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ON THE LEFT REGULAR *po*-Γ-SEMIGROUPS

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Abstract. We consider the ordered Γ-semigroups in which $x\gamma x$ $(x \in M, \gamma \in \Gamma)$ are left elements. We show that this *po*-Γ-semigroup is left regular if and only if M is a union of left simple sub-Γ-semigroups of M.

The concept of left regular *poe*-semigroups has been introduced in [2] and extends the concept of left regular *po*-semigroups not having the greatest element "e" in [1]. In [5], Lee and Jung showed that a *poe*-semigroup S in which every $x^2 (x \in S)$ is a left ideal element is left regular if and only if there exists a family $\{S_\alpha | \alpha \in Y\}$ of left simple subsemigroups of S such that $S = \cup\{S_\alpha | \alpha \in Y\}$. Recently, Kwon([3]) showed that a *poe*-Γ-semigroup is left regular if and only if M is a union of left simple sub-Γ-semigroups of M.

Now we consider a *po*-Γ-semigroups which does not necessarily have a greatest element "e". In this paper we prove that a *po*-Γ-semigroup M in which every $x\gamma x (x \in M, \gamma \in \Gamma)$ is a left ideal element is left regular if and only if M is a union of left simple sub-Γ-semigroups of M.

M. K. Sen ([6]) introduced Γ-semigroups in 1981. M. K. Sen and N. K. Saha ([7],[8]) introduced Γ-semigroups different from the first definition of Γ-semigroups in the sense of Sen (1981). From Sen ([6]) we recall the following definition of Γ-semigroup.

Let M and Γ be any two non-empty sets. M is called a Γ-*semigroup* if

(1) $M\Gamma M \subseteq M, \Gamma M\Gamma \subseteq \Gamma$.

(2) $(a\gamma b)\mu c = a(\gamma b\mu)c = a\gamma(b\mu c)$

for all $a, b, c \in M$ and $\gamma, \mu \in \Gamma$.

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EXAMPLE 1. Let M be the set of all integers of the form $4n+1$ where n is an integer and Γ denote the set of all integers of the form $4n+3$. If $a\gamma b$ is $a+\gamma+b$, $\gamma a\mu$ is $\gamma+a+\mu$ (usual sum of the integers) for all $a,b\in M$ and $\gamma,\mu\in\Gamma$, then M is a Γ-semigroup.

A *po*-Γ-semigroup(: partially ordered Γ-semigroup)([5]) is an ordered set M at the same time a Γ-semigroup such that:

$$a\leq b\Longrightarrow a\gamma c\leq b\gamma c \text{ and } c\mu a\leq c\mu b$$

$\forall a,b,c\in M$ and $\forall\gamma,\mu\in\Gamma$.

Notation. For subsets A,B of M, let

$$A\Gamma B:=\{a\gamma b|a\in A,b\in B \text{ and } \gamma\in\Gamma\}.$$

DEFINITION 1. Let M be a *po*-Γ-semigroup and A a nonempty subset of M. A is called a *left ideal* of M if
(1) $M\Gamma A\subseteq A$.
(2) $a\in A, b\leq a(b\in M)\Longrightarrow b\in A$.

DEFINITION 2. A *po*-Γ-semigroup M is called *left*(*right*) *regular* if for every $a\in M$ there exists $x\in M$ such that $a\leq x\gamma(a\mu a)$(resp.$a\leq(a\gamma a)\mu x$) for some $\gamma,\mu\in\Gamma$.

DEFINITION 3. Let M be a *po*-Γ-semigroup and T a nonempty subset of M. T is called *semiprime* if $a\in M, a\gamma a\in T(\gamma\in\Gamma)\Longrightarrow a\in T$.

DEFINITION 4. Let M be a *po*-Γ-semigroup. An element t of M is called *semiprime* if $a\in M, a\gamma a\leq t(\gamma\in\Gamma)$ implies $a\leq t$.

DEFINITION 5. Let M be a *po*-Γ-semigroup. A sub-Γ-semigroup T of M is called *left simple* if for every left ideal L of T we have $L=T$.

