

REMARKS ON THE REIDEMEISTER NUMBER OF A G -MAP

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ABSTRACT. For a G -map $\phi : X \rightarrow X$, we define and characterize the Reidemeister number $R_G(\phi)$ of ϕ . Also, we prove that $R_G(\phi)$ is a G -homotopy invariance and we obtain a lower bound of $R_G(\phi)$.

1. Introduction

For a self map ϕ of a compact connected polyhedron X , let $\pi_1(X, x_0)$ and $L(\phi)$ denote the fundamental group of X based at $x_0 \in X$ and the set of all liftings of ϕ on a universal covering space, respectively. For two $\tilde{f}_1, \tilde{f}_2 \in L(\phi)$, set $\tilde{f}_1 \sim \tilde{f}_2$ if there exists an $[\alpha] \in \pi_1(X, x_0)$ such that $\tilde{f}_2 = [\alpha] \circ \tilde{f}_1 \circ [\alpha]^{-1}$. This is an equivalence relation on $L(\phi)$. The cardinal number of the set of all equivalence classes is called the Reidemeister number of ϕ and denoted by $R(\phi)$.

The following theorem is well known.

THEOREM. (1) *The number $R(\phi)$ is a homotopy invariance.*

(2) *There exists a homomorphism $\tilde{f}_\#$ of $\pi_1(X, x_0)$ induced by an element \tilde{f} of $L(\phi)$. For any two $[\alpha], [\beta] \in \pi_1(X, x_0)$, set $[\alpha] \sim [\beta]$ if there exists a $[\gamma] \in \pi_1(X, x_0)$ such that $[\beta] = [\gamma] \circ [\alpha] \circ \tilde{f}_\#([\gamma]^{-1})$. Then this is an equivalence relation on $\pi_1(X, x_0)$ and the cardinal number of the set of equivalence classes is equal to $R(\phi)$.*

(3) *There exists a homomorphism $H_1(\phi)$ of the 1-st homology group $H_1(X)$ of X induced by ϕ such that $R(\phi) \geq |\text{Coker}(i_{H_1(X)} \cdot H_1(\phi)')|$, where $i_{H_1(X)} \cdot H_1(\phi)'$ is a function of $H_1(X)$ defined by $(i_{H_1(X)} \cdot H_1(\phi)'([z])) = [z] \cdot H_1(\phi)([z]^{-1})$ for $[z] \in H_1(X)$.*

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The purpose of this note is to define the Reidemeister number $R_G(\phi)$ of a G -map ϕ and to generalize the above theorem to the case of a G -map.

We shall assume throughout this note that X is a connected, locally path connected, and semi-locally 1-connected space, that G is a topological group acting effectively on X , and that $\phi : X \rightarrow X$ is a G -map.

2. Main results

Let $p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$ be a based universal covering projection. For each $g \in G$, let $\theta_g : X \rightarrow X$ be the homeomorphism defined by $\theta_g(x) = gx$ for $x \in X$. Since \tilde{X} is simply connected, the map $\theta_g \circ \phi$ of X can be covered by maps of \tilde{X} . Let $L(\theta_g \circ \phi)$ be the set of all such liftings of $\theta_g \circ \phi$ and let $\mathcal{L}(\phi)$ be the disjoint union of the collection $\{L(\theta_g \circ \phi) | g \in G\}$. Clearly, $\mathcal{L}(i_X)$ is a subgroup of the group of all homeomorphisms of \tilde{X} .

DEFINITION 1. Two liftings $\tilde{f}_1, \tilde{f}_2 \in \mathcal{L}(\phi)$ are said to be *conjugate* if there exists $\tilde{l} \in \mathcal{L}(i_X)$ such that $\tilde{f}_2 = \tilde{l} \circ \tilde{f}_1 \circ \tilde{l}^{-1}$. The equivalence classes by conjugacy are called *G -lifting classes* and the G -lifting classes of \tilde{f} is denoted by $[\tilde{f}]_G$. If $\mathcal{L}'(\phi)$ is the set of all G -lifting classes, then $R_G(\phi) = |\mathcal{L}'(\phi)|$, the order of $\mathcal{L}'(\phi)$, is called the *Reidemeister number* of ϕ .

