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A STUDY ON NILPOTENT LIE GROUPS

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Abstract. We briefly discuss the Lie groups, it's nilpotency and representations of a nilpotent Lie groups. Dixmier and Kirillov proved that simply connected nilpotent Lie groups over $\mathbb{R}$ are monomial. We reformulate the above result at the Lie algebra level.

1. Introduction

Standard references are in Helgason [He], Varadarajan [V] and Serre [Se].

An *analytic group* G is a topological group with the structure of a connected smooth manifold such that its multiplication from $G \times G$ to G given by $(x, y) \mapsto xy$ and the inverse map of G to G given by $x \mapsto x^{-1}$ $(x \in G)$ are both smooth mappings. Such a group is locally compact; it is generated by any compact neighborhood of the identity element e. It also has a countable base. A *Lie group* is a locally compact topological group with a countable base such that the identity component is an analytic group.

Now we let G be an analytic group and let $L_x : G \to G$ be the *left translation by* $x \in G$ given by

$$L_x(y) := xy, \quad y \in G.$$

A vector field X on G is *left invariant* if it commutes with left translations as an operator on functions, i.e., if

$$dL_x(y)(X(y)) = X(xy) \quad \text{for all } x \in G,$$

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where $dL_x(y)$ denotes the differential of L_x at $y \in G$. We recall that if $\dim G = n$ and (U, φ) is a chart at $g \in G$ with $\varphi(g) = (x_1(g), \cdots, x_n(g))$ for some choice of coordinates $x_1, \cdots, x_n$ in $\mathbb{R}^n$, we can write a vector field X as

$$X(g) = \sum_{j=1}^{n} X_j(g)(d\varphi)^{-1}\left(\frac{\partial}{\partial x_j}\right), \quad g \in G.$$

Then X acts on $C^\infty(G)$, the *space of smooth functions* on G, via

$$Xf(g) = \sum_{j=1}^{n} X_j(g)\frac{\partial(f \circ \varphi^{-1})}{\partial x_j} \circ \varphi(g), \tag{1.1}$$

where $f \in C^\infty(G)$ and $g \in G$. Using the equation (1.1), we can define the *product* of two vector fields X and Y through their composition action on $C^\infty(G)$:

$$XY(f)(g) := X(Y(f))(g), \quad f \in C^\infty(G),\ g \in G.$$

The bracket $[X, Y] := XY - YX$ of two vector fields X and Y on G is also a vector field on G. It is easy to show that for any vector fields X, Y, Z on G, the following properties

(1.2)

(i) the bracket operation is bilinear,

(ii) $[X, X] = 0 \Leftrightarrow [X, Y] = -[Y, X]$, i.e., $[\cdot, \cdot]$ is skew-symmetric,

(iii) $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$

hold. The last identity (iii) is known as the *Jacobi identity.* A vector space over any field F with a bracket operation $[\cdot, \cdot]$ satisfying the conditions (i), (ii), (iii) in (1.2) is called a *Lie algebra* over F. In this section, $F = \mathbb{R}$ or $\mathbb{C}$. The vector space $\mathfrak{X}(G)$ of all vector fields on G forms a Lie algebra over $\mathbb{R}$ with the bracket operation $[\cdot, \cdot]$. It is easy to check that the set $\mathfrak{g}$ of all left-invariant vector fields on G is a Lie algebra over $\mathbb{R}$ with the bracket operation $[\cdot, \cdot]$ and $\dim_{\mathbb{R}} \mathfrak{g} = n$.

Definition 1.1. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$ with the bracket operation $[\cdot, \cdot]$. A subspace I of $\mathfrak{g}$ is called an *ideal* of $\mathfrak{g}$ if $[I, \mathfrak{g}] \subseteq I$.

A Lie algebra is said to be *simple* if it has no nontrivial ideals. A Lie algebra is said to be *semisimple* if it can be written as a direct sum of simple Lie algebras.

One can associate the Lie algebra $\mathfrak{g}$ of all left-invariant vector fields on G with the tangent space $T_e(G)$ of G at e via

$$X \mapsto X(e), \quad X \in \mathfrak{g}. \tag{1.3}$$

It is easy to see that the mapping (1.3) is an isomorphism of vector spaces. The Lie algebra structure of $\mathfrak{g}$ carries over $T_e(G)$ via the mapping (1.3) and so we identify $\mathfrak{g}$ with $T_e(G)$ as Lie algebras. We will say that G is *semisimple* if its Lie algebra $\mathfrak{g}$ is semisimple and G has a finite center.

