(1) *$f$ is a fuzzy $r$-semiclosed map.*
(2) $\mathrm{scl}(f(\rho), r) \leq f(\mathrm{cl}(\rho, r))$ *for each fuzzy set $\rho$ of $X$.*

*Proof.* (1) $\Rightarrow$ (2) Let $\rho$ be any fuzzy set of $X$. Clearly $\mathrm{cl}(\rho, r)$ is fuzzy $r$-closed in $X$. Since $f$ is fuzzy $r$-semiclosed, $f(\mathrm{cl}(\rho, r))$ is fuzzy $r$-semiclosed in $Y$. Thus we have

$$\mathrm{scl}(f(\rho), r) \leq \mathrm{scl}(f(\mathrm{cl}(\rho, r)), r) = f(\mathrm{cl}(\rho, r)).$$

(2) $\Rightarrow$ (1) Let $\rho$ be any fuzzy $r$-closed of $X$. Then $\mathrm{cl}(\rho, r) = \rho$. By (2),

$$\mathrm{scl}(f(\rho), r) \leq f(\mathrm{cl}(\rho, r)) = f(\rho) \leq \mathrm{scl}(f(\rho), r).$$

Thus $f(\rho) = \mathrm{scl}(f(\rho), r)$ and hence $f(\rho)$ is fuzzy $r$-semiclosed in $Y$. Therefore $f$ is fuzzy $r$-semiclosed. □

THEOREM 3.11. *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a bijection and $r \in I_0$. Then $f$ is a fuzzy $r$-semiclosed map if and only if $f^{-1}(\mathrm{scl}(\mu, r)) \leq \mathrm{cl}(f^{-1}(\mu), r)$ for each fuzzy set $\mu$ of $Y$.*

*Proof.* Let $f$ be a fuzzy $r$-semiclosed map and $\mu$ any fuzzy set of $Y$. Then $f^{-1}(\mu)$ is a fuzzy set of $X$. Since $f$ is onto, we have

$$\mathrm{scl}(\mu, r) = \mathrm{scl}(ff^{-1}(\mu), r) \le f(\mathrm{cl}(f^{-1}(\mu), r)).$$

Since $f$ is one-to-one, we have

$$f^{-1}(\mathrm{scl}(\mu, r)) \le f^{-1}f(\mathrm{cl}(f^{-1}(\mu), r)) = \mathrm{cl}(f^{-1}(\mu), r).$$

Conversely, let $\rho$ be fuzzy $r$- closed of $X$. Then $\mathrm{cl}(\rho, r) = \rho$. Since $f$ is one-to-one,

$$f^{-1}(\mathrm{scl}(f(\rho), r)) \le \mathrm{cl}(f^{-1}f(\rho), r) = \mathrm{cl}(\rho, r) = \rho.$$

Since $f$ is onto, we have

$$\mathrm{scl}(f(\rho), r) = ff^{-1}(\mathrm{scl}(f(\rho), r)) \le f(\rho) \le \mathrm{scl}(f(\rho), r).$$

Thus $f(\rho) = \mathrm{scl}(f(\rho), r)$ and hence $f(\rho)$ is fuzzy $r$-semiclosed in $Y$. Therefore $f$ is fuzzy $r$-semiclosed. □

THEOREM 3.12. *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ and $g : (Y, \mathcal{U}) \to (Z, \mathcal{V})$ be maps and $r \in I_0$. Then the following statements are true.*

(1) *If $f$ is fuzzy $r$-semicontinuous and $g$ is fuzzy $r$-continuous then $g \circ f$ is fuzzy $r$-semicontinuous.*
(2) *If $f$ is fuzzy $r$-open and $g$ is fuzzy $r$-semiopen then $g \circ f$ is fuzzy $r$-semiopen.*
(3) *If $f$ is fuzzy $r$-closed and $g$ is fuzzy $r$-semiclosed then $g \circ f$ is fuzzy $r$-semiclosed.*

*Proof.* Straightforward. □

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