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AN APPLICATION OF FRACTIONAL DERIVATIVE OPERATOR TO A NEW CLASS OF ANALYTIC AND MULTIVALENT FUNCTIONS

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Abstract. Making use of a certain operator of fractional derivative, a new subclass $L_p(\alpha,\beta,\gamma,\lambda)$ of analytic and p-valent functions is introduced in the present paper. Apart from various coefficient bounds, many interesting and useful properties of this class of functions are given, some of these properties involve, for example, linear combinations and modified Hadamard product of several functions belonging to the class introduced here.

1. Introduction and definitions.

Let S_p denote the class of functions defined by

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}, \qquad p \in N = \{1, 2, \cdots\} \tag{1.1}$$

which are analytic and p-valent in the unit disk $U = \{z : |z| < 1\}$. Also, let T_p denote the subclass of S_p consisting of analytic and p-valent functions of the form

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n}, \qquad (a_{p+n} \geq 0,\ \ p \in N). \tag{1.2}$$

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The object of the present paper is to investigate systematically a new class $L_p(\alpha,\ \beta,\gamma,\lambda)$ of analytic and p-valent functions $f(z)$ belonging to the class T_p and satisfying the condition

$$\left|\frac{\frac{\Gamma(1+p-\lambda)}{\Gamma(1+p)}D^{\lambda-p}D_z^{\lambda}f(z)-1}{\alpha\frac{\Gamma(1+p-\lambda)}{\Gamma(1+p)}D^{\lambda-p}D_z^{\lambda}f(z)+(1-\gamma)}\right|<\beta,\qquad z\in U \tag{1.3}$$

where and throughout this paper, parameters α,β,γ and λ are restricted as follows:

$$0\le\alpha\le 1,\ \ 0<\beta\le 1,\ \ 0\le\gamma<1 \text{ and } 0\le\lambda\le 1.$$

Further, $D_z^{\lambda}f(z)$ denotes the fractional derivative of $f(z)$ of order λ, as defined below, with

$$D_z^0 f(z)=f(z) \text{ and } D_z^1 f(z)=f'(z).$$

We note that such type of classes have been rather extensively studied by Kim and Lee[3], Gupta and Jain[2], Srivastava and Aouf[8] and by Srivastava and Owa[10]. Several essentially equivalent definitions of fractional derivative and fractional integral have been given in the literature(c.f. [1], [6], [7]). We find it to be convenient to restrict ourselves to the following definition used recently by Owa[5] (and also by Srivastava and Owa[9]).

Defintion 1. The fractional integral of order γ is defined, for a function $f(z)$, by

$$D_z^{-\lambda}f(z)=\frac{1}{\Gamma(\lambda)}\int_0^z\frac{f(\xi)}{(z-\xi)^{1-\lambda}}d\zeta \tag{1.4}$$

where $\lambda>0$, $f(z)$ is an analytic function in a simply connected region of the z-plane containg the origin, and the multiplicity of $(z-\xi)^{\lambda-1}$ is removed by requiring $log(z-\xi)$ to be real when $z-\xi>0$.

Definition 2. The fractional derivative of order λ is defined, for a function $f(z)$, by

$$D_z^{\lambda}f(z)=\frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\int_0^z\frac{f(\xi)}{(z-\xi)^{\lambda}}d\xi, \tag{1.5}$$

where $0\le\lambda<1$, $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origion and the multiplicity of $(z-\xi)^{-\lambda}$ is removed as in Definition 1 above.

DEFINITION 3. Under the hypothesis of Definition 2, the fractional derivative of order $n+\lambda$ is defined, for a function $f(z)$, by

$$D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n} D_z^{\lambda} f(z), \qquad (0 \le \lambda < 1,\ n \in N_0). \tag{1.6}$$

In the present paper, we have obtained sharp result, involving coefficients and distortion theorems, and theorems involving modified Hadamard products.

2. Coefficient estimates.

THEOREM 1. *A function $f(z)$ defined by (1.2) is in the class $L_p(\alpha, \beta, \gamma; \lambda)$ if and only if*

$$\sum_{n=1}^{\infty} \frac{\Gamma(n+1+p)\Gamma(1+p-\lambda)}{\Gamma(1+p)\Gamma(n+1+p-\lambda)} (1+\alpha\beta) a_{p+n} \le \beta(\alpha+1-\gamma). \tag{2.1}$$

The result (2.1) is sharp.