An important link between a Lie group and its Lie algebra is the existence of the *exponential map*. Let G be a Lie group with its Lie algebra $\mathfrak{g}$. For each $X \in \mathfrak{g}$, we let $\tilde{X}$ be the corresponding left-invariant vector field on G. Then there exists a unique map, called the exponential map

$$\exp : \mathfrak{g} \to G \tag{1.4}$$

such that for each $X \in \mathfrak{g}$, $t \mapsto \exp tX$ $(t \in \mathbb{R})$ is a one-parameter subgroup of G and conversely every one-parameter subgroup has this form. We will say that $t \mapsto \exp tX$ is the one-parameter subgroup of G generated by X in $\mathfrak{g}$. Also we may refer to X as the *infinitesimal generator* of $\exp tX$.

A fundamental fact is that the exponential map $\exp : \mathfrak{g} \to G$ is a local diffeomorphism. It is well known that for any analytic group homomorphism ϕ of a Lie group G to another Lie group G' with Lie algebra $\mathfrak{g}'$, the following diagram is *commutative* :

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "d\phi"] \arrow[d, "\exp"'] & \mathfrak{g}' \arrow[d, "\exp"] \\ G \arrow[r, "\phi"] & G' \end{tikzcd}$$

diagram 1-1.

For each $x \in G$, we denote by

$$\mathrm{Ad}(x) : \mathfrak{g} \to \mathfrak{g} \tag{1.5}$$

the automorphism of $\mathfrak{g}$ which is the differential of the inner automorphism I_x of G. The action of G on $\mathfrak{g}$ given by $\mathrm{Ad} : G \to GL(\mathfrak{g})$ is called the *adjoint action* of G or the *adjoint representation* of G on $\mathfrak{g}$. It is obvious that the following diagrams are commutative :

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\mathrm{Ad}(x)"] \arrow[d, "\exp"'] & \mathfrak{g} \arrow[d, "\exp"] \\ G \arrow[r, "I_x"] & G \end{tikzcd}$$

diagram 1-2.

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\mathrm{ad}"] \arrow[d, "\exp"'] & \mathrm{End}(\mathfrak{g}) \arrow[d, "\exp"] \\ G \arrow[r, "\mathrm{Ad}"] & GL(\mathfrak{g}) \end{tikzcd}$$

diagram 1-3.

Here ad denotes the differential of Ad.

For a later use, we introduce the coadjoint action of G. Let $\mathfrak{g}^*$ be the real vector space of all $\mathbb{R}$-linear forms on $\mathfrak{g}$. For each $x \in G$, we denote by

$$\mathrm{Ad}^*(x) : \mathfrak{g}^* \to \mathfrak{g}^* \tag{1.6}$$

the *contragredient* of the adjoint mapping $\mathrm{Ad}(x) : \mathfrak{g} \to \mathfrak{g}$, i.e., the transpose of the $\mathbb{R}$-linear mapping $\mathrm{Ad}(x^{-1}) : \mathfrak{g} \to \mathfrak{g}$. These mappings $\mathrm{Ad}^*(x)$ $(x \in G)$ give rise to a linear representation of G in $\mathfrak{g}$, which is called the *coadjoint* representation of G in $\mathfrak{g}$. Obviously the following diagram is commutative :

$$\begin{tikzcd} \mathfrak{g}^* \arrow[r, "\mathrm{ad}^*"] \arrow[d, "\exp"'] & \mathrm{End}(\mathfrak{g}^*) \arrow[d, "\exp"] \\ G \arrow[r, "\mathrm{Ad}^*"] & GL(\mathfrak{g}^*) \end{tikzcd}$$

diagram 1-4.

Here ad^* denotes the differential of the map Ad^*.

