A STUDY ON NILPOTENT LIE GROUPS

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Abstract. We briefly discuss the Lie groups, its nilpotency and representations of a nilpotent Lie groups. Dixmier and Kirillov proved that simply connected nilpotent Lie groups over $\mathbb{R}$ are monomial. We reformulate the above result at the Lie algebra level.

1. Introduction

Standard references are in Helgason [He], Varadarajan [V] and Serre [Se].

An analytic group $G$ is a topological group with the structure of a connected smooth manifold such that its multiplication from $G \times G$ to $G$ given by $(x, y) \mapsto xy$ and the inverse map of $G$ to $G$ given by $x \mapsto x^{-1}$ ($x \in G$) are both smooth mappings. Such a group is locally compact; it is generated by any compact neighborhood of the identity element $e$. It also has a countable base. A Lie group is a locally compact topological group with a countable base such that the identity component is an analytic group.

Now we let $G$ be an analytic group and let $L_x : G \to G$ be the left translation by $x \in G$ given by

$$L_x(y) := xy, \quad y \in G.$$ 

A vector field $X$ on $G$ is left invariant if it commutes with left translations as an operator on functions, i.e., if

$$dL_x(y)(X(y)) = X(xy) \quad \text{for all } x \in G,$$ 

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where $dL_x(y)$ denotes the differential of $L_x$ at $y \in G$. We recall that if $\dim G = n$ and $(U, \varphi)$ is a chart at $g \in G$ with $\varphi(g) = (x_1(g), \ldots, x_n(g))$ for some choice of coordinates $x_1, \ldots, x_n$ in $\mathbb{R}^n$, we can write a vector field $X$ as

$$X(g) = \sum_{j=1}^{n} X_j(g)(d\varphi)^{-1}\left(\frac{\partial}{\partial x_j}\right), \quad g \in G.$$  

Then $X$ acts on $C^\infty(G)$, the space of smooth functions on $G$, via

$$(1.1) \quad Xf(g) = \sum_{j=1}^{n} X_j(g) \frac{\partial(f \circ \varphi^{-1})}{\partial x_j} \circ \varphi(g),$$

where $f \in C^\infty(G)$ and $g \in G$. Using the equation (1.1), we can define the product of two vector fields $X$ and $Y$ through their composition action on $C^\infty(G)$:

$$XY(f)(g) := X(Y(f))(g), \quad f \in C^\infty(G), \quad g \in G.$$  

The bracket $[X, Y] := XY - YX$ of two vector fields $X$ and $Y$ on $G$ is also a vector field on $G$. It is easy to show that for any vector fields $X, Y, Z$ on $G$, the following properties

$$(1.2) \quad \begin{align*}
(i) & \quad \text{the bracket operation is bilinear,} \\
(ii) & \quad [X, X] = 0 \iff [X, Y] = -[Y, X], \quad \text{i.e.,} \ [\cdot, \cdot] \text{ is skew-symmetric,} \\
(iii) & \quad [[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0 
\end{align*}$$

hold. The last identity (iii) is known as the Jacobi identity. A vector space over any field $F$ with a bracket operation $[\cdot, \cdot]$ satisfying the conditions (i), (ii), (iii) in (1.2) is called a Lie algebra over $F$. In this section, $F = \mathbb{R}$ or $\mathbb{C}$. The vector space $\mathfrak{X}(G)$ of all vector fields on $G$ forms a Lie algebra over $\mathbb{R}$ with the bracket operation $[\cdot, \cdot]$. It is easy to check that the set $\mathfrak{g}$ of all left-invariant vector fields on $G$ is a Lie algebra over $\mathbb{R}$ with the bracket operation $[\cdot, \cdot]$ and $\dim_{\mathbb{R}} \mathfrak{g} = n$.

**Definition 1.1.** Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$ with the bracket operation $[\cdot, \cdot]$. A subspace $I$ of $\mathfrak{g}$ is called an ideal of $\mathfrak{g}$ if $[I, \mathfrak{g}] \subseteq I$. 

A Lie algebra is said to be simple if it has no nontrivial ideals. A Lie algebra is said to be semisimple if it can be written as a direct sum of simple Lie algebras.