THEOREM 2. *If two maps $\phi_1, \phi_2 : X \rightarrow X$ are G -homotopic, then $R_G(\phi_1) = R_G(\phi_2)$.*

Proof. Let $H : X \times I \rightarrow X$ be a G -homotopy from ϕ_1 to ϕ_2 . If $\tilde{f} \in L(\theta_g \circ \phi_1)$, then $(p \circ \tilde{f})(\tilde{x}) = (\theta_g \circ \phi_1)(p(\tilde{x})) = (\theta_g \circ H)(p(\tilde{x}), 0)$ for all $\tilde{x} \in \tilde{X}$. Since $\tilde{X} \times I$ is connected, there exists a unique lifting $\tilde{F} : \tilde{X} \times I \rightarrow \tilde{X}$ of $\theta_g \circ H$ such that $\tilde{F}(\cdot, 0) = \tilde{f}$ and $p \circ \tilde{F} = \theta_g \circ H \circ (p \times i_I)$. Let $\tilde{F}(\cdot, 1) = \tilde{f}'$. Since $\tilde{f}' \in L(\theta_g \circ \phi_2)$, we have a function $\Phi_g : L(\theta_g \circ \phi_1) \rightarrow L(\theta_g \circ \phi_2)$ defined by $\Phi_g(\tilde{f}) = \tilde{f}'$.

Claim 1. Φ_g is one-to-one: Let \tilde{f}_1 and \tilde{f}_2 be distinct elements of $L(\theta_g \circ \phi_1)$ and let \tilde{F}_1 and \tilde{F}_2 be the liftings of $\theta_g \circ H$ such that $\tilde{F}_1(\cdot, 0) = \tilde{f}_1$ and $\tilde{F}_2(\cdot, 0) = \tilde{f}_2$. If $\tilde{F}_1(\tilde{x}_0, 1) = \tilde{F}_2(\tilde{x}_0, 1)$, then $\tilde{F}_1 = \tilde{F}_2$. This is a contradiction. Thus $\Phi_g(\tilde{f}_1) = \tilde{F}_1(\cdot, 1) \neq \tilde{F}_2(\cdot, 1) = \Phi_g(\tilde{f}_2)$.

Claim 2. Φ_g is onto: Let $f' \in L(\theta_g \circ \phi_2)$. If we define $H' : X \times I \rightarrow X$ by $H'(x, t) = H(x, 1 - t)$ for $(x, t) \in X \times I$, then $\theta_g \circ H'$ is a homotopy from $\theta_g \circ \phi_2$ to $\theta_g \circ \phi_1$. Let $\tilde{F}' : \tilde{X} \times I \rightarrow \tilde{X}$ be the lifting of $\theta_g \circ H'$ with $\tilde{F}'(\cdot, 0) = f'$. Define $\tilde{F} : \tilde{X} \times I \rightarrow \tilde{X}$ by $\tilde{F}(\tilde{x}, t) = \tilde{F}'(\tilde{x}, 1 - t)$ for $(\tilde{x}, t) \in \tilde{X} \times I$. Since $p \circ \tilde{F}(\tilde{x}, t) = p \circ \tilde{F}'(\tilde{x}, 1 - t) = \theta_g \circ H'(p(\tilde{x}), 1 - t) = \theta_g \circ H(p(\tilde{x}), t)$ for $(\tilde{x}, t) \in \tilde{X} \times I$ and $\tilde{F}(\tilde{x}, 0) \in L(\theta_g \circ \phi_1)$, we have \tilde{F} is the lifting of $\theta_g \circ H$ such that $\tilde{F}(\tilde{x}, 1) = f' = \Phi_g(\tilde{F}(\tilde{x}, 0))$.

Let $\Phi : \mathcal{L}(\phi_1) \rightarrow \mathcal{L}(\phi_2)$ be the function whose restriction to $L(\theta_g \circ \phi_1)$ is equal to Φ_g for every $g \in G$. By Claim 1 and Claim 2, Φ is one-to-one and onto.