A Lie group that can be realized as a closed subgroup of $GL(n,\mathbb{R})$ for some n will be called a *linear Lie group*. If G is a linear Lie group, then the Lie algebra of G can be thought of as a Lie algebra of matrices, and the exponential mapping is given by the exponential mapping for matrices :

$$\exp X := \sum_{l=0}^{\infty} \frac{1}{l!} X^l = I + X + \frac{X^2}{2!} + \frac{X^3}{3!} + \cdots . \tag{1.7}$$

LEMMA 1.2. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let $\exp : \mathfrak{g} \to G$ be the exponential mapping of $\mathfrak{g}$ into G. Then, if $X, Y \in \mathfrak{g}$,*

(a) $\exp tX \cdot \exp tY = \exp\left\{ t(X+Y) + \frac{t^2}{2}[X,Y] + O(t^3) \right\}$,

(b) $\exp(-tX) \cdot \exp(-tY) \cdot \exp tX \cdot \exp tY = \exp\{t^2[X,Y] + O(t^3)\}$,

(c) $\exp tX \cdot \exp tY \cdot \exp(-tX) = \exp\{tY + t^2[X,Y] + O(t^3)\}$.

In each case $O(t^3)$ denotes a vector in $\mathfrak{g}$ with the property : there exists an $\epsilon > 0$ such that $\dfrac{O(t^3)}{t^3}$ is bounded and analytic for $|t| < \epsilon$.

For the proof we refer to Helgason [He], pp. 96–97.

THEOREM 1.3. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. The exponential mapping $\exp : \mathfrak{g} \to G$ has the differential*

$$d\exp X = d(L_{\exp X})_e \circ \frac{1 - e^{-\mathrm{ad}\ X}}{\mathrm{ad}\ X} \quad (X \in \mathfrak{g}). \tag{1.8}$$

As usual, $\mathfrak{g}$ is here identified with the tangent space $\mathfrak{g}_X$.

The proof of Theorem 1.3 can be found in Helgason [He], pp. 95–96.

DEFINITION 1.4. Let $\mathfrak{g}$ be a Lie algebra over a field F, where $F = \mathbb{R}$ or $\mathbb{C}$ with the Lie bracket $[\ ,\ ]$. The bilinear form $B_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to F$ defined by

$$B_{\mathfrak{g}}(X,Y) := Tr(\mathrm{ad}\, X \circ \mathrm{ad}\, Y), \quad X, Y \in \mathfrak{g} \tag{1.9}$$

is called the *Killing form* of $\mathfrak{g}$. Here $\operatorname{ad} X : \mathfrak{g} \to \mathfrak{g}$ is the linear mapping of $\mathfrak{g}$ into $\mathfrak{g}$ defined by

$$\operatorname{ad} X(Y) := [X, Y], \quad Y \in \mathfrak{g}. \tag{1.10}$$

If σ is an automorphism of $\mathfrak{g}$, then $\operatorname{ad}(\sigma X) = \sigma \circ \operatorname{ad} X \circ \sigma^{-1}$. Thus we have

$$B_{\mathfrak{g}}(\sigma X, \sigma Y) = B_{\mathfrak{g}}(X, Y), \quad \sigma \in \operatorname{Aut}(\mathfrak{g}), \quad X, Y \in \mathfrak{g}, \tag{1.11}$$

$$B_{\mathfrak{g}}(X, [Y, Z]) = B_{\mathfrak{g}}(Y, [Z, X]) = B_{\mathfrak{g}}(Z, [X, Y]), \quad X, Y, Z \in \mathfrak{g}. \tag{1.12}$$

EXAMPLE 1.5. $G = GL(n, \mathbb{R})$; the general linear group over $\mathbb{R}$. $\mathbb{R}^{(n,n)}$ is regarded as the Lie algebra of the Lie group $GL(n, \mathbb{R})$ with the bracket operation

$$[X, Y] = XY - YX, \quad X, Y \in \mathbb{R}^{(n,n)}.$$

EXAMPLE 1.6. Orthogonal Groups;
Let $p, q \in \mathbb{Z}^+$ with $p \geq q > 0$ and $p + q = n > 0$. Let $I_{p,q}$ be the quadratic form on $\mathbb{R}^n$ given by

$$I_{p,q}(x_1, \cdots, x_n) = \sum_{i=1}^{p} x_i^2 - \sum_{j=p+1}^{n} x_j^2. \tag{1.13}$$