One can associate the Lie algebra $\mathfrak{g}$ of all left-invariant vector fields on $G$ with the tangent space $T_e(G)$ of $G$ at $e$ via

$$\tag{1.3} X \mapsto X(e), \quad X \in \mathfrak{g}.$$  

It is easy to see that the mapping (1.3) is an isomorphism of vector spaces. The Lie algebra structure of $\mathfrak{g}$ carries over $T_e(G)$ via the mapping (1.3) and so we identify $\mathfrak{g}$ with $T_e(G)$ as Lie algebras. We will say that $G$ is semisimple if its Lie algebra $\mathfrak{g}$ is semisimple and $G$ has a finite center.

An important link between a Lie group and its Lie algebra is the existence of the exponential map. Let $G$ be a Lie group with its Lie algebra $\mathfrak{g}$. For each $X \in \mathfrak{g}$, we let $\tilde{X}$ be the corresponding left-invariant vector field on $G$. Then there exists a unique map, called the exponential map

$$\tag{1.4} \exp : \mathfrak{g} \to G$$

such that for each $X \in \mathfrak{g}$, $t \mapsto \exp tX$ ($t \in \mathbb{R}$) is a one-parameter subgroup of $G$ and conversely every one-parameter subgroup has this form. We will say that $t \mapsto \exp tX$ is the one-parameter subgroup of $G$ generated by $X$ in $\mathfrak{g}$. Also we may refer to $X$ as the infinitesimal generator of $\exp tX$.

A fundamental fact is that the exponential map $\exp : \mathfrak{g} \to G$ is a local diffeomorphism. It is well known that for any analytic group homomorphism $\phi$ of a Lie group $G$ to another Lie group $G'$ with Lie algebra $\mathfrak{g}'$, the following diagram is commutative:

$$\begin{array}{ccc}
\mathfrak{g} & \xrightarrow{d\phi} & \mathfrak{g}' \\
\exp \downarrow & & \downarrow \exp \\
G & \xrightarrow{\phi} & G'
\end{array}$$

diagram 1-1.
For each $x \in G$, we denote by
\begin{equation}
\text{Ad}(x) : \mathfrak{g} \rightarrow \mathfrak{g}
\end{equation}
the automorphism of $\mathfrak{g}$ which is the differential of the inner automorphism $I_x$ of $G$. The action of $G$ on $\mathfrak{g}$ given by $\text{Ad} : G \rightarrow GL(\mathfrak{g})$ is called the \textit{adjoint action} of $G$ or the \textit{adjoint representation} of $G$ on $\mathfrak{g}$. It is obvious that the following diagrams are commutative:

\begin{align*}
\begin{array}{ccc}
\mathfrak{g} & \xrightarrow{\text{Ad}(x)} & \mathfrak{g} \\
\exp & \downarrow & \exp \\
G & \xrightarrow{I_x} & G \\
\end{array}
\end{align*}

\text{diagram 1-2.}

\begin{align*}
\begin{array}{ccc}
\mathfrak{g} & \xrightarrow{ad} & \text{End}(\mathfrak{g}) \\
\exp & \downarrow & \exp \\
G & \xrightarrow{\text{Ad}} & GL(\mathfrak{g}) \\
\end{array}
\end{align*}

\text{diagram 1-3.}

Here $ad$ denotes the differential of $\text{Ad}$.

For a later use, we introduce the coadjoint action of $G$. Let $\mathfrak{g}^*$ be the real vector space of all $\mathbb{R}$-linear forms on $\mathfrak{g}$. For each $x \in G$, we denote by
\begin{equation}
\text{Ad}^*(x) : \mathfrak{g}^* \rightarrow \mathfrak{g}^*
\end{equation}
the \textit{contragredient} of the adjoint mapping $\text{Ad}(x) : \mathfrak{g} \rightarrow \mathfrak{g}$, i.e., the transpose of the $\mathbb{R}$-linear mapping $\text{Ad}(x^{-1}) : \mathfrak{g} \rightarrow \mathfrak{g}$. These mappings $\text{Ad}^*(x) (x \in G)$ give rise to a linear representation of $G$ in $\mathfrak{g}$, which is called the \textit{coadjoint representation} of $G$ in $\mathfrak{g}$. Obviously the following diagram is commutative:

\begin{align*}
\begin{array}{ccc}
\mathfrak{g}^* & \xrightarrow{ad^*} & \text{End}(\mathfrak{g}^*) \\
\exp & \downarrow & \exp \\
G & \xrightarrow{\text{Ad}^*} & GL(\mathfrak{g}^*) \\
\end{array}
\end{align*}
A study on nilpotent Lie groups

Here $\text{ad}^*$ denotes the differential of the map $\text{Ad}^*$.