Claim 3. For any $\tilde{l} \in \mathcal{L}(i_X)$ and any $\tilde{f} \in \mathcal{L}(\phi_1)$, $\Phi(\tilde{f} \circ \tilde{l}) = \Phi(\tilde{f}) \circ \tilde{l}$ and $\Phi(\tilde{l} \circ \tilde{f}) = \tilde{l} \circ \Phi(\tilde{f})$: Assume $\tilde{l} \in L(\theta_{g'} \circ i_X)$ and $\tilde{f} \in L(\theta_g \circ \phi_1)$. Clearly, $\tilde{f} \circ \tilde{l} \in L(\theta_{gg'} \circ \phi_1)$. Let \tilde{F} and \tilde{K} be the liftings of $\theta_g \circ H$ and $\theta_{gg'} \circ H$, respectively, such that $\tilde{F}(\cdot, 0) = \tilde{f}$ and $\tilde{K}(\cdot, 0) = \tilde{f} \circ \tilde{l}$. Define $\tilde{F}' : \tilde{X} \times I \rightarrow \tilde{X}$ by $\tilde{F}'(\tilde{x}, t) = \tilde{F}(\tilde{l}(\tilde{x}), t)$ for $(\tilde{x}, t) \in \tilde{X} \times I$. Because H is a G -homotopy, \tilde{F}' is the lifting of $\theta_{gg'} \circ H$ with $\tilde{F}'(\cdot, 0) = \tilde{f} \circ \tilde{l}$. By the uniqueness of lifting, $\tilde{F}' = \tilde{K}$. Therefore, $\Phi(\tilde{f} \circ \tilde{l}) = \tilde{K}(\cdot, 1) = \tilde{F}'(\cdot, 1) = \tilde{F}(\tilde{l}(\cdot), 1) = \Phi(\tilde{f}) \circ \tilde{l}$. Similarly, we can prove $\Phi(\tilde{l} \circ \tilde{f}) = \tilde{l} \circ \Phi(\tilde{f})$.

Now, assume that for $\tilde{f}_1, \tilde{f}_2 \in \mathcal{L}(\phi_1)$, $\tilde{f}_2 = \tilde{l} \circ \tilde{f}_1 \circ \tilde{l}^{-1}$. By Claim 3, $\Phi(\tilde{f}_2) = \Phi(\tilde{l} \circ \tilde{f}_1 \circ \tilde{l}^{-1}) = \tilde{l} \circ \Phi(\tilde{f}_1 \circ \tilde{l}^{-1}) = \tilde{l} \circ \Phi(\tilde{f}_1) \circ \tilde{l}^{-1}$. Thus Φ induces a one-to-one onto function from $\mathcal{L}'(\phi_1)$ to $\mathcal{L}'(\phi_2)$, and hence we have $R_G(\phi_1) = R_G(\phi_2)$. \square

LEMMA 3. For any fixed $\tilde{f}_0 \in L(\theta_{g_0} \circ \phi)$, there exists a function $\Psi : \mathcal{L}(\phi) \rightarrow \mathcal{L}(i_X)$ induced by \tilde{f}_0 such that $\Psi(\tilde{l} \circ \tilde{f}_0) = \tilde{l}$ for every $\tilde{l} \in \mathcal{L}(i_X)$.

Proof. Assume $\tilde{f} \in L(\theta_g \circ \phi)$. Since $(\theta_{gg_0^{-1}} \circ p)(\tilde{f}_0(\tilde{x}_0)) = (p \circ \tilde{f})(\tilde{x}_0)$, there exists a unique map $\tilde{l} \in L(\theta_{gg_0^{-1}} \circ i_X)$ such that $\tilde{l}(\tilde{f}_0(\tilde{x}_0)) = \tilde{f}(\tilde{x}_0)$. Obviously, $\tilde{l} \circ \tilde{f}_0 \in L(\theta_g \circ \phi)$. By the uniqueness of lifting, $\tilde{l} \circ \tilde{f}_0 = \tilde{f}$. This induces a function $\Psi_g : L(\theta_g \circ \phi) \rightarrow L(\theta_{gg_0^{-1}} \circ i_X)$ defined by $\Psi_g(\tilde{f}) = \tilde{l}$, and hence we have a function $\Psi : \mathcal{L}(\phi) \rightarrow \mathcal{L}(i_X)$ whose restriction to $L(\theta_g \circ \phi)$ is equal to Ψ_g for every $g \in G$. Now, let $\tilde{l} \in L(\theta_{g'} \circ i_X)$. Then $\Psi(\tilde{l} \circ \tilde{f}_0) \in L(\theta_{g'g_0g_0^{-1}} \circ i_X) = L(\theta_{g'} \circ i_X)$ because $\tilde{l} \circ \tilde{f}_0 \in L(\theta_{g'g_0} \circ \phi)$. Since $\tilde{l} \circ \tilde{f}_0 = \Psi(\tilde{l} \circ \tilde{f}_0) \circ \tilde{f}_0$, we have $\Psi(\tilde{l} \circ \tilde{f}_0) = \tilde{l}$. \square

