ON THE BEHAVIOR OF $L^2$ HARMONIC FORMS ON COMPLETE MANIFOLDS AT INFINITY AND ITS APPLICATIONS

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Abstract. We investigate the behavior of $L^2$ harmonic one forms on complete manifolds and as an application, we show the space of $L^2$ harmonic one forms on a complete Riemannian manifold of nonnegative Ricci curvature outside a compact set with bounded $n/2$-norm of Ricci curvature satisfying the Sobolev inequality is finite dimensional.

1. Introduction

In this paper, all manifolds are complete, oriented and Riemannian unless explicitly stated otherwise. Let $(M, g)$ be a complete Riemannian manifold and $H^p L_2(M)$ denote the space of $L^2$ harmonic $p$-forms on $M$, i.e., $p$-forms $\omega \in \Omega^p(M)$ such that

\[(1.1) \quad \Delta \omega = 0, \quad \int_M \omega \wedge \star \omega = \int_M |\omega|^2 \, dv_g < \infty.\]

It is clear that $H^p L_2(M)$ is naturally isomorphic to $H^{n-p} L_2(M)$ under the Hodge star operator $\star$, $n = \text{dim}(M)$.

The basic fact for the $L^2$ harmonic forms is the following

Lemma 1.1 ([Gro1], [Ya1]). If $\omega$ is $L^2$ harmonic $p$-form on a complete Riemannian manifold $M$, then it is closed and co-closed, i.e., $d \omega = \delta \omega = 0$.

Received June 1, 1998.

1991 Mathematics Subject Classification: 58C35, 58G05, 53C20.

Key words and phrases: $L^2$ harmonic form, Ricci curvature, Sobolev inequality.

Supported in part by BSRI-98-1434 and KOSEF 960-7010-201-3.
Obviously harmonic 0-forms are harmonic functions and there are several well-known results for harmonic functions on a complete noncompact Riemannian manifold. For instance, from Lemma 1.1 it is easy to see that every harmonic $L^2$-function on a complete Riemannian manifold $M$ is constant. For harmonic 1-forms, if $M$ has nonnegative Ricci curvature, then there is no $L^2$ harmonic one forms on $M$ ([Ya3]). However, if $M$ has nonnegative Ricci curvature outside a compact set, then there should be a nontrivial $L^2$ harmonic one form ([Don]). In fact, Donnelly has proved the following

**Theorem 1.2** ([Don]). Assume $M$ is a complete Riemannian manifold of nonnegative Ricci curvature outside a compact set. Then for fixed $p \geq 2$, the space of $L^{4p}$ harmonic one forms on $M$ is finite dimensional.

Note that $L^2$ harmonic one forms are not necessarily in the class of $L^{4p}$, $p \geq 2$. In this article, we will investigate the behavior of $L^2$ harmonic forms and prove finiteness of dimension for the space of $L^2$ harmonic 1-forms on a complete noncompact Riemannian manifold as an application.

### 2. Behavior of harmonic one forms at infinity

Assume $M$ is a complete noncompact Riemannian $n$-manifold ($n \geq 3$) of nonnegative Ricci curvature outside a compact set $K$ satisfying the Sobolev inequality, i.e.,

\[
(\int_M \phi^{2\mu})^{\frac{1}{n}} \leq C_s \int_M |\nabla \phi|^2 \quad \text{for all} \quad \phi \in C^1_c(M),
\]

with some positive constant $C_s$ and $\mu = \frac{n}{n-2}$.

Suppose $K$ is contained in a geodesic ball $B(p, r_o)$ of radius $r_o > 0$ for a point $p \in M$. The volume comparison theorem shows for $r \geq r_o$ sufficiently large

\[
\text{vol}(B(p, r)) \leq V r^n,
\]

where $V > 0$ is a positive constant. We denote $M \setminus B(p, r)$ by $D(r)$ for $r > r_o$.

In this section, we shall prove the following
Theorem 2.1. Let $M$ be a complete Riemannian $n$-manifold ($n \geq 3$) of nonnegative Ricci curvature outside a compact set satisfying the Sobolev inequality. Suppose furthermore

$$\int_M |\text{Ric}|^{n/2} < \infty.$$ 

If $\omega$ is an $L^2$ harmonic one form on $M$, then for $r > r_o$,

$$\sup_{D(r)} |\omega| \leq C \cdot r^{-n/2},$$

where $C > 0$ is a positive constant depending only on $C_s, K, V, n$ and $k = \max |\text{Ric}|$.