Proof. Assume that the inequality (2.1) holds true and let $|z| = 1$. Then we obtain

$$\begin{aligned}
&\left| \frac{\frac{\Gamma(1+p-\lambda)}{\Gamma(1+p)} D^{\lambda-p} D_z^{\lambda} f(z) - 1}{\alpha \frac{\Gamma(1+p-\lambda)}{\Gamma(1+p)} D^{\lambda-p} D_z^{\lambda} f(z) + (1-\gamma)} \right| \\
&\quad = \left| - \sum_{n=1}^{\infty} \frac{\Gamma(n+1+p)\Gamma(1+p-\lambda)}{\Gamma(1+p)\Gamma(n+1+p-\lambda)} a_{p+n} z^n \right| \\
&\qquad - \beta \left| \alpha - \alpha \sum_{n=1}^{\infty} a_{p+n} \frac{\Gamma(n+1-p)\Gamma(1+p-\lambda)}{\Gamma(1+p)\Gamma(n+1+p-\lambda)} z^n + (1-\gamma) \right| \\
&\quad \le \left| \sum_{n=1}^{\infty} \frac{\Gamma(n+1+p)\Gamma(1+p-\lambda)}{\Gamma(1+p)\Gamma(n+1+p-\lambda)} a_{p+n} z^n - \beta(\alpha+1-\gamma) \right. \\
&\qquad \left. + \alpha\beta \sum_{n=1}^{\infty} \frac{\Gamma(n+1+p)\Gamma(1+p-\lambda)}{\Gamma(1+p)\Gamma(n+1+p-\lambda)} a_{p+n} z^n \right| \\
&\quad \le \sum_{n=1}^{\infty} (1+\alpha\beta) \frac{\Gamma(n+1+p)\Gamma(1+p-\lambda)}{\Gamma(1+p)\Gamma(n+1+p-\lambda)} a_{p+n} - \beta(\alpha+1-\gamma) \\
&\quad \le 0.
\end{aligned}$$

Thus we have that $f(z)$ is in the class $L_p(\alpha, \beta, \gamma; \lambda)$.

Conversely, assume that $f(z)$ is in the class $L_p(\alpha, \beta, \gamma; \lambda)$. Then it has

(2.2)
$$\left|\frac{\frac{\Gamma(1+p-\lambda)}{\Gamma(1+p)} D^{\lambda-p} D_z^{\lambda} f(z) - 1}{\alpha\frac{\Gamma(1+p-\lambda)}{\Gamma(1+p)} D^{\lambda-p} D_z^{\lambda} f(z) + (1-\gamma)}\right|$$
$$= \left|\sum_{n=1}^{\infty} \frac{\Gamma(n+1+p)\Gamma(1+p-\lambda)}{\Gamma(1+p)\Gamma(n+1+p-\lambda)} a_{p+n} z^n\right| \left|(\alpha+1-\gamma) - \alpha \sum_{n=1}^{\infty} \frac{\Gamma(n+1+p)\Gamma(1+p-\lambda)}{\Gamma(1+p)\Gamma(n+1+p-\lambda)} a_{p+n} z^n\right|^{-1} < \beta.$$

Since $|Re(z)| \leq |z|$ for any z, we find from (2.2) that

(2.3)
$$Re\left\{\left(\sum_{n=1}^{\infty} \frac{\Gamma(n+1+p)\Gamma(1+p-\lambda)}{\Gamma(1+p)\Gamma(n+1+p-\lambda)} a_{p+n} z^n\right)\left((\alpha+1-\gamma) -\alpha \sum_{n=1}^{\infty} \frac{\Gamma(n+1-p)\Gamma(1+p-\lambda)}{\Gamma(1+p)\Gamma(n+1+p-\lambda)} a_{p+n} z^n\right)\right\}^{-1} < \beta$$

Choose values of z on the real axis so that $\frac{\Gamma(1+p-\lambda)}{\Gamma(1+p)} D_z^{\lambda-p} f(z)$ is real. Upon clearing the denominator in (2.3) and letting $z \to 1^-$ through real values, we have

$$\sum_{n=1}^{\infty} \frac{\Gamma(n+1+p)\Gamma(1+p-\lambda)}{\Gamma(1+p)\Gamma(n+1+p-\lambda)} a_{p+n} \leq \beta(\alpha+1-\gamma) - \alpha\beta \sum_{n=1}^{\infty} \frac{\Gamma(n+1+p)\Gamma(1+p-\lambda)}{\Gamma(1+p)\Gamma(n+1+p-\lambda)} a_{p+n},$$

which gives the desired assertion (2.1).