A Lie group that can be realized as a closed subgroup of $GL(n, \mathbb{R})$ for some $n$ will be called a linear Lie group. If $G$ is a linear Lie group, then the Lie algebra of $G$ can be thought of as a Lie algebra of matrices, and the exponential mapping is given by the exponential mapping for matrices:

$$(1.7) \quad \exp X := \sum_{l=0}^{\infty} \frac{1}{l!} X^l = I + X + \frac{X^2}{2!} + \frac{X^3}{3!} + \cdots.$$ 

**Lemma 1.2.** Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$ and let $\exp : \mathfrak{g} \to G$ be the exponential mapping of $\mathfrak{g}$ into $G$. Then, if $X, Y \in \mathfrak{g}$,

(a) $\exp tX \cdot \exp tY = \exp \left\{ t(X + Y) + \frac{t^2}{2} [X,Y] + O(t^3) \right\},$

(b) $\exp(-tX) \cdot \exp(-tY) \cdot \exp tX \cdot \exp tY = \exp \left\{ t^2[X,Y] + O(t^3) \right\},$

(c) $\exp tX \cdot \exp tY \cdot \exp(-tX) = \exp \left\{ tY + t^2[X,Y] + O(t^3) \right\}.$

In each case $O(t^3)$ denotes a vector in $\mathfrak{g}$ with the property: there exists an $\epsilon > 0$ such that $\frac{O(t^3)}{t^3}$ is bounded and analytic for $|t| < \epsilon$.

For the proof we refer to Helgason [He], pp. 96–97.

**Theorem 1.3.** Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$. The exponential mapping $\exp : \mathfrak{g} \to G$ has the differential

$$(1.8) \quad d \exp X = d(L_{\exp X})_e \circ \frac{1 - e^{-\text{ad} X}}{\text{ad} X} \quad (X \in \mathfrak{g}).$$

As usual, $\mathfrak{g}$ is here identified with the tangent space $\mathfrak{g}_X$.

The proof of Theorem 1.3 can be found in Helgason [He], pp. 95–96.

**Definition 1.4.** Let $\mathfrak{g}$ be a Lie algebra over a field $F$, where $F = \mathbb{R}$ or $\mathbb{C}$ with the Lie bracket $[\cdot, \cdot]$. The bilinear form $B_\mathfrak{g} : \mathfrak{g} \times \mathfrak{g} \to F$ defined by

$$(1.9) \quad B_\mathfrak{g}(X,Y) := \text{Tr}(\text{ad} X \circ \text{ad} Y), \quad X, Y \in \mathfrak{g}$$
is called the Killing form of $\mathfrak{g}$. Here $ad : \mathfrak{g} \rightarrow \mathfrak{g}$ is the linear mapping of $\mathfrak{g}$ into $\mathfrak{g}$ defined by

$$ad X(Y) := [X, Y], \quad Y \in \mathfrak{g}.$$ 

If $\sigma$ is an automorphism of $\mathfrak{g}$, then $ad(\sigma X) = \sigma \circ ad X \circ \sigma^{-1}$. Thus we have

$$(1.11) \quad B_{\mathfrak{g}}(\sigma X, \sigma Y) = B_{\mathfrak{g}}(X, Y), \quad \sigma \in \text{Aut}(\mathfrak{g}), \quad X, Y \in \mathfrak{g},$$

$$(1.12) \quad B_{\mathfrak{g}}(X, [Y, Z]) = B_{\mathfrak{g}}(Y, [Z, X]) = B_{\mathfrak{g}}(Z, [X, Y]), \quad X, Y, Z \in \mathfrak{g}.$$  

**Example 1.5.** $G = GL(n, \mathbb{R})$; the general linear group over $\mathbb{R}$. $\mathbb{R}^{(n,n)}$ is regarded as the Lie algebra of the Lie group $GL(n, \mathbb{R})$ with the bracket operation

$$[X, Y] = XY - YX, \quad X, Y \in \mathbb{R}^{(n,n)}.$$  

**Example 1.6.** Orthogonal Groups;

Let $p, q \in \mathbb{Z}^+$ with $p \geq q > 0$ and $p + q = n > 0$. Let $I_{p,q}$ be the quadratic form on $\mathbb{R}^n$ given by

$$(1.13) \quad I_{p,q}(x_1, \ldots, x_n) = \sum_{i=1}^{p} x_i^2 - \sum_{j=p+1}^{n} x_j^2.$$  