DEFINITION 6. An element t of M is called a *left ideal element* if $x\gamma x\leq t$ for all $x\in M$ and $\gamma\in\Gamma$.

Notation. Let M be a *po*-Γ-semigroup. For $H\subseteq M$,

$$(H]=\{t\in M|t\leq h \text{ for some } h\in H\}.$$

We denote by $L(x)$ the left ideal of M generated by $x (x \in M)$. For a *po*-Γ-semigroup M we can easily prove that :

$$L(x) = \{t \in M | t \leq x \text{ or } t \leq a\gamma x \text{ for some } a \in M \text{ and } \gamma \in \Gamma\}$$
$$= (x \cup M\Gamma x], \forall x \in M.$$

We define a relation "$\mathcal{L}$" on M as follows:

$$a\mathcal{L}b \iff L(a) = L(b).$$

Then $\mathcal{L}$ is a right congruence on M i.e. it is an equivalence relation on M such that

$$a\mathcal{L}b \Longrightarrow (a\gamma c)\mathcal{L}(b\gamma c), \forall c \in M, \forall \gamma \in \Gamma.$$

Indeed: Let $a\mathcal{L}b$. If $t \in L(a\gamma c)$, then $t \leq a\gamma c$ or $t \leq x\mu(a\gamma c)$ for some $x \in M$ and $\mu \in \Gamma$. Since $a \in L(a) = L(b)$, we have $a \leq b$ or $a \leq y\delta b$ for some $y \in M$ and $\delta \in \Gamma$. If $a \leq b$, then $t \leq b\gamma c$ or $t \leq x\mu(b\gamma c)$ i.e. $t \in L(b\gamma c)$. If $a \leq y\delta b (y \in M, \delta \in \Gamma)$, then $t \leq (y\delta b)\gamma c = y\delta(b\gamma c)$ or $t \leq (x\mu y)\delta(b\gamma c)$ i.e. $t \in L(b\gamma c)$. Thus we have $L(a\gamma c) \subseteq L(b\gamma c)$. By symmetry, $L(b\gamma c) \subseteq L(a\gamma c)$.

Theorem 1. *Let M be a po-Γ-semigroup. The following are equivalent:*

(1) *M is left regular.*
(2) $L(a) \subseteq L(a\gamma a), \forall a \in M, \forall \gamma \in \Gamma.$
(3) $a\mathcal{L}(a\gamma a),\ \forall a \in M, \forall \gamma \in \Gamma.$

Proof. (1) $\Longrightarrow$ (2). Let M be left regular. If $t \in L(a)$, then

$$t \leq a \text{ or } t \leq x\gamma a$$

for some $x \in M$ and $\gamma \in \Gamma$. Since M is left regular, $a \leq y\mu(a\gamma a)$ for some $y \in M$ and $\mu, \gamma \in \Gamma$.

If $t \leq a$, then $t \leq a \leq y\mu(a\gamma a)(y \in M, \mu, \gamma \in \Gamma)$.

If $t \leq x\gamma a$, then $t \leq x\gamma a \leq x\gamma(y\mu a\gamma a) = (x\gamma y)\mu(a\gamma a)$.

In any case, $t \leq z\mu(a\gamma a)$ for some $z \in M$. Hence $t \in L(a\gamma a)$, and so $L(a) \subseteq L(a\gamma a)$.

(2) $\Longrightarrow$ (3). Let $a \in M$. Then

$$\begin{aligned} t \in L(a\gamma a) &\Rightarrow t \leq a\gamma a(\forall\gamma \in \Gamma) \text{ or } t \leq x\mu(a\gamma a)(x \in M, \mu \in \Gamma) \\ &\Rightarrow t \leq z\gamma a \text{ for some } z \in M. \\ &\Rightarrow t \in L(a). \end{aligned}$$