Finally, we note that the assertion (2.1) of Theorem 1, is sharp, the extremal function being

$$f(z) = z^p - \frac{\beta(\alpha+1-\gamma)\Gamma(1+p)\Gamma(n+1+p-\lambda)}{(1+\alpha\beta)\Gamma(n+1+p)\Gamma(1+p-\lambda)} z^{n+p}. \quad (2.4)$$

□

Corollary 1. *Let the function $f(z)$ given by (1.2) belong to the class $L_p(\alpha, \beta, \gamma; \lambda)$. Then*

$$a_{p+n} \leq \frac{\beta(\alpha+1-\lambda)\Gamma(1+p)\Gamma(n+1+p-\lambda)}{(1+\alpha\beta)\Gamma(n+1+p)\Gamma(1+p-\lambda)}, \tag{2.5}$$

for every integer $n \in N$.

3. Distortion theorem

Theorem 2. *Let the function $f(z)$ defined by (1.2) be in the class $L_(\alpha, \beta, \gamma; \lambda)$. Then*

$$\begin{aligned} |z|^p - |z|^{p+1}\frac{(1+p-\lambda)\beta(\alpha+1-\gamma)}{(1+\alpha\beta)(1+p)} &\\ \leq |f(z)| &\\ \leq |z|^p + |z|^{p+1}\frac{(1+p-\lambda)\beta(\alpha+1-\gamma)}{(1+\alpha\beta)(1+p)} \end{aligned} \tag{3.1}$$

Furthermore

$$\begin{aligned} \frac{\Gamma(1+p)}{\Gamma(1+p-\lambda)}|z|^{p-\lambda} - \frac{\beta(\alpha+1-\gamma)\Gamma(1+p)}{(1+\alpha\beta)\Gamma(1+p-\lambda)}|z|^{p+1-\lambda} &\\ \leq |D_z^\lambda f(z)| &\\ \leq \frac{\Gamma(1+p)}{\Gamma(1+p-\lambda)}|z|^{p-\lambda} + \frac{\beta(\alpha+1-\gamma)\Gamma(1+p)}{(1+\alpha\beta)\Gamma(1+p-\lambda)}|z|^{p+1-\lambda} \end{aligned} \tag{3.2}$$

whenever $z \in U$.

Proof. Since $f(z) \in L_p(\alpha, \beta, \gamma; \lambda)$, in view of Theorem 1, we have

$$\begin{aligned} \frac{(1+p)(1+\alpha\beta)}{(1+p+\lambda)}\sum_{n=1}^{\infty} a_{n+p} &\\ \leq \sum_{n=1}^{\infty}\frac{\Gamma(n+1+p)\Gamma(1+p-\lambda)}{\Gamma(1+p)\Gamma(n+1+p-\lambda)}(1+\alpha\beta)a_{p+n} &\\ \leq \beta(\alpha+1-\gamma), \end{aligned} \tag{3.3}$$

which evidently yields

$$\sum_{n=1}^{\infty} a_{n+p} \leq \frac{\beta(\alpha+1-\gamma)(1+p-\gamma)}{(1+p)(1+\alpha\beta)}.$$

Consequently, we obtain

$$\begin{aligned}|f(z)| &\geq |z|^p - |z|^{p+1}\sum_{n=1}^{\infty} a_{n+p} \\ &\geq |z|^p - |z|^{p+1}\frac{\beta(\alpha+1-\gamma)(1+p-\lambda)}{(1+\alpha\beta)(1+p)},\end{aligned}$$

and

$$\begin{aligned}|f(z)| &\leq |z|^p + |z|^{p+1}\sum_{n=1}^{\infty} a_{n+p} \\ &\leq |z|^p + |z|^{p+1}\frac{\beta(\alpha+1-\gamma)(1+p-\lambda)}{(1+\alpha\beta)(1+p)},\end{aligned}$$

which prove the assertion (3.1).