LEMMA 4. Let $\tilde{f} \in \mathcal{L}(\phi)$ be fixed. Then there exists a homomorphism $\tilde{f}_\# : \mathcal{L}(i_X) \rightarrow \mathcal{L}(i_X)$ induced by \tilde{f} .

Proof. By the above lemma, there exists a function $\Psi : \mathcal{L}(\phi) \rightarrow \mathcal{L}(i_X)$ induced by \tilde{f} such that $\Psi(\tilde{f} \circ \tilde{l}) \circ \tilde{f} = \tilde{f} \circ \tilde{l}$ for every $\tilde{l} \in \mathcal{L}(i_X)$. This induces a homomorphism $\tilde{f}_\# : \mathcal{L}(i_X) \rightarrow \mathcal{L}(i_X)$ defined by $\tilde{f}_\#(\tilde{l}) = \Psi(\tilde{f} \circ \tilde{l})$. In fact, if $\tilde{l}_1, \tilde{l}_2 \in \mathcal{L}(i_X)$, then

$$\begin{aligned} \Psi(\tilde{f} \circ (\tilde{l}_1 \circ \tilde{l}_2)) &= \Psi((\tilde{f} \circ \tilde{l}_1) \circ \tilde{l}_2) \\ &= \Psi(\Psi(\tilde{f} \circ \tilde{l}_1) \circ (\tilde{f} \circ \tilde{l}_2)) \\ &= \Psi(\Psi(\tilde{f} \circ \tilde{l}_1) \circ \Psi(\tilde{f} \circ \tilde{l}_2) \circ \tilde{f}) \\ &= (\Psi(\tilde{f} \circ \tilde{l}_1) \circ \Psi(\tilde{f} \circ \tilde{l}_2)), \end{aligned}$$

so $\tilde{f}_\#(\tilde{l}_1 \circ \tilde{l}_2) = \Psi(\tilde{f} \circ (\tilde{l}_1 \circ \tilde{l}_2)) = \Psi(\tilde{f} \circ \tilde{l}_1) \circ \Psi(\tilde{f} \circ \tilde{l}_2) = \tilde{f}_\#(\tilde{l}_1) \circ \tilde{f}_\#(\tilde{l}_2)$. \square

THEOREM 5. For any two $\tilde{l}_1, \tilde{l}_2 \in \mathcal{L}(i_X)$, set $\tilde{l}_1 \sim \tilde{l}_2$ if there exists an $\tilde{l} \in \mathcal{L}(i_X)$ such that $\tilde{l}_2 = \tilde{l} \circ \tilde{l}_1 \circ \tilde{f}_\#(\tilde{l}^{-1})$. This is an equivalence relation on $\mathcal{L}(i_X)$. Let $[\tilde{l}]_{\tilde{f}}$ be the equivalence class of \tilde{l} . If $\mathcal{L}'(i_X)$ is the set of all equivalence classes, then $R_G(\phi) = |\mathcal{L}'(i_X)|$.

Proof. Define $\Psi' : \mathcal{L}(i_X) \rightarrow \mathcal{L}(\phi)$ by $\Psi'(\tilde{l}) = \tilde{l} \circ \tilde{f}$ for $\tilde{l} \in \mathcal{L}(i_X)$. Since $(\Psi \circ \Psi')(\tilde{l}) = \Psi(\tilde{l} \circ \tilde{f}) = \tilde{l}$ for $\tilde{l} \in \mathcal{L}(i_X)$ and $(\Psi' \circ \Psi)(f') = \Psi(f') \circ \tilde{f} = f'$ for $f' \in \mathcal{L}(\phi)$, the map Ψ is one-to-one and onto.