Example. Let $(S^n, g_1), \ n \geq 3$, be the standard sphere and $(\mathbb{R}^n, g_0)$ the Euclidean flat space. Let $M = S^n \# \mathbb{R}^n$ be a connected sum of $S^n$ and $\mathbb{R}^n$. The Riemannian metric $g$ on $M$ is obtained from $g_1$ and $g_0$ by smoothing on the gluing part. Then obviously one has $\text{Ric}(M, g) \geq -k$ for some constant $k > 0$ and $\text{Ric}(M, g) \geq 0$ outside a compact set. In fact, if letting $K$ be a compact set containing the part $S^n$, then $\text{Ric} \equiv 0$ on $M - K$ and so

$$\int_M |\text{Ric}|^{n/2} = \int_K |\text{Ric}|^{n/2} < \infty.$$ 

It is also well-known (cf. [Aub]) that such a manifold $M$ satisfies the Sobolev inequality (2.1). It follows from Theorem 3.1 that the space of $L^2$ harmonic one-forms is finite dimensional. Furthermore it is known ([Don]) that such a manifold admits a non-constant bounded harmonic function.

First we shall prove some a priori estimates for a nonnegative function $u$ which satisfies

$$\Delta u \geq -fu \quad \text{on} \quad M,$$

with nonnegative function $f \leq k$, $k$ constant.
Lemma 2.2. Suppose \( f \in L^{n/2}(M) \), and \( u \in L^2(M) \). Then one has

\[
(2.4) \quad \left( \int_{D(2r)} u^{2\mu} \right)^{\frac{1}{\mu}} \leq C(r^{-2} + 1) \int_{D(r)} u^2,
\]

where \( C > 0 \) depends only on \( C_s, k \) and \( V \).

Proof. For any constant \( \alpha \geq 1 \), multiplying (2.3) by \( \phi^2 u^{2\alpha - 1} \), we have

\[
k \int \phi^2 u^{2\alpha} \geq - \int \phi^2 u^{2\alpha - 1} \Delta u,
\]

for any compactly supported Lipschitz function \( \phi \) on \( M \). Integration by parts gives

\[
- \int \phi^2 u^{2\alpha - 1} \Delta u = 2 \int \phi u^{2\alpha - 1} \langle \nabla \phi, \nabla u \rangle + (2\alpha - 1) \int \phi^2 u^{2\alpha - 2} |\nabla u|^2 \geq 2 \int \phi u^{2\alpha - 1} \langle \nabla \phi, \nabla u \rangle + \alpha \int \phi^2 u^{2\alpha - 2} |\nabla u|^2.
\]

By (2.1), and using the following identity

\[
\int |\nabla (\phi u^{\alpha})|^2 = \int |\nabla \phi|^2 u^{2\alpha} + 2\alpha \int \phi u^{2\alpha - 1} \langle \nabla \phi, \nabla u \rangle + \alpha^2 \int \phi^2 u^{2\alpha - 2} |\nabla u|^2,
\]

we get

\[
(2.5) \quad k \alpha \int \phi^2 u^{2\alpha} + \int |\nabla \phi|^2 u^{2\alpha} \geq C_s^{-1} \left( \int (\phi^2 u^{2\alpha})^\mu \right)^{\frac{1}{\mu}}.
\]

Let us now choose \( \phi(r) \) to be the cut-off function such that \( 0 \leq \phi \leq 1, \phi = 0 \) in \( B(p, r) \cup D(2r') \) and \( \phi = 1 \) in \( D(2r) \setminus D(2r') \) with \( |\nabla \phi| \leq C_1 (r^{-1} + r'^{-1}) \) for \( 2r < r' \). Substituting this \( \phi \) into (2.5), one gets

\[
C_s^{-1} \left( \int_{D(2r) \setminus D(r')} u^{2\alpha \mu} \right)^{\frac{1}{\mu}} \leq C_1 \left( \frac{1}{r} + \frac{1}{r'} \right)^2 \int_{\text{supp} |\nabla \phi|} u^{2\alpha} + k \alpha \int_{D(r) \setminus D(2r')} u^{2\alpha}.
\]

Letting \( \alpha = 1 \) and \( r' \to \infty \), one gets (2.4) \( \square \)
Lemma 2.3. Suppose \( u \in L^2(M) \). Then
\[
\sup_{D(r) \setminus D(2r)} u \leq C r^{-n/2} \int_{D(r/2)} u^2.
\]

Proof. Let \( \beta \leq \alpha \). From (2.5) one has
\[
\beta \int \phi^2 u^{2\beta} + \int |\nabla \phi|^2 u^{2\beta} \geq C_s^{-1} \left( \int (\phi^2 u^{2\beta})^\mu \right)^{1/\mu}.
\]

