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ON THE GROWTH OF ENTIRE FUNCTIONS WITH APPLICATIONS TO LINEAR DIFFERENTIAL EQUATIONS

Ki-Ho Kwon and Jeong-Heon Kim

Abstract. Let $\rho(A)$ and $\rho(B)$ denote the orders of entire functions $A(z)$ and $B(z)$ respectively. Suppose that $\rho(A) > 1$ and $0 < \rho(B) \leq \frac{1}{2}$, and that $\rho(A)$ is not an integer. Then it is shown that every nonconstant solution f of $f'' + A(z)f' + B(z)f = 0$ is of infinite order if all the zeros of $A(z)$ lie in a certain angular sector depending on its genus. In addition, we investigate some growth properties of $A(z)$.

1. Statements of results

Let $\rho(g)$ denote the order of an entire function g. Consider the second order linear differential equation

$$f'' + A(z)f' + B(z)f = 0 \tag{1}$$

where $A(z)$ and $B(z)$ are entire functions. It is known that if $\rho(B) < \rho(A) \leq \frac{1}{2}$, then every nonconstant solution of (1) is of infinite order [2]. In the case that $\rho(A) > \frac{1}{2}$ and $\rho(B) < \rho(A)$, the possibility of finite order solution of (1) remains open.

Recently we proved the following result in [4].

Theorem A. *Suppose that $A(z)$ is an entire function of finite non-integral order with $\rho(A) > 1$, and that all the zeros of $A(z)$ lie in the angular sector $\theta_1 \leq arg z \leq \theta_2$ satisfying*

$$\theta_2 - \theta_1 < \frac{\pi}{q+1}$$

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if q is odd, and

$$\theta_2 - \theta_1 < \frac{(2q-1)\pi}{2q(q+1)}$$

if q is even, where q is the genus of $A(z)$. Let $B(z)$ be an entire function with $0 < \rho(B) < \frac{1}{2}$. Then every nonconstant solution f of (1) has infinite order.

In this paper, we extend Theorem A by proving

THEOREM 1. *Let $A(z)$ satisfy the conditions of Theorem A. If $B(z)$ is an entire function with $0 < \rho(B) \leq \frac{1}{2}$, then every nonconstant solution f of (1) has infinite order.*

Thus our contribution is to treat the case $\rho(B) = \frac{1}{2}$. The main ingredient in the proof is the growth of entire functions having their zeros in certain angular sectors.

THEOREM 2. *Suppose that $f(z)$ is an entire function of finite non-integral order with genus $q \geq 1$, and that for given $\epsilon > 0$, all the zeros of $f(z)$ lie in the angular sector $\theta_1 \leq argz \leq \theta_2$ satisfying*

$$\theta_2 - \theta_1 \leq \frac{\pi}{q+1} - \epsilon$$

if q is odd, and

$$\theta_2 - \theta_1 \leq \frac{(2q-1)\pi}{2q(q+1)} - \epsilon$$

if q is even. Then for any $c > 1$, β_1 and β_2 with $0 < \beta_2 - \beta_1 < \epsilon$, there exists a real number R, such that

$$\log |f(re^{i\theta})| \leq -cr^q$$

for all $r \geq R$ and for all $\theta \in [\beta_1, \beta_2]$.

THEOREM 3. *Suppose that $f(z)$ is an entire function of finite non-integral order with genus q, and that for given $\epsilon > 0$, all the zeros of $f(z)$ lie in the angular sector $\theta_1 \leq argz \leq \theta_2$ satisfying*

$$\theta_2 - \theta_1 \leq \frac{\pi}{q+1} - \epsilon$$

if q is even, and

$$\theta_2 - \theta_1 \leq \frac{(2q-1)\pi}{2q(q+1)} - \epsilon$$

if q is odd. Then for any $c > 1$, β_1 and β_2 with $0 < \beta_2 - \beta_1 < \epsilon$, there exists a real number R such that

$$\log |f(re^{i\theta})| \geq cr^q, \quad q \geq 1;$$
$$\log |f(re^{i\theta})| \geq c \log r, \quad q = 0$$

for all $r \geq R$, and for all $\theta \in [\beta_1, \beta_2]$.