The corresponding bilinear form $B_{p,q}$ is given by

$$B_{p,q}((x_i), (y_i)) = \sum_{i=1}^{p} x_i y_i - \sum_{j=p+1}^{n} x_j y_j, \tag{1.14}$$

where $x = (x_i), y = (y_i) \in \mathbb{R}^n$. This form is definite if $q = 0$ and indefinite if $q > 0$. The general linear group $GL(n, \mathbb{R})$ acts on $\mathbb{R}^n$ in the usual way by matrix multiplication. We define the orthogonal group $O(p, q)$ by

$$O(p, q) := \{ g \in GL(n, \mathbb{R}) \mid I_{p,q}(g \cdot x) = I_{p,q}(x) \quad \text{for all } x \in \mathbb{R}^n \}. \tag{1.15}$$

If $q=0$, we write $O(p)$ instead of $O(p,0)$. It is easy to see that $O(p)$ is compact. But if $q>0$, $O(p,q)$ is not compact and it has four connected components. We denote by $\mathfrak{o}(p,q)$ the Lie algebra of $O(p,q)$. Then

$$\mathfrak{o}(p,q)=\{A\in\mathbb{R}^{(n,n)}\mid B_{p,q}(Au,v)+B_{p,q}(u,Av)=0\quad\text{for all } u,v\in\mathbb{R}^n\}.$$

A simple computation shows that each element of $\mathfrak{o}(p,q)$ may be put in the form

$$\begin{bmatrix}-A & B\\ C & D\end{bmatrix},$$

where $A=-{}^tA\in\mathbb{R}^{(p,p)}$, $D=-{}^tD\in\mathbb{R}^{(q,q)}$ and $B={}^tC$ is a $p\times q$ matrix.

2. Nilpotent groups

Let G be a group with the identity element e. For two subsets A,B of G, we denote by $[A,B]$ the subgroup of G generated by the set of commutators

$$\{\ [x,y]:=(xy)(yx)^{-1}=xyx^{-1}y^{-1}\mid x,y\in G\ \}.$$

We observe that if A and B are normal subgroups of G, then $[A,B]$ is a normal subgroup of G. In particular, the *derived group* $D^1G:=[G,G]$ of G is a *normal* subgroup of G.

Let $l\in\mathbb{Z}^+$ be a positive integer. We define the *descending central series* $(\mathcal{C}^lG)_{l\geq 0}$ of G recursively via

$$\mathcal{C}^oG:=G,\quad \mathcal{C}^{l+1}:=[G,\mathcal{C}^lG],\qquad l=0,1,2,\cdots. \tag{2.1}$$

Then we get the following descending filtration of *normal subgroup of* G :

$$G\hookleftarrow\mathcal{C}^1G\supset\mathcal{C}^2G\supset\cdots\supset\mathcal{C}^lG\supset\cdots\supset\{e\}. \tag{2.2}$$

The group G is said to be *nilpotent* if there exists a positive integer $m\in\mathbb{Z}^+$ such that $\mathcal{C}^mG=\{e\}$. If $\mathcal{C}^{m-1}G\neq\{e\}$ and $\mathcal{C}^mG=\{e\}$ for $m\geqq 1$, then the number m is called the *length* of the nilpotent group G and G is called a *m-step nilpotent* group.

The ascending central series $(\mathcal{C}_l G)_{l\geq 0}$ of the group G is defined recursively according to the rules

$$\begin{cases} \mathcal{C}_0 G := \{e\}, & \\ \mathcal{C}_{l+1} G := & \text{the preimage of the center of } G/\mathcal{C}_l G \\ & \text{under the canonical epimorphism } G \to G/\mathcal{C}_l G. \end{cases}$$

Obviously $\mathcal{C}_1 G$ is the center Z of G and we have

$$\mathcal{C}_{l+1} G = \{x \in G \mid [x,y] \in \mathcal{C}_l(G) \quad \text{for all } y \in G\}. \tag{2.3}$$

We have the following ascending filtration of *normal* subgroups of G :

$$\{e\} \subset \mathcal{C}_1 G \subset \mathcal{C}_2 G \subset \cdots \subset \mathcal{C}_l G \subset \cdots \subset G. \tag{2.4}$$

Let G be a group with the identity element e and let $m \geqq 1$ be a sufficiently large positive integer. Then it is easy to show that the following conditions are mutually pairwise equivalent :

(a) $\mathcal{C}^m G = \{e\}$;

(b) There exists a sequence $(G_l)_{0\leq l\leq m}$ of subgroups of G such that $G_0 = G$, $G_m = \{e\}$,

$$G = G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_l \supset \cdots \supset G_m = \{e\}$$

and

$$[G, G_l] \subseteq G_{l+1} \quad \text{for all } l \text{ with } 0 \leq l \leq m-1;$$

(c) $\mathcal{C}_m G = G$.