By (2), $L(a) = L(a\gamma a)$. Thus we have $a\mathcal{L}(a\gamma a)(\forall a \in M, \forall\gamma \in \Gamma)$.
(3) $\Longrightarrow$ (1). Let $a \in M$. Since $a\mathcal{L}(a\gamma a)(\forall\gamma \in \Gamma)$, we have

$$a \in L(a) = L(a\gamma a) \Rightarrow a \leq a\gamma a \text{ or } a \leq x\mu(a\gamma a)(\mu \in \Gamma, x \in M).$$

If $a \leq a\gamma a(\gamma \in \Gamma)$, then $a\gamma a \leq a\gamma(a\gamma a)$ and so $a \leq a\gamma(a\gamma a)$. In any case a is left regular, and so M is left regular. □

Theorem 2. *Let M be a po-Γ-semigroup in which every $x\gamma x(x \in M, \gamma \in \Gamma)$ is a left ideal element. The following are equivalent:*

(1) *M is left regular.*
(2) *Every left ideal element of M is semiprime.*
(3) *Every left ideal of M is semiprime.*

Proof. (1) $\Longrightarrow$ (2). Let t be a left ideal element of $M, a \in M$ and $a\gamma a \leq t(\gamma \in \Gamma)$. Since M is left regular, $a \leq x\mu(a\gamma a) \leq x\mu t \leq t(x \in M, \mu \in \Gamma)$. Thus t is semiprime.

(2) $\Longrightarrow$ (3). Let L be a left ideal of $M, a \in M$ and $a\gamma a \in L(\gamma \in \Gamma)$. Since $a\gamma a \leq a\gamma a$, and $a\gamma a$ is a left ideal element and so it is semiprime, we have $a \leq a\gamma a(\gamma \in \Gamma)$. And since L is a left ideal, $a \in L$. Hence L is semiprime.

(3) $\Longrightarrow$ (1). Let $a \in M$. Since a left ideal $L(a\gamma a)(\gamma \in \Gamma)$ is semiprime and $a\gamma a \in L(a\gamma a)$, we have $a \in L(a\gamma a)$. Thus $L(a) \subseteq L(a\gamma a)(\gamma \in \Gamma)$. By Theorem 1, M is left regular. □

Theorem 3. *Let M be a po-Γ-semigroup in which every $a\gamma a(a \in M, \gamma \in \Gamma)$ is a left ideal element. Then we have that M is left regular if and only if there exists a family $\{M_\alpha | \alpha \in Y\}$ of left simple sub-Γ-semigroups of M such that $M = \cup\{M_\alpha | \alpha \in Y\}$.*

Proof. Assume that M is left regular. We denote by $\mathcal{L}(x)$ the $\mathcal{L}$-class of M containing $x(x \in M)$.

Then $\mathcal{L}(x)$ is a left simple sub-Γ-semigroup of $M, \forall x \in M$.
In fact, since $x \in \mathcal{L}(x)$, $\mathcal{L}(x)$ is nonempty.

Let $a, b \in \mathcal{L}(x)$. Then $a\mathcal{L}x$ and $x\mathcal{L}b$. Since $\mathcal{L}$ is a right congruence on M, we have $(a\gamma b)\mathcal{L}(x\gamma b)$ and $(x\gamma b)\mathcal{L}(b\gamma b)$ $\forall \gamma \in \Gamma$. Since M is left regular, by Theorem 1, $(b\gamma b)\mathcal{L}b$. Hence we have $(a\gamma b)\mathcal{L}b$ and so $a\gamma b \in \mathcal{L}(b) = \mathcal{L}(x)(\forall \gamma \in \Gamma)$. Thus $\mathcal{L}(x)$ is sub-Γ-semigroup of M.

Let L be a left ideal of $\mathcal{L}(x)$ and $z \in L$. If $y \in \mathcal{L}(x)$, then $z \in L \subseteq \mathcal{L}(x) = \mathcal{L}(y)$. Since M is left regular, by Theorem 1, we have $y \in L(y) = L(z) = L(z\gamma z)(\forall \gamma \in \Gamma)$. Then $y \leq z\gamma z$ or $y \leq t\mu(z\gamma z)(t \in M, \gamma, \mu \in \Gamma)$.