2. Proofs of Theorem 2 and 3

Proof of Theorem 2. Rotating properly the axes of the complex plane, we may assume that all the zeros of $f(z)$ have their arguments in the set

$$S(q, \epsilon) = \{\theta : |\theta| \leq \frac{\pi}{2(q+1)} - \frac{\epsilon}{2}\}$$

if q is odd, and

$$S(q, \epsilon) = \{\theta : \frac{\pi}{2q} + \frac{\epsilon}{2} \leq |\theta| \leq \frac{3\pi}{2(q+1)} - \frac{\epsilon}{2}\}$$

if q is even.

Let a_n be nonzero zeros of $f(z)$. Then we may set

$$f(z) = z^m e^{P(z)} g(z) \tag{2}$$

for some nonnegative integer m, a polynomial $P(z)$ with $\deg P \leq q$ and $g(z) = \prod E(\frac{z}{a_n}, q)$, where $E(w, q)$ is an elementary factor with genus q. Note that $a_n = r_n e^{i\theta_n}$ for some $r_n > 0$ and $\theta_n \in S(q, \epsilon)$.

Let $z = re^{i\phi}$ with $r > 0$, $|\phi| < \pi$. Then, from the well-known representation due to Valiron[5], we have

$$\log E(-\frac{z}{r_n}, q) = (-1)^q \int_{r_n}^{\infty} \frac{z^{q+1}}{t^{q+1}(z+t)} dt.$$

Taking the real parts, we obtain

$$\log|E(-\frac{z}{r_n},q)| = (-1)^q r^{q+1}\int_{r_n}^{\infty}\frac{t\cos(q+1)\phi + r\cos q\phi}{t^{q+1}(t^2+2tr\cos\phi+r^2)}dt. \tag{3}$$

Set $\delta = \frac{\epsilon-(\beta_2-\beta_1)}{2}$. If $\phi \in S(q,\delta)$, then there is a positive real number d such that

$$(-1)^q\cos(q+1)\phi \le -d \quad \text{and} \quad (-1)^q\cos q\phi \le -d. \tag{4}$$

If $|\theta-\pi| \le \frac{\beta_2-\beta_1}{2}$ and $\theta_n \in S(q,\epsilon)$, then $\theta-\pi-\theta_n \in S(q,\delta)$. Hence it follows from (3) and (4) that for all $r>0$ and for all θ satisfying $|\theta-\pi| \le \frac{\beta_2-\beta_1}{2}$, we have

$$\begin{aligned}\log|g(re^{i\theta})| &= \sum\log|E(\frac{re^{i\theta}}{a_n},q)|\\ &= \sum\log|E(-\frac{re^{i(\theta-\pi-\theta_n)}}{r_n},q)|\\ &\le -dr^{q+1}\sum\int_{r_n}^{\infty}\frac{dt}{t^{q+1}(t+r)}\\ &= -dr^{q+1}\int_0^{\infty}\frac{n(t)dt}{t^{q+1}(t+r)},\end{aligned} \tag{5}$$

where $n(t)$ is the number of zeros of $g(z)$ in the disk $|z|\le t$.

Since $\deg P \le q$, there is real numbers $b>1$ and $R_1>0$ such that $|P(z)| \le br^q$ for all z satisfying $|z|=r\ge R_1$. Since $\rho(g)>q$ and

$$\limsup_{r\to\infty}\frac{\log n(r)}{\log r} = \rho(g),$$

there is an $R_2>R_1$ such that

$$n(R_2) \ge \frac{12bcqR_2^q}{d}. \tag{6}$$

Hence it follows from (6) that for any $r \geq 2R_2$,

$$\begin{aligned}\int_0^\infty \frac{n(t)dt}{t^{q+1}(t+r)} &\geq \int_{R_2}^r \frac{n(t)dt}{t^{q+1}(t+r)} \\ &\geq \frac{n(R_2)}{2r}\int_{R_2}^r \frac{dt}{t^{q+1}} \\ &\geq \frac{n(R_2)}{4qrR_2^q} \geq \frac{3bc}{dr}.\end{aligned} \tag{7}$$

Thus (5) and (7) imply that

$$\log|g(re^{i\theta})| \leq -3bcr^q$$

for all $r \geq 2R_2$, and for all θ satisfying $|\theta - \pi| \leq \frac{\beta_2-\beta_1}{2}$.