Therefore G is a nilpotent group if it satisfies one of the above three conditions. If G is a nilpotent Lie group, then it is easy to see that

(i) each subgroup H of G is nilpotent ;

(ii) for each normal subgroup H of G, the quotient group G/H is nilpotent.

Let $G \neq \{e\}$ be a nilpotent group. Then the center Z of G is nontrivial.

Definition 2.1. The *derived series* $(D^l G)_{l\geq 0}$ of the group G is defined recursively via the prescriptions.

$$D^0 G = G,\ D^{l+1} G = [D^l G, D^l G], \quad l = 0, 1, 2, \cdots . \tag{2.5}$$

The derived series yields the descending filtration of *normal subgroups* of G :

$$G \supset D^1 G \supset D^2 G \supset \cdots \supset D^l G \supset \cdots \supset \{e\}. \tag{2.6}$$

We observe that $D^1 G = \mathcal{C}^1 G = [G, G]$ is the derived group of G. The group G is said to be *solvable* if there exists a sufficiently large positive integer $m > 0$ such that

$$D^m G = \{e\}$$

holds. If $D^{m-1} G \neq \{e\}$ and $D^m G = \{e\}$ for $m \geq 1$, then the number m is called the *length* of the solvable group G. Obviously we have $\mathcal{C}^l G \supseteq D^l G$ for all $l \in \mathbb{Z}^+$. Thus every nilpotent group is solvable, but the converse is false.

The following remarkable fact concerning finite dimensional representations of solvable groups can be easily established (c.f. [K2]).

Theorem 2.2. *Let G be a connected solvable locally compact topological group. Suppose $(\pi, \mathcal{H})$ is a finite dimensional irreducible unitary representation of G. Then* $\dim_{\mathbb{C}} \mathcal{H} = 1$.

Corollary 2.3. *A compact connected solvable topological group G is abelian.*

Proof. It is well known that every irreducible unitary representation of a compact connected topological group is finite dimensional. Therefore any topologically irreducible unitary representation of G is finite dimensional. According to Theorem 2.2, it is one-dimensional. Hence G is abelian. □

Remark 2.4. In view of Theorem 2.2, the unitary dual $\hat{G}$ of a connected solvable locally compact topological group G consists of two types of equivalence classes of

(I) continuous unitary characters of G;
(II) equivalence classes of infinite dimensional, topological irreducible unitary representations of G.

It is well known that the identity component G_0 of G is a nilpotent Lie group if and only if the Lie algebra $\mathfrak{g}$ of G is a nilpotent Lie algebra over $\mathbb{R}$. We recall the notion of the nilpotency of a Lie algebra $\mathfrak{g}$. We define the descending central series $(\mathcal{C}^l\mathfrak{g})_{l\geq 0}$ of $\mathfrak{g}$ recursively via the prescriptions

$$\mathcal{C}^0\mathfrak{g} = \mathfrak{g}, \quad \mathcal{C}^{l+1}\mathfrak{g} = [\mathfrak{g}, \mathcal{C}^l\mathfrak{g}], \qquad l = 0, 1, 2, \cdots . \tag{2.7}$$

A Lie algebra $\mathfrak{g}$ is said to be *nilpotent* if there exists a positive integer $m \in \mathbb{Z}^+$ such that $\mathcal{C}^m\mathfrak{g} = \{0\}$. It is well known that if G is a nilpotent Lie group over $\mathbb{R}$, then the exponential mapping $\exp : \mathfrak{g} \to G$ is a diffeomorphism of $\mathfrak{g}$ onto G, where $\mathfrak{g}$ denotes the Lie algebra of G (cf.[He]).

3. Representations of a nilpotent Lie group

First we recall that a real Lie group is said to be *monomial* if each topologically irreducible unitary representation of G can be unitarily induced by a unitary character of some closed subgroup H of G.

Using the Mackey machinery, Dixmier and Kirillov proved the following important result :

THEOREM 3.1. *(Dixmier-Kirillov) The simply connected nilpotent Lie groups over* $\mathbb{R}$ *are monomial.*

REMARK 3.2. More generally, it can be proved that a simply connected real Lie group whose exponential mapping $\exp : \mathfrak{g} = \mathrm{Lie}(G) \to G$ is a global diffeomorphism is monomial.