Next, by using second inequality in (3.3), we observe that

$$\begin{aligned}&\left|z^{\lambda}\frac{\Gamma(1+p-\lambda)}{\Gamma(1+p)}D_z^{\lambda}f(z)\right| \\ &\geq |z|^p - \sum_{n=1}^{\infty}\frac{\Gamma(n+1+p)\Gamma(1+p-\lambda)}{\Gamma(1+p)\Gamma(n+1+p-\lambda)}a_{n+p}|z|^{n+p} \\ &\geq |z|^p - |z|^{p+1}\sum_{n=1}^{\infty}\frac{\Gamma(n+1+p)\Gamma(1+p-\lambda)}{\Gamma(1+p)\Gamma(n+1+p-\lambda)}a_{n+p} \\ &\geq |z|^p - \frac{\beta(\alpha+1-\lambda)}{(1+\alpha\beta)}|z|^{p+1},\end{aligned}$$

and

$$\begin{aligned}
&\left|z^{\lambda}\frac{\Gamma(1+p-\lambda)}{\Gamma(1+p)}D_z^{\lambda}f(z)\right| \\
&\le |z|^p + \sum_{n=1}^{\infty}\frac{\Gamma(n+1+p)\Gamma(1+p-\lambda)}{\Gamma(1+p)\Gamma(n+1+p-\lambda)}a_{n+p}|z|^{n+p} \\
&\le |z|^p + |z|^{p+1}\sum_{n=1}^{\infty}\frac{\Gamma(n+1+p)\Gamma(1+p-\lambda)}{\Gamma(1+p)\Gamma(n+1+p-\lambda)}a_{n+p} \\
&\le |z|^p + \frac{\beta(\alpha+1-\lambda)}{(1+\alpha\beta)}|z|^{p+1},
\end{aligned}$$

which prove the assertion (3.2) of Theorem 2. □

4. Theorems involving modified Hadamard products.

Let $f(z)$ be defined by (1.2), and let

$$g(z) = z^p - \sum_{n=1}^{\infty} b_{p+n}z^{p+n}, \;\; b_{p+n} \ge 0, \;\; p \in N. \tag{4.1}$$

The modified Hadamard product of $f(z)$ and $g(z)$ is defined here by

$$f * g(z) = z^p - \sum_{n=1}^{\infty} a_{p+n}b_{p+n}z^{p+n}. \tag{4.2}$$

We first prove

THEOREM 3. *Let the functions $f_j(z) (j = 1, 2, \cdots, m)$ defined by*
(4.3)

$$f_j(z) = z^p - \sum_{n=1}^{\infty} C_{n+p,j}z^{n+p} \;\; (C_{n+p,j} \ge 0, \;\; j = 1, 2, \cdots, m; \;\; p \in N)$$

be in the class $L_p(\alpha_j, \beta_j, \gamma_j; \lambda)$, $(j = 1, 2, \cdots, m)$, respectively. Also, let

$$\frac{2\lambda}{1+p} + \min_{1\le j\le m}\{\alpha_j\beta_j\} \ge 1.$$

Then

$$f_1 * f_2 * \cdots * f_m(z) \in L_p(\Pi_{j=1}^m\alpha_j, \Pi_{j=1}^m\beta_j, \Pi_{j=1}^m\gamma_j; \lambda). \tag{4.5}$$

Proof. Since $f_j(z) \in L_p(\alpha_j, \beta_j, \gamma_j; \lambda)(j = 1, 2, \cdots, m)$, by using Theorem 1, we have

$$\sum_{n=1}^{\infty} \frac{\Gamma(n+1+p)\Gamma(1+p-\lambda)}{\Gamma(1+p)\Gamma(n+1+p-\lambda)}(1+\alpha_j\beta_j)C_{n+p,j} \leq \beta_j(\alpha_j+1-\gamma_j) \tag{4.6}$$

and

$$\sum_{n=1}^{\infty} C_{n+p,j} \leq \frac{\beta_j(\alpha_j+1-\gamma_j)(1+p-\lambda)}{(1+\alpha_j\beta_j)(1+p)} \tag{4.7}$$