Therefore, setting $R = 2R_2$, we have

$$\begin{aligned}\log|f(re^{i\theta})| &= m\log r + \Re P(re^{i\theta}) + \log|g(re^{i\theta})| \\ &\leq -bcr^q < -cr^q\end{aligned}$$

for all $r \geq R$ and for all θ satisfying $|\theta - \pi| \leq \frac{\beta_2-\beta_1}{2}$. Hence Theorem 2 is proved.

Proof of Theorem 3. To prove Theorem 3, we set

$$S(q,\epsilon) = \{\theta : |\theta| \leq \frac{\pi}{2(q+1)} - \frac{\epsilon}{2}\}$$

if q is even, and

$$S(q,\epsilon) = \{\theta : \frac{\pi}{2q} + \frac{\epsilon}{2} \leq |\theta| \leq \frac{3\pi}{2(q+1)} - \frac{\epsilon}{2}\}$$

if q is odd. Then for $q \geq 1$, we obtain

$$(-1)^q\cos(q+1)\phi \geq d \text{ and } (-1)^q\cos q\phi \geq d$$

instead of (4). Hence we have

$$\log|g(re^{i\theta})| \geq dr^{q+1}\int_0^\infty \frac{n(t)dt}{t^{q+1}(t+r)} \tag{8}$$

for all $r > 0$ and for all θ satisfying $|\theta - \pi| \leq \frac{\beta_2-\beta_1}{2}$. Therefore the result of Theorem 3 follows from (6), (7) and (8) by the same reasoning as in the proof of Theorem 2.

If $q = 0$, it follows from (8) that

$$\begin{aligned}\log|g(re^{i\theta})| &\geq dr\int_0^\infty \frac{n(t)dt}{t(t+r)}\\ &\geq dn(R_2)\int_{R_2}^\infty \frac{rdt}{t(t+r)}\\ &= dn(R_2)\ln\frac{R_2+r}{R_2}.\end{aligned}$$

Since $n(R_2) \to \infty$ as $R_2 \to \infty$, there exists $R \geq R_2$ such that

$$\log|g(re^{i\theta})| \geq c\log r$$

for all $r \geq R$ and for all θ satisfying $|\theta - \pi| \leq \frac{\beta_2-\beta_1}{2}$. Hence Theorem 3 is proved.

3. Proof of Theorem 1

For the proof of the theorem, we need the following results as well as Theorem 2. Before stating these results, we recall the concepts of density and logarithmic density of the subset of $[1,\infty)$. For $E \subset [1,\infty)$, let $m(E)$ denote the Lebesgue measure of E and define the logarithmic measure of E by

$$m_l(E) = \int_E \frac{dt}{t}.$$

The upper density and upper logarithmic density of E are defined by

$$\begin{aligned}\overline{dens}\, E &= \limsup_{r\to\infty}\frac{m(E\cap[1,r])}{r}\\ \overline{\log dens}\, E &= \limsup_{r\to\infty}\frac{m_l(E\cap[1,r])}{\log r}.\end{aligned}$$

The lower density and lower logarithmic density are defined similarly with lim sup replaced by lim inf. It is easy to verify

$$0 \le \underline{dens}\, E \le \underline{\log dens}\, E \le \overline{\log dens}\, E \le \overline{dens}\, E \le 1$$

for any $E \subset [1, \infty)$.