The corresponding bilinear form $B_{p,q}$ is given by

$$(1.14) \quad B_{p,q}((x_i), (y_i)) = \sum_{i=1}^{p} x_i y_i - \sum_{j=p+1}^{n} x_j y_j,$$

where $x = (x_i), y = (y_i) \in \mathbb{R}^n$. This form is definite if $q = 0$ and indefinite if $q > 0$. The general linear group $GL(n, \mathbb{R})$ acts on $\mathbb{R}^n$ in the usual way by matrix multiplication. We define the orthogonal group $O(p, q)$ by

$$(1.15) \quad O(p, q) := \{g \in GL(n, \mathbb{R}) \mid I_{p,q}(g \cdot x) = I_{p,q}(x) \quad \text{for all} \ x \in \mathbb{R}^n\}.$$
If \( q = 0 \), we write \( O(p) \) instead of \( O(p, 0) \). It is easy to see that \( O(p) \) is compact. But if \( q > 0 \), \( O(p, q) \) is not compact and it has four connected components. We denote by \( \mathfrak{o}(p, q) \) the Lie algebra of \( O(p, q) \). Then

\[
\mathfrak{o}(p, q) = \{ A \in \mathbb{R}^{(n,n)} \mid B_{p,q}(Au,v) + B_{p,q}(u,Av) = 0 \text{ for all } u, v \in \mathbb{R}^n \}.
\]

A simple computation shows that each element of \( \mathfrak{o}(p, q) \) may be put in the form

\[
\begin{bmatrix}
-A & B \\
C & D
\end{bmatrix},
\]

where \( A = -tA \in \mathbb{R}^{(p,p)} \), \( D = -tD \in \mathbb{R}^{(q,q)} \) and \( B = tC \) is a \( p \times q \) matrix.

## 2. Nilpotent groups

Let \( G \) be a group with the identity element \( e \). For two subsets \( A, B \) of \( G \), we denote by \( [A, B] \) the subgroup of \( G \) generated by the set of commutators

\[
\{ [x, y] := (xy)(yx)^{-1} = xyyx^{-1}y^{-1} \mid x, y \in G \}.
\]

We observe that if \( A \) and \( B \) are normal subgroups of \( G \), then \( [A, B] \) is a normal subgroup of \( G \). In particular, the derived group \( D^1G := [G, G] \) of \( G \) is a normal subgroup of \( G \).

Let \( l \in \mathbb{Z}^+ \) be a positive integer. We define the descending central series \( \{ C^lG \}_{l \geq 0} \) of \( G \) recursively via

\[
C^0G := G, \quad C^{l+1} := [G, C^lG], \quad l = 0, 1, 2, \cdots.
\]

Then we get the following descending filtration of normal subgroup of \( G \):

\[
G \hookrightarrow C^1G \supset C^2G \supset \cdots \supset C^lG \supset \cdots \supset \{ e \}.
\]

The group \( G \) is said to be nilpotent if there exists a positive integer \( m \in \mathbb{Z}^+ \) such that \( C^mG = \{ e \} \). If \( C^{m-1}G \neq \{ e \} \) and \( C^mG = \{ e \} \) for \( m \geq 1 \), then the number \( m \) is called the length of the nilpotent group \( G \) and \( G \) is called a \( m \)-step nilpotent group.
The ascending central series \((C_lG)_{l \geq 0}\) of the group \(G\) is defined recursively according to the rules

\[
\begin{align*}
C_0G &:= \{e\}, \\
C_{l+1}G &:= \text{the preimage of the center of } G/C_lG \\
&\quad \text{under the canonical epimorphism } G \to G/C_lG.
\end{align*}
\]

Obviously \(C_1G\) is the center \(Z\) of \(G\) and we have

\[(2.3) \quad C_{l+1}G = \{x \in G \mid [x, y] \in C_l(G) \text{ for all } y \in G\}.
\]

We have the following ascending filtration of normal subgroups of \(G\):

\[(2.4) \quad \{e\} \subset C_1G \subset C_2G \subset \cdots \subset C_lG \subset \cdots \subset G.
\]

Let \(G\) be a group with the identity element \(e\) and let \(m \geq 1\) be a sufficiently large positive integer. Then it is easy to show that the following conditions are mutually pairwise equivalent:

(a) \(C^mG = \{e\}\);
(b) There exists a sequence \((G_l)_{0 \leq l \leq m}\) of subgroups of \(G\) such that \(G_0 = G, \ G_m = \{e\}, \ G = G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_l \supset \cdots \supset G_m = \{e\} \)
and

\([G, G_l] \subseteq G_{l+1} \text{ for all } l \text{ with } 0 \leq l \leq m - 1; \)

(c) \(C_mG = G\).