Let $\tilde{l}_1, \tilde{l}_2 \in \mathcal{L}(i_X)$. Then $[\tilde{l}_1 \circ \tilde{f}]_G = [\tilde{l}_2 \circ \tilde{f}]_G$ if and only if there exists an $\tilde{l} \in \mathcal{L}(i_X)$ such that

$$\begin{aligned} \tilde{l}_2 \circ \tilde{f} &= \tilde{l} \circ (\tilde{l}_1 \circ \tilde{f}) \circ \tilde{l}^{-1} \\ &= (\tilde{l} \circ \tilde{l}_1) \circ (\tilde{f} \circ \tilde{l}^{-1}) \\ &= (\tilde{l} \circ \tilde{l}_1) \circ (\tilde{f}_\#(\tilde{l}^{-1}) \circ \tilde{f}) \end{aligned}$$

if and only if $\tilde{l}_2 = \tilde{l} \circ \tilde{l}_1 \circ \tilde{f}_\#(\tilde{l}^{-1})$ if and only if $[\tilde{l}_2]_{\tilde{f}} = [\tilde{l}_1]_{\tilde{f}}$. Thus Ψ induces a one-to-one correspondence between $\mathcal{L}'(\phi)$ and $\mathcal{L}'(i_X)$. \square

In [4], F. Rhodes defined the fundamental group $\sigma(X, x_0, G)$ of a G -space X , which is a generalization of the concept of the fundamental group of a topological space.

LEMMA 6. $\mathcal{L}(i_X)$ is isomorphic to $\sigma(X, x_0, G)$

Proof. For any $\tilde{l} \in L(\theta_g \circ i_X) \subset \mathcal{L}(i_X)$, choose a path $\tilde{\gamma}$ in \tilde{X} from \tilde{x}_0 to $\tilde{l}(\tilde{x}_0)$. Define $\iota : \mathcal{L}(i_X) \rightarrow \sigma(X, x_0, G)$ by $\iota(\tilde{l}) = [p\tilde{\gamma}; g]$. Since \tilde{X} is simply connected, ι is well defined. To show that ι is a homomorphism, let $\tilde{l}_1 \in L(\theta_{g_1} \circ i_X)$ and $\tilde{l}_2 \in L(\theta_{g_2} \circ i_X)$. If $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are paths from \tilde{x}_0 to $\tilde{l}_1(\tilde{x}_0)$ and $\tilde{l}_2(\tilde{x}_0)$, respectively, then $\tilde{l}_2 \circ \tilde{\gamma}_1$ is a path in \tilde{X} from $\tilde{l}_2(\tilde{x}_0)$ to $(\tilde{l}_2 \circ \tilde{l}_1)(\tilde{x}_0)$ and $p(\tilde{l}_2 \circ \tilde{\gamma}_1) = g_2(p \circ \tilde{\gamma}_1)$. Thus, if $\tilde{\gamma}_3$ is a path from \tilde{x}_0 to $(\tilde{l}_2 \circ \tilde{l}_1)(\tilde{x}_0)$, we have $\iota(\tilde{l}_2 \circ \tilde{l}_1) = [p \circ \tilde{\gamma}_3; g_2 g_1] = [p \circ \tilde{\gamma}_2 * g_2(p \circ \tilde{\gamma}_1); g_2 g_1] = [p \circ \tilde{\gamma}_2; g_2] * [p \circ \tilde{\gamma}_1; g_1]$. This shows that ι is a homomorphism.