For \( r_1 < r_2 < r_3 < r_4 \) with \( r_1 - r_2 = r_3 - r_4 \), we take \( \phi \) so that
\[
0 \leq \phi \leq 1, \phi \equiv 0 \text{ in } B(p, r) \cup D(r_4), \phi = 1 \text{ in } D(r_2) \setminus D(r_3) \text{ with } |\nabla \phi| \leq C_2 (r_2 - r_1)^{-1}. \]
Then substituting this \( \phi \) into (2.6) we get
\[
C_s^{-1} \left( \int_{D(r_2) \setminus D(r_3)} u^{2\beta \mu} \right)^{\frac{1}{\mu}} \leq C_2 \left( \beta + (r_2 - r_1)^{-2} \right) \int_{D(r_1) \setminus D(r_4)} u^{2\beta}.
\]
We set
\[
\Gamma(\beta, r, r') = \left( \int_{D(r) \setminus D(r')} u^\beta \right)^{1/\beta}.
\]
We have
\[
\Gamma(2\beta \mu, r_2, r_3) \leq \{C(\beta + (r_2 - r_1)^{-2})\}^{1/2\beta} \Gamma(2\beta, r_1, r_4).
\]
Taking \( r_{1,m} = (1 - 2^{-m})r, r_{2,m} = r_{1,m+1}, r_{4,m} = (2 + 2^{-m})r, r_{3,m} = r_{4,m+1} \) and \( \beta_m = 2\beta \mu^m \), we obtain
\[
\Gamma(\beta_{m+1}, r_{1,m+1}, r_{4,m+1}) \leq \{C(\beta + 4 \cdot 4^m r^{-2})\}^{1/2\beta \mu^m} \Gamma(\beta_m, r_{1,m}, r_{4,m}).
\]
Inductively we have
\[
\Gamma(\beta_m, r_{1,m}, r_{4,m}) \leq \left( \prod_{m=1}^{\infty} \{C(\beta + 4 \cdot 4^m \cdot r^{-2})\}^{1/2\beta \mu^m} \right) \Gamma(2, r/2, 5r/2).
\]
Since $\sum m\mu^{-m} < \infty$ and $\sum \mu^{-m} = n/2$, one has
\[
\prod_{m=1}^{\infty} \left\{ C(\beta + 4 \cdot 4^m \cdot r^{-2}) \right\}^{1/2 \beta \mu^m} \leq \prod_{m=1}^{\infty} \left\{ C(\beta + 4r^{-2})4^m \right\}^{1/2 \beta \mu^m} \leq C_3(\beta + 4r^{-2})^{n/4\beta}
\]
Hence, by setting $\beta = 1$, we get
\[
\sup_{D(r) \setminus D(2r)} u \leq C \cdot r^{-n/2} \int_{D(r/2)} u^2. \quad \square
\]

**Proposition 2.4.** Under the same hypotheses as above, one has
\[
\sup_{D(2r)} u \leq C \cdot r^{-n/2} \int_{D(r)} u^2.
\]

**Proof.** It follows from Lemma 2.3 directly. In fact, one has
\[
\sup_{D(2r) \setminus D(4r)} u \leq C \cdot (2r)^{-n/2} \int_{D(r)} u^2.
\]
and $\int_{D(r)} u^2$ is bounded by $\int_M u^2$ which is independent of $r$. So the radius is becoming larger, the supremum is getting smaller. \quad \square

**Proof of Theorem 2.1.** It follows from Proposition 2.4. \quad \square

### 3. Applications

As an application, we shall prove the following

**Theorem 3.1.** Let $M$ be a complete Riemannian $n$-manifold ($n \geq 3$) of nonnegative Ricci curvature outside a compact set satisfying the Sobolev inequality. If
\[
\int_M |\text{Ric}|^{n/2} < \infty,
\]
then the vector space of $L^2$ harmonic one forms on $M$ is finite dimensional.

The proof follows from a priori estimates of the behavior of $L^2$ harmonic forms at infinity. Since Theorem 3.1 is trivial if $M$ is compact, we may assume $M$ is noncompact.
Proof of Theorem 3.1. Let \( \omega \) be a differential one form on \( M \). From the Bochner-Weitzenböck formula, we have

\[
\frac{1}{2} \Delta |\omega|^2 = |D\omega|^2 + \text{Ric}(\omega^\#, \omega^\#).
\]

On the other hand, since

\[
\frac{1}{2} \Delta |\omega|^2 = |\omega| \Delta |\omega| + |\nabla |\omega||^2,
\]

from Kato’s inequality

\[ |\nabla |\omega|| \leq |D\omega|, \]

one has

\[
\Delta |\omega| \geq -|\text{Ric}||\omega|.
\]

We may assume from the hypothesis that

\[ \text{Ric}(M) \geq -k, \]

where \( k > 0 \) depends on the compact set \( K \) and assume \( K \subset B(p, r_o) \). Let \( \omega \) be a \( L^2 \) harmonic one form on \( M \). Then one has from (3.3)

\[ \Delta |\omega| \geq -k|\omega|. \]

Applying Theorem 2.1, one has for \( r_o < r \)

\[
\int_M |\omega|^{4p} = \int_{B(p,r)} |\omega|^{4p} + \int_{D(r)} |\omega|^{4p} \leq \int_{B(p,r)} |\omega|^{4p} + C \cdot r^{-2np} \left( \int_{D(r/2)} |\omega|^2 \right)^{4p}.
\]

This implies \( H^1 L^2(M) \subset H^1 L^4p(M) \) and so the proof follows from Theorem 1.2. \( \square \)
References

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