for each $j = 1, 2, \cdots, m$. Using (4.6) for any j_0 and (4.7) for the rest we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{\Gamma(n+1+p)\Gamma(1+p-\lambda)}{\Gamma(1+p)\Gamma(n+1+p-\lambda)}(1+\Pi_{j=1}^m\alpha_j\Pi_{j=1}^m\beta_j)\Pi_{j=1}^m C_{n+p,j} \\
&\leq \frac{\left(\frac{2(1+p-\lambda)}{1+p}\right)^{m-1}\Pi_{j=1}^m\beta_j(\Pi_{j=1}^m\alpha_j+1-\Pi_{j=1}^m\gamma_j)}{\Pi_{j=1,j\neq j_0}^m(1+\alpha_j\beta_j)} \\
&\leq \frac{\left(\frac{2(1+p-\lambda)}{1+p}\right)^{m-1}\Pi_{j=1}^m\beta_j(\Pi_{j=1}^m\alpha_j+1-\Pi_{j=1}^m\gamma_j)}{(1+\min\limits_{1\leq j\leq m}\{\alpha_j\beta_j\})^{m-1}} \\
&\leq \Pi_{j=1}^m\beta_j(\Pi_{j=1}^m\alpha_j+1-\Pi_{j=1}^m\gamma_j),
\end{aligned}$$

since

$$\frac{2-\frac{2\lambda}{1+p}}{1+\min\limits_{1\leq j\leq m}\{\alpha_j\beta_j\}} \leq 1. \tag{4.8}$$

Cosequently, we have the assertion (4.3) with the aid of Theorem 1. □

Theorem 4.. *Let the functions $f_j(z)(j = 1, 2)$ defined by (4.2) be in the class $L_p(\alpha, \beta, \gamma; \lambda)$. Then*

$$f_1 * f_2(z) \in L_p(\mu(\alpha, \beta, \gamma, \lambda), \beta, \gamma, \lambda) \tag{4.9}$$

where

$$\mu(\alpha, \beta, \gamma, \lambda) = \frac{(\alpha+1)(1+\alpha\beta) - \beta(\alpha+1-\gamma)^2(1+p-\lambda)}{(1+\alpha\beta)(1+p)}. \tag{4.10}$$

The result is sharp.

Proof. It is sufficient to prove that

$$\sum_{n=1}^{\infty} \frac{(1+\alpha\beta)\Gamma(n+1+p)\Gamma(1+p-\lambda)}{\beta(\alpha+1-\gamma)\Gamma(1+p)\Gamma(n+1+p-\lambda)} C_{n,1}C_{n,2} \leq 1 \tag{4.11}$$

for $\mu \leq \mu(\alpha, \beta, \gamma, \lambda)$. By using Cauchy-Schwarrz inequality, it follows from (2.1) that

$$\sum_{n=1}^{\infty} \frac{(1+\alpha\beta)\Gamma(n+1+p)\Gamma(1+p-\lambda)}{\beta(\alpha+1-\gamma)\Gamma(1+p)\Gamma(n+1+p-\lambda)} \sqrt{C_{n,1}C_{n,2}}. \leq 1 \tag{4.12}$$

Thus we need to find the largest μ such that

$$\sum_{n=1}^{\infty} \frac{(1+\alpha\beta)\Gamma(n+1+p)\Gamma(1+p-\lambda)}{\beta(\alpha+1-\mu)\Gamma(1+p)\Gamma(n+1+p-\lambda)} C_{n,1}C_{n,2}$$
$$\leq \sum_{n=1}^{\infty} \frac{(1+\alpha\beta)\Gamma(n+1+p)\Gamma(1+p-\lambda)}{\beta(\alpha+1-\gamma)\Gamma(1+p)\Gamma(n+1+p-\lambda)} \sqrt{C_{n,1}C_{n,2}}$$

or equivalently, that

$$\sqrt{C_{n,1}C_{n,2}} \leq \frac{\alpha+1-\mu}{\alpha+1-\gamma}, \quad n \in N. \tag{4.13}$$

In view of (4.12), it is sufficient to find the largest μ such that

$$\frac{\beta(\alpha+1-\gamma)\Gamma(1+p)\Gamma(n+1+p-\lambda)}{(1+\alpha\beta)\Gamma(n+1+p)\Gamma(1+p-\lambda)} \leq \frac{\alpha+1-\mu}{\alpha+1-\gamma}. \tag{4.14}$$

The inequality (4.14) yields

$$\mu \leq \frac{(\alpha+1)(1+\alpha\beta) - \beta(\alpha+1-\mu)^2}{1+\alpha\beta} \psi(n), \quad n \in N, \tag{4.15}$$

where

$$\psi(n) = \frac{\Gamma(1+p)\Gamma(n+1+p-\lambda)}{\Gamma(n+1+p)\Gamma(1+p-\lambda)}. \tag{4.16}$$