Therefore \(G\) is a nilpotent group if it satisfies one of the above three conditions. If \(G\) is a nilpotent Lie group, then it is easy to see that

(i) each subgroup \(H\) of \(G\) is nilpotent;
(ii) for each normal subgroup \(H\) of \(G\), the quotient group \(G/H\) is nilpotent.

Let \(G \neq \{e\}\) be a nilpotent group. Then the center \(Z\) of \(G\) is nontrivial.
Definition 2.1. The derived series \((D^l G)_{l \geq 0}\) of the group \(G\) is defined recursively via the prescriptions.

\[
D^0 G = G, \quad D^{l+1} G = [D^l G, D^l G], \quad l = 0, 1, 2, \ldots .
\]

The derived series yields the descending filtration of normal subgroups of \(G\):

\[
G \supset D^1 G \supset D^2 G \supset \cdots \supset D^l G \supset \cdots \supset \{e\}.
\]

We observe that \(D^1 G = C^1 G = [G, G]\) is the derived group of \(G\). The group \(G\) is said to be solvable if there exists a sufficiently large positive integer \(m > 0\) such that

\[
D^m G = \{e\}
\]

holds. If \(D^{m-1} G \neq \{e\}\) and \(D^m G = \{e\}\) for \(m \geq 1\), then the number \(m\) is called the length of the solvable group \(G\). Obviously we have \(C^l G \supset D^l G\) for all \(l \in \mathbb{Z}^+\). Thus every nilpotent group is solvable, but the converse is false.

The following remarkable fact concerning finite dimensional representations of solvable groups can be easily established (c.f. [K2]).

Theorem 2.2. Let \(G\) be a connected solvable locally compact topological group. Suppose \((\pi, \mathcal{H})\) is a finite dimensional irreducible unitary representation of \(G\). Then \(\dim_{\mathbb{C}} \mathcal{H} = 1\).

Corollary 2.3. A compact connected solvable topological group \(G\) is abelian.

Proof. It is well known that every irreducible unitary representation of a compact connected topological group is finite dimensional. Therefore any topologically irreducible unitary representation of \(G\) is finite dimensional. According to Theorem 2.2, it is one-dimensional. Hence \(G\) is abelian.

Remark 2.4. In view of Theorem 2.2, the unitary dual \(\hat{G}\) of a connected solvable locally compact topological group \(G\) consists of two types of equivalence classes of

(I) continuous unitary characters of \(G\);

(II) equivalence classes of infinite dimensional, topological irreducible unitary representations of \(G\).
It is well known that the identity component \( G_0 \) of \( G \) is a nilpotent Lie group if and only if the Lie algebra \( g \) of \( G \) is a nilpotent Lie algebra over \( \mathbb{R} \). We recall the notion of the nilpotency of a Lie algebra \( g \). We define the descending central series \((C^l g)_{l \geq 0}\) of \( g \) recursively via the prescriptions

\[
C^0 g = g, \quad C^{l+1} g = [g, C^l g], \quad l = 0, 1, 2, \ldots .
\]

A Lie algebra \( g \) is said to be nilpotent if there exists a positive integer \( m \in \mathbb{Z}^+ \) such that \( C^m g = \{0\} \). It is well known that if \( G \) is a nilpotent Lie group over \( \mathbb{R} \), then the exponential mapping \( \exp : g \to G \) is a diffeomorphism of \( g \) onto \( G \), where \( g \) denotes the Lie algebra of \( G \) (cf.\([He]\)).

### 3. Representations of a nilpotent Lie group

First we recall that a real Lie group is said to be monomial if each topologically irreducible unitary representation of \( G \) can be unitarily induced by a unitary character of some closed subgroup \( H \) of \( G \).

Using the Mackey machinery, Dixmier and Kirillov proved the following important result:

**Theorem 3.1. (Dixmier-Kirillov)** The simply connected nilpotent Lie groups over \( \mathbb{R} \) are monomial.

**Remark 3.2.** More generally, it can be proved that a simply connected real Lie group whose exponential mapping \( \exp : g = \text{Lie}(G) \to G \) is a global diffeomorphism is monomial.