Now, consider the following diagram

$$\begin{array}{ccccccccc} 0 & \rightarrow & \pi_1(X, x_0) & \rightarrow & \mathcal{L}(i_X) & \rightarrow & G & \rightarrow & 0 \\ & & \downarrow & & \iota \downarrow & & \downarrow & & \\ 0 & \rightarrow & \pi_1(X, x_0) & \rightarrow & \sigma(X, x_0, G) & \rightarrow & G & \rightarrow & 0, \end{array}$$

where each map is defined naturally. It is easy to show that each square is commutative and each row is exact. By the five lemma, ι is an isomorphism. \square

Let $\phi_{\#} = \iota \circ \tilde{f}_{\#} \circ \iota^{-1}$. Then $\phi_{\#}$ is a homomorphism of $\sigma(X, x_0, G)$. By Lemma 6, it is possible to restate Theorem 5 as follows:

THEOREM 7. For any two $[\alpha_1; g_1], [\alpha_2; g_2] \in \sigma(X, x_0, G)$, set $[\alpha_1; g_1] \sim [\alpha_2; g_2]$ if there exists a $[\beta; h] \in \sigma(X, x_0, G)$ such that $[\alpha_2; g_2] = [\beta; h] * [\alpha_1; g_1] * \phi_{\#}([\beta; h]^{-1})$. This is an equivalence relation on $\sigma(X, x_0, G)$ and the order of the set of equivalence classes is equal to $R_G(\phi)$.

THEOREM 8. If $\sigma(X, x_0, G)$ is abelian, then the subset $N = \{[\beta; h] * \phi_{\#}([\beta; h]^{-1}) \mid [\beta; h] \in \sigma(X, x_0, G)\}$ is a normal subgroup of $\sigma(X, x_0, G)$ and $R_G(\phi) = |\sigma(X, x_0, G)/N|$.

Proof. It is clear that N is a subgroup of $\sigma(X, x_0, G)$, and hence normal. Let $[\alpha; g] \in \sigma(X, x_0, G)$. From the fact that

$$\begin{aligned} & \{[\beta; h] * [\alpha; g] * \phi_{\#}([\beta; h]^{-1}) \mid [\beta; h] \in \sigma(X, x_0, G)\} \\ &= \{[\alpha; h] * ([\beta; g] * \phi_{\#}([\beta; h]^{-1})) \mid [\beta; h] \in \sigma(X, x_0, G)\} \\ &= [\alpha; g] * N, \end{aligned}$$

we have

$$R_G(\phi) = |\{[\alpha; g] * N | [\alpha; g] \in \sigma(X, x_0, G)\}| = |\sigma(X, x_0, G)/N|. \quad \square$$

DEFINITION 9. ([3]) Let $f : G \rightarrow G$ be a group homomorphism. If there exists a positive integer n such that $f^n(G)$ is abelian, then f is said to be *eventually commutative*.

Clearly, every homomorphism of an abelian group is eventually commutative.

DEFINITION 10. ([3]) For any group (G, \cdot) and any homomorphism $f : G \rightarrow G$, the *Reidemeister operator* of f on G is the left action of G on itself given by

$$(g_1, g_2) \rightarrow g_1 \cdot g_2 \cdot f(g_1^{-1}).$$

We write the set of orbits of the operation as $Coker(i_G \cdot f')$ with elements $\langle g \rangle$ for $g \in G$.

LEMMA 11. ([3]) Let $q : A \rightarrow B$ be a surjective homomorphism. If the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & A \\ q \downarrow & & q \downarrow \\ B & \xrightarrow{g} & B \end{array}$$

of groups and homomorphisms is commutative, then $|Coker(i_A \circ f)| \geq |Coker(i_B \cdot g)|$.

LEMMA 12. Let i_σ be the identity on $\sigma(X, x_0, G)$. Then $R_G(\phi) = |Coker(i_\sigma \cdot \phi'_\#)|$.

Proof. If $j : \sigma(X, x_0, G) \rightarrow Coker(i_\sigma \cdot \phi'_\#)$ is the function defined by $j([\alpha; g]) = \langle [\alpha; g] \rangle$ for $[\alpha; g] \in \sigma(X, x_0, G)$, then $j([\alpha_1; g_1]) = j([\alpha_2; g_2])$ if and only if there exists a $[\beta; h] \in \sigma(X, x_0, G)$ such that $[\alpha_2; g_2] = [\beta; h] * [\alpha_1; g_1] * \phi'_\#([\beta; h]^{-1})$. By Theorem 7, the desired result follows. \square