LEMMA A [1]. *Let $f(z)$ be a nonconstant entire function, and let $\alpha > 1$ and $\epsilon > 0$ be given constants. Then the following two statements hold:*

(i) *There exist a constant $c > 0$ and a set $E_1 \subset [0, \infty)$ having finite Lebesgue measure such that for all z satisfying $|z| = r \notin E_1$ we have*

$$\left| \frac{f^{(k)}(z)}{f(z)} \right| \le c[T(\alpha r, f) r^{\epsilon} \log T(\alpha r, f)]^k, \quad k \in N.$$

(ii) *There exist a constant $c > 0$ and a set $E_2 \subset [0, 2\pi)$ having Lebesgue measure zero such that if $\phi_0 \in [0, 2\pi) - E_2$, then there is a constant $R_0 = R_0(\phi_0) > 0$ so that for all z satisfying $\arg z = \phi_0$ and $|z| = r \ge R_0$, we have*

$$\left| \frac{f^{(k)}(z)}{f(z)} \right| \le c[T(\alpha r, f) \log T(\alpha r, f)]^k, \quad k \in N.$$

LEMMA B [3]. *Suppose that $g(z)$ is entire with $\rho(g) \le \frac{1}{2}$ and $\rho < \rho(g)$. Then either there exists $\{r_m\}$ such that $r_m \to \infty$ and*

$$\min_{|z|=r_m} \log |g(z)| > r_m^{\rho},$$

or, if

$$K_r(\rho) = K_r = \{\theta \in [0, 2\pi] : \log |g(re^{i\theta})| < r^{\rho}\},$$

there exists a set $E(1) \subset [1, \infty)$ of lower logarithmic density 1 such that for $r \in E(1)$, K_r satisfies

$$m(K_r) \to 0, \text{ as } r \to \infty.$$

Proof of Theorem 1. If f is a nonconstant solution of (1), It follows from (1) that

$$|B(z)| \leq \left|\frac{f''(z)}{f(z)}\right| + |A(z)| \left|\frac{f'(z)}{f(z)}\right|. \tag{9}$$

The proof is divided into two cases depending on the behavior of the minimum modulus of $B(z)$ on $|z| = r$ by Lemma B. First, we assume that there exists $r_m \to \infty$ such that

$$\log |B(r_m e^{i\theta})| > r_m^\rho \tag{10}$$

for all $\theta \in [0, 2\pi]$, and for any positive real number $\rho < \rho(B)$. Applying Theorem 2 to $A(z)$, we get real numbers β_1, $\beta_2 (\beta_1 < \beta_2)$ and R such that

$$|A(re^{i\theta})| < 1 \tag{11}$$

for all $r \geq R$, and for all $\theta \in [\beta_1, \beta_2]$. Furthermore, by Lemma A*(ii)*, there exists $\theta_0 \in [\beta_1, \beta_2]$ such that if $z = re^{i\theta_0}$,

$$\left|\frac{f^{(k)}(z)}{f(z)}\right| \leq T(2r, f)^3; \quad k = 1, 2 \tag{12}$$

for all sufficiently large r. Hence from (9), (10), (11) and (12), we have

$$exp(r_m^\rho) \leq 2T(2r_m, f)^3$$

for all sufficiently large m. Therefore f has infinite order and the theorem is proved in the first case.

Now, we assume that there is a set $E(1)$ of lower logarithmic density 1 such that for $r \in E(1)$, we have

$$m(K_r) \to 0, \text{ as } r \to \infty, \tag{13}$$

where

$$K_r = K_r(\rho) = \{\theta : \log |B(re^{i\theta})| < r^\rho\}. \tag{14}$$

By Lemma A*(i)* and (13), we have a set $E \subset E(1)$ with lower logarithmic density 1 such that if $z = re^{i\theta}$,

$$\left|\frac{f^{(k)}(z)}{f(z)}\right| \leq rT(2r,f)^3, \quad k = 1,2 \tag{15}$$

holds for all $r \in E$, and for all $\theta \in [0, 2\pi]$. Therefore it follows from (9), (11), (14) and (15) that

$$exp(r^\rho) \leq 2rT(2r,f)^3$$

for all $r \in E$, and for all $\theta \in [\beta_1, \beta_2] - K_r$. Since $[\beta_1, \beta_2] - K_r$ is nonempty for all sufficiently large $r \in E$ by (13), we conclude that f has infinite order. Hence the proof of the theorem is complete.

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Department of Mathematics
Korea Military Academy
Seoul 139-799, Korea