Now we reformulate Theorem 3.1 at the Lie algebra level. Let \((\pi, \mathcal{H})\) be a topologically irreducible, unitary representation of the simply connected, nilpotent real Lie group \( G \) in the complex Hilbert space \( \mathcal{H} \). Let \( H \) be a connected closed subgroup of \( G \) with a unitary character \( \chi \) such that

\[
(\pi, \mathcal{H}) = \text{Ind}_H^G(\chi, \mathbb{C}).
\]

Let \( g \) be the Lie algebra of \( G \) and let \( h \) the Lie subalgebra of \( g \) corresponding to the Lie subgroup \( H \) of \( G \). Let \( l_0 \) be the differential of \( \chi \).
Then we have the following commutative diagram:

\[
\begin{array}{c}
\mathfrak{h} \\
\exp_H \downarrow \\
H \end{array} \longrightarrow \begin{array}{c}
\mathbb{R} \\
\mathbb{C}^\times_1 \\
\chi
\end{array}
\]

Diagram 3-1.

Here \( \exp_H : \mathfrak{h} \to H \) denotes the exponential mapping of \( \mathfrak{h} \) to \( H \). Thus we have

\[ \chi(\exp_H X) = e^{2\pi i <X,l_0>}, \quad X \in \mathfrak{h} \]

and

\[ <[X,Y],l_0> = 0 \quad \text{for all } X,Y \in \mathfrak{h}. \]

The relation (3.2) follows from the fact that \( \exp_H [X,Y] \in [H,H] \) and \( \chi(g) = 1 \) for all \( g \in [H,H] \). Indeed, according to (3.1), we have

\[ 1 = \chi(\exp_H [X,Y]) = e^{2\pi i<[X,Y],l_0>}, \quad X,Y \in \mathfrak{h}. \]

Therefore \( <[X,Y],l_0> \) is an integer. We have \( <[X,Y],l_0> = 0 \). Otherwise, \( <[X,Y],l_0> = n \in \mathbb{Z} \) with \( n \neq 0 \). Then for sufficiently small positive real numbers \( t \), we have

\[ 1 = \chi(\exp_H [tX,Y]) = e^{2\pi itn} \neq 1 \]

because \( tn \notin \mathbb{Z} \). This leads to the contradiction.

Now we let \( l \in \mathfrak{g}^* \) be any \( \mathbb{R} \)-linear form which extends \( l_0 \) to the whole Lie algebra \( \mathfrak{g} \). Then \( \mathfrak{h} \) forms a totally isotropic vector subspace of \( \mathfrak{g} \) relative to the skew-symmetric \( \mathbb{R} \)-bilinear form \( B_l : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R} \) given by

\[ B_l(X,Y) := <[X,Y],l>, \quad X,Y \in \mathfrak{g} \]

associated with \( l \) on \( \mathfrak{g} \). More precisely, \( B_l|_{\mathfrak{g} \times \mathfrak{g}} = 0 \). In this case we say that the Lie subalgebra \( \mathfrak{h} \) of \( \mathfrak{g} \) is subordinate to \( l \). We introduce the notation

\[ \chi_{l,\mathfrak{h}} := \chi \]

and the monomial representation of \( G \)

\[ (\pi_{l,\mathfrak{h}},\mathcal{H}) := \text{Ind}_H^G(\chi_{l,\mathfrak{h}},\mathbb{C}). \]

Then we obtain the following theorem 3.3 which can be considered as an another version of theorem 7.2 a in [K4]
**Theorem 3.3.** Let $G$ be a simply connected nilpotent real Lie group with Lie algebra $\mathfrak{g}$. Assume that there is given a topologically irreducible unitary representation $(\pi, \mathcal{H})$ of $G$. Then there exists a linear form $l \in \mathfrak{g}^*$ and a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ subordinate to $l$ such that

$$(\pi, \mathcal{H}) = (\pi_{l, \mathfrak{h}}, \mathcal{H}) := \text{Ind}^G_H(\chi_{l, \mathfrak{h}}, \mathbb{C}).$$

We note that $\chi_{l, \mathfrak{h}}$ is a unitary character of $H$ such that

$$(3.6) \quad \chi_{l, \mathfrak{h}}(\exp_H X) = e^{2\pi i \langle X, l \rangle}, \quad X \in \mathfrak{h},$$

where $H$ is the simply connected closed subgroup of $G$ corresponding to a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.

**References**


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