Now we reformulate Theorem 3.1 at the Lie algebra level. Let $(\pi, \mathcal{H})$ be a topologically irreducible, unitary representation of the simply connected, nilpotent real Lie group G in the complex Hilbert space $\mathcal{H}$. Let H be a connected closed subgroup of G with a unitary character χ such that

$$(\pi, \mathcal{H}) = \mathrm{Ind}_H^G(\chi, \mathbb{C}).$$

Let $\mathfrak{g}$ be the Lie algebra of G and let $\mathfrak{h}$ the Lie subalgebra of $\mathfrak{g}$ corresponding to the Lie subgroup H of G. Let l_0 be the differential of χ.

Then we have the following *commutative* diagram :

$$\begin{tikzcd} \mathfrak{h} \arrow[r, "l_0"] \arrow[d, "\exp_H"'] & \mathbb{R} \arrow[d, "e^{2\pi i}"] \\ H \arrow[r, "\chi"] & \mathbb{C}_1^\times \end{tikzcd}$$

diagram 3-1.

Here $\exp_H : \mathfrak{h} \to H$ denotes the exponential mapping of $\mathfrak{h}$ to H. Thus we have

$$(3.1) \qquad \chi(\exp_H X) = e^{2\pi i < X, l_0 >}, \quad X \in \mathfrak{h}$$

and

$$(3.2) \qquad < [X, Y], l_0 >= 0 \quad \text{for all } X, Y \in \mathfrak{h}.$$

The relation (3.2) follows from the fact that $\exp_H[X, Y] \in [H, H]$ and $\chi(g) = 1$ for all $g \in [H, H]$. Indeed, according to (3.1), we have

$$1 = \chi(\exp_H[X, Y]) = e^{2\pi i < [X,Y], l_0 >}, \quad X, Y \in \mathfrak{h}.$$

Therefore $< [X, Y], l_0 >$ is an integer. We have $< [X, Y], l_0 >= 0$. Otherwise, $< [X, Y], l_0 >= n \in \mathbb{Z}$ with $n \neq 0$. Then for sufficiently small positive real numbers t, we have

$$1 = \chi(\exp_H[tX, Y]) = e^{2\pi i t n} \neq 1$$

because $tn \notin \mathbb{Z}$. This leads to the contradiction.

Now we let $l \in \mathfrak{g}^*$ be any $\mathbb{R}$-linear form which extends l_0 to the whole Lie algebra $\mathfrak{g}$. Then $\mathfrak{h}$ forms a *totally isotropic* vector subspace of $\mathfrak{g}$ relative to the *skew-symmetric* $\mathbb{R}$*-bilinear form* $B_l : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ given by

$$(3.3) \qquad B_l(X, Y) := < [X, Y], l >, \quad X, Y \in \mathfrak{g}$$

associated with l on $\mathfrak{g}$. More precisely, $B_l|_{\mathfrak{g}\times\mathfrak{g}} = 0$. In this case we say that the Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is *subordinate to* l. We introduce the notation

$$(3.4) \qquad \chi_{l,\mathfrak{h}} := \chi$$

and the monomial representation of G

$$(3.5) \qquad (\pi_{l,\mathfrak{h}}, \mathcal{H}) := \mathrm{Ind}_H^G(\chi_{l,\mathfrak{h}}, \mathbb{C}).$$

Then we obtain the following theorem 3.3 which can be considered as an another version of theorem 7.2 a in [K4]

THEOREM 3.3. *Let G be a simply connected nilpotent real Lie group with Lie algebra $\mathfrak{g}$. Assume that there is given a topologically irreducible unitary representation $(\pi, \mathcal{H})$ of G. Then there exists a linear form $l \in \mathfrak{g}^*$ and a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ subordinate to l such that*

$$(\pi, \mathcal{H}) = (\pi_{l,\mathfrak{h}}, \mathcal{H}) := \mathrm{Ind}_H^G(\chi_{l,\mathfrak{h}}, \mathbb{C}).$$

We note that $\chi_{l,\mathfrak{h}}$ is a unitary character of H such that

$$\chi_{l,\mathfrak{h}}(\exp_H X) = e^{2\pi i <X,l>}, \quad X \in \mathfrak{h}, \tag{3.6}$$

where H is the simply connected closed subgroup of G corresponding to a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.

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