THEOREM 13. *Let $H_1(\phi_{\#}) : H_1(\sigma(X, x_0, G)) \rightarrow H_1(\sigma(X, x_0, G))$ be the homomorphism induced by $\phi_{\#} : \sigma(X, x_0, G) \rightarrow \sigma(X, x_0, G)$, where H_1 is the first integral homology functor from groups to abelian groups. Then*

$$R_G(\phi) \geq |\text{Coker}(i_{H_1(\sigma)} \cdot H_1(\phi_{\#})')|,$$

where $i_{H_1(\sigma)}$ is the identity on $H_1(\sigma(X, x_0, G))$. Furthermore, if $\phi_{\#}$ is eventually commutative, then

$$R_G(\phi) = |\text{coker}(i_{H_1(\sigma)} \cdot H_1(\phi_{\#})')|.$$

Proof. Let $q : \sigma(X, x_0, G) \rightarrow H_1(\sigma(X, x_0, G))$ be the natural projection. Since the following diagram commutes

$$\begin{array}{ccc} \sigma(X, x_0, G) & \xrightarrow{\phi_{\#}} & \sigma(X, x_0, G) \\ q \downarrow & & q \downarrow \\ H_1(\sigma(X, x_0, G)) & \xrightarrow{H_1(\phi_{\#})} & H_1(\sigma(X, x_0, G)), \end{array}$$

we have, by Lemma 11 and Lemma 12,

$$R_G(\phi) = |\text{Coker}(i_{\sigma} \cdot \phi'_{\#})| \geq |\text{Coker}(i_{H_1(\sigma)} \cdot H_1(\phi_{\#})')|.$$

In the case $\phi_{\#}$ is eventually commutative, the result follows from Corollary 1.15 of [3]. \square

REMARK. It is well-known that for a group G , $H_1(G)$ is isomorphic to G/G' where G' is the commutator subgroup of G . Thus, in the above theorem, $H_1(\sigma(X, x_0, G))$ and $\text{Coker}(i_{H_1(\sigma)} \cdot H_1(\phi_{\#})')$ can be replaced by $\sigma(X, x_0, G)/\sigma'(X, x_0, G)$ and $\text{Coker}(i_{\sigma/\sigma'} \cdot (\bar{\phi}_{\#})')$, respectively, where $i_{\sigma/\sigma'}$ is the identity homomorphism on $\sigma(X, x_0, G)/\sigma'(X, x_0, G)$ and $\bar{\phi}_{\#}$ is the homomorphism of $\sigma(X, x_0, G)/\sigma'(X, x_0, G)$ induced by $\phi_{\#}$.

THEOREM 14. *If $N = \{[\alpha; g] * \phi_{\#}([\alpha; g]^{-1}) | [\alpha; g] \in \sigma(X, x_0, G)\}$, then*

$$R_G(\phi) \geq |\text{Coker}(i_{H_1(\sigma)} \cdot H_1(\phi_{\#})')| = |\sigma(X, x_0, G)/\sigma'(X, x_0, G)N|.$$

Proof. By the same method as in the proof of Lemma 5 in [2], we can prove that $\sigma'(X, x_0, G)N$ is a normal subgroup of $\sigma(X, x_0, G)$. Consider the commutative diagram in the proof of Theorem 13 and the composition $\eta \circ q$

$$\sigma(X, x_0, G) \xrightarrow{q} H_1(\sigma(X, x_0, G)) \xrightarrow{\eta} \text{Coker}(i_{H_1(\sigma)} \cdot H_1(\phi_{\#})'),$$

where η is the natural projection (Note that $\text{Coker}(i_{H_1(\sigma)} \cdot H_1(\phi_{\#})')$ is a group because $H_1(\sigma(X, x_0, G))$ is abelian). Since $\eta \circ q$ is an epimorphism, $\text{Coker}(i_{H_1(\sigma)} \cdot H_1(\phi_{\#})')$ is isomorphic to $\sigma(X, x_0, G)/\text{Ker}(\eta \circ q)$. Now, applying the method used in the proof of Theorem 3 in [2], we have $\text{Ker}(\eta \circ q) = \sigma'(X, x_0, G)N$. Therefore, we obtain the desired result. \square

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