Since $\psi(n)$ defined by (4.16) is a decreasing function of n, for fixed λ, we have
(4.17)

$$\mu \le \mu(\alpha,\beta,\gamma,\lambda) = \frac{(\alpha+1)(1+\alpha\beta)-\beta(\alpha+1-\gamma)^2\Gamma(2+p-\lambda)}{(1+\alpha\beta)\Gamma(2+p)\Gamma(1+p-\lambda)}$$

that is

$$\mu \le \mu(\alpha,\beta,\gamma,\lambda) = \frac{(\alpha+1)(1+\alpha\beta)-\beta(\alpha+1-\gamma)^2(1+p-\lambda)}{(1+\alpha\beta)(1+p)}$$

which evidently proves the assertion (4.9) under constraint (4.10).
Finally, by taking the functions

$$f_j(z) = z^p - \frac{\beta(\alpha+1-\lambda)(1+p-\lambda)}{(1+\alpha\beta)(1+p)} z^{p+1}, \quad j = 1, 2.$$

We can see that the result in Theorem 4 is sharp. □

5. Linear combination of functions in the class $L_p(\alpha,\beta,\gamma;\lambda)$

Finally, we prove

THEOREM 5.. *Let each of the functions $f_j(z)(j=1,2,\cdots,m)$ defined by (4.3) be in the class $L_p(\alpha,\beta,\gamma;\lambda)$. Then the function $h(z)$ given by*

$$h(z) = \frac{1}{m}\sum_{j=1}^{m} f_j(z) \tag{5.1}$$

is also in the class $L_p(\alpha,\beta,\gamma,\lambda)$.

Proof. By the definition (5.1) of $h(z)$, we have

$$h(z) = z^p - \sum_{n=1}^{\infty}\left(\frac{1}{m}\sum_{j=1}^{m} C_{n+p,j}\right) z^{n+p}. \tag{5.2}$$

Since $f_j(z) \in L_p(\alpha, \beta, \gamma; \lambda)(j = 1, 2, \cdots, m)$, by using Theorem 1, we obtain

(5.3)

$$\sum_{n=1}^{\infty} \frac{\Gamma(n+1+p)\Gamma(1+p-\lambda)}{\Gamma(1+p)\Gamma(n+1+p-\lambda)}(1+\alpha\beta)\left(\frac{1}{m}\sum_{j=1}^{m} C_{n+p,j}\right) \leq \beta(\alpha+1-\gamma),$$

which, in view of Theorem 1, yields Theorem 5. □

References

1. A. Eredelyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Tables of integral transformations, Vol II*, MacGraw-Hill, New York/Toronto/London, 1954.
2. V. P. Gupta and P. K. Jain, *Certain classes of univalent functions with negative coefficients II*, Bull. Austral. Math. Soc. **15** (1976), 467–473.
3. K. S. Kim and S. K. Lee, *Some classes of univalent functions*, Math. Japon. **32** (1987), 781–796.
4. K. Nishimoto, *Fractional derivative and integral, Part I*, J. College Engrg. Nihon Univ. Ser., B, **17** (1976), 11–19.
5. S. Owa, *On the distortion theorems I*, Kyungpook Math. J. **18** (1978), 53–59.
6. B. Ross, *A brief history and exposition of the fundamental theory of fractional calculus in fractional calculus and its applications(B. Ross, ed.)* (1975), Springer-verlag, New york/Heidelberg/Berin, 1–36.
7. M. Saigo, *A remark on integral operators in Gauss hypergeometric functions*, Math. Rep. College General Ed. Kyushu Univ. **11** (1978), 135–143.
8. H. M. Srivastava and M. K. Aouf, *A certain fractional derivative operator and its applications to a new class of analytic and multivalent functions with negative coefficients*, J. Math. Anal. Appl. **171** (1992), 1–13.
9. H. M. Srivastava and S. Owa, *An application of the fractional derivative*, Math. Japon. **29** (1984), 383–389.
10. H. M. Srivastava and S. Owa, *A new class of analytic functions with negative coefficients*, Comm. Math. Univ. Sancti. Pauli. **35** (1986), 175–188.

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