If $y \leq z\gamma z$ then, since L is a left ideal of $\mathcal{L}(x)$, we have $y \leq z\gamma z \in \mathcal{L}(x)\Gamma L \subseteq L$, and $y \in L$. And if $y \leq t\mu(z\gamma z)$, then $y \leq z\gamma z$ since every $z\gamma z(z \in M, \gamma \in \Gamma)$ is a left ideal element. In any case, $y \in L$ and so $L = \mathcal{L}(x)$. Hence every $\mathcal{L}(x)$ is a left simple sub-Γ-semigroup of M. Now $M = \cup\{\mathcal{L}(x) | x \in M\}$.

Conversely, suppose that $M = \cup\{M_\alpha | \alpha \in Y\}$ where M_α is a left simple sub-Γ-semigroup of $M, \forall \alpha \in Y$. Let L be a left ideal of $M, a \in M$ and $a\gamma a \in L(\gamma \in \Gamma)$. Then $a \in M_\alpha$ for some $\alpha \in Y$ and $L \cap M_\alpha$ is a left ideal of M_α.

Indeed: Since $a\gamma a \in L$ and $a\gamma a \in M_\alpha$, $L \cap M_\alpha$ is nonempty and $L \cap M_\alpha \subseteq M_\alpha$. Furthermore

$$M_\alpha\Gamma(L \cap M_\alpha) \subseteq M_\alpha\Gamma L \cap M_\alpha\Gamma M_\alpha \subseteq M\Gamma L \cap M_\alpha \subseteq L \cap M_\alpha.$$

Let $x \in I \cap M_\alpha$ and $y \leq x(y \in M_\alpha)$. Since $x \in L$ and $y \leq x$, $y \in L$. Thus $y \in L \cap M_\alpha$. Hence $L \cap M_\alpha$ is a left ideal of M_α. Since M_α is left simple, we have $I \cap M_\alpha = M_\alpha$, and $a \in L$. Hence L is semiprime. By Theorem 2, M is left regular. □

REMARK. If $\mathcal{L}(x)$ is a sub-Γ-semigroup of $M, \forall x \in M$, then M is left regular.

In fact: Since $x\gamma x \in \mathcal{L}(x)(\gamma \in \Gamma)$, we have $(x\gamma x)\mathcal{L}x, \forall x \in M$. By Lemma, M is left regular.

THEOREM 4. *For a po-Γ-semigroup M, the following conditions are equivalent:*

(1) *M is left regular.*
(2) *Every $\mathcal{L}$-class of M is a left simple sub-Γ-semigroup of M.*
(3) *Every $\mathcal{L}$-class of M is a sub-Γ-semigroup of M.*
(4) *M is a union of disjoint left simple sub-Γ-semigroups of M.*
(5) *M is a union of left simple sub-Γ-semigroups of M.*

Proof. From the proof of the Theorem 3, we have (1) $\Rightarrow$ (2) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (1). On the other hand, (2) $\Rightarrow$ (3) is obvious and (3) $\Rightarrow$ (1) by the Remark. □

References

1. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups Vol. I*, Amer. Math. Soc., Math. Surveys 7 ,Providence (1964).
2. N. Kehayoupulu, *On intra-regular Ve-semigroups*, Semigroup Forum **19** (1980), 111-121.
3. Y. I. Kwon, *The left regular ordered* Γ*-semigroups*, preprint.
4. Y. I. Kwon and S. K. Lee, *Some special elements on ordered* Γ*-semigroups*, Kyungpook Math. J. **35(3)** (1996), 679-685.
5. S. K. Lee and J. H. Jung, *On the left regular po-semigroups*, Comm.Korea Math.Soc., **13** (1997), 1-5.
6. M. K. Sen, *On* Γ*-semigroups*, Proc. of the Int. Conf. on Algebra and it's Appl., Decker Publication, New York 301.
7. M. K. Sen and N. K. Saha, *On* Γ*-semigroup I*, Bull. Cal. Math. Soc., **78** (1986), 180-186.
8. N. K. Saha, *On* Γ*-semigroup II*, Bull. Cal. Math. Soc., **79** (1987), 331-335.

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