

**SOME CHARACTERIZATIONS OF  
 $CR$ -SUBMANIFOLDS WITH  $(n - 1)$   $CR$ -DIMENSION  
IN A COMPLEX PROJECTIVE SPACE**

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ABSTRACT. The purpose of this paper is to give some characterizations of  $n$ -dimensional  $CR$ -submanifolds with  $(n - 1)$   $CR$ -dimension immersed in a complex projective space  $CP^{\frac{n+p}{2}}$ , in terms of the Riemannian curvature tensor  $R$ .

### 1. Introduction

Let  $M$  be a connected real  $n$ -dimensional submanifold of real codimension  $p$  of a complex manifold  $\overline{M}$  with complex structure  $J$ . If the maximal  $J$ -invariant subspace  $JT_xM \cap T_xM$  of  $T_xM$  has constant dimension for any  $x$  in  $M$ , then  $M$  is called a  $CR$ -submanifold and the constant is called the  $CR$ -dimension of  $M$  ([8,9]). Now let  $M$  be an  $n$ -dimensional  $CR$ -submanifold of  $(n - 1)$   $CR$ -dimension of  $\overline{M}$ . Then  $M$  admits an induced almost contact structure ([11,15,16]). A typical example of an  $n$ -dimensional  $CR$ -submanifold of  $(n - 1)$   $CR$ -dimension is a real hypersurface. When the ambient manifold  $\overline{M}$  is a complex projective space, real hypersurfaces are investigated by many authors ([2,4,5,6,7,10,12,13,14]) in connection with the shape operator and the induced almost contact structure.

Recently, from these results, the several authors ([8,11]) studied about an  $n$ -dimensional  $CR$ -submanifold of  $(n - 1)$   $CR$ -dimension in a complex projective space  $CP^{\frac{n+p}{2}}$ . Especially, by using the Erbacher's

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reduction theorem ([3]), Okumura and Vanhecke [11] proved the following theorem, which is focused on the induced almost contact metric structure  $F$  on  $M$  and  $A_1$  a special kind of shape operators.

**THEOREM A.** *Let  $M$  be an  $n$ -dimensional  $CR$ -submanifold of  $(n - 1)$   $CR$ -dimension immersed in a complex projective space  $CP^{\frac{n+p}{2}}$ . If the normal vector field  $\xi_1 := \xi$  appeared in (2.1) is parallel with respect to the normal connection and if  $F$  and  $A_1$  commute, then  $\pi^{-1}(M)$  is locally a product of  $M_1 \times M_2$ , where  $M_1$  and  $M_2$  belong to some odd-dimensional spheres ( $\pi$  is the Hopf-fibration  $S^{n+p+1}(1) \rightarrow CP^{\frac{n+p}{2}}$ ).*

The purpose of this paper is to give some characterizations of  $CR$ -sub-manifolds of  $(n - 1)$   $CR$ -dimension in  $CP^{\frac{n+p}{2}}$ , in terms of the Riemannian curvature tensor  $R$ . We first have a classification of  $CR$ -submanifold of  $(n - 1)$   $CR$ -dimension in  $CP^{\frac{n+p}{2}}$  satisfying  $\mathcal{L}_{U_1}R = 0$ , where  $\mathcal{L}_{U_1}$  denotes the Lie derivative in the direction of the structure vector field  $U_1$ .

**THEOREM 1.** *Let  $M$  be an  $n$ -dimensional  $CR$ -submanifold of  $(n - 1)$   $CR$ -dimension immersed in  $CP^{\frac{n+p}{2}}$  and let there exist an orthonormal basis  $\{\xi_\alpha\}_{\alpha=1,\dots,p}$  ( $\xi_1 := \xi$ ) of normal vectors to  $M$  each of which is parallel with respect to the normal connection. If  $\mathcal{L}_{U_1}R = 0$ , then  $\pi^{-1}(M)$  is locally a product of  $M_1 \times M_2$ , where  $M_1$  and  $M_2$  belong to some odd-dimensional spheres.*

Next, we also have a classification of  $CR$ -submanifold of  $(n - 1)$   $CR$ -dimension in  $CP^{\frac{n+p}{2}}$  satisfying  $\nabla_{U_1}R = 0$ , where  $\nabla_{U_1}R$  denotes the covariant derivative in the direction of the structure vector field  $U_1$ . Namely, we prove the following theorem

**THEOREM 2.** *Let  $M$  be as in Theorem 1 with  $n \geq 3$ . If  $\nabla_{U_1}R = 0$  and  $g(A_1U_1, U_1) \neq 0$ , then  $\pi^{-1}(M)$  is locally a product of  $M_1 \times M_2$ , where  $M_1$  and  $M_2$  belong to some odd-dimensional spheres.*

## 2. Preliminaries

Let  $M$  be an  $n$ -dimensional Riemannian manifold isometrically immersed in a complex space form  $\overline{M} = M^{\frac{n+p}{2}}(c)$  and denote by  $(J, \overline{g})$  the Kähler structure on  $\overline{M}$ . For  $x$  of  $M$  we denote by  $T_xM$  and  $T_xM^\perp$

the tangent space and normal space of  $M$  at  $x$ , respectively. From now on we assume that  $M$  is an  $n$ -dimensional  $CR$ -submanifold of  $(n - 1)$   $CR$ -dimension, that is,  $\dim(JT_xM \cap T_xM) = n - 1$ . This implies that  $\dim M$  is odd ([11]).

Note that the definition of  $CR$ -submanifold of  $(n - 1)$   $CR$ -dimension meets the definition of  $CR$ -submanifold in the sense of Bejancu [1].

Furthermore, our hypothesis implies that *there exists a unit vector field  $\xi_1$  normal to  $M$  such that  $JTM \subset TM \oplus \text{Span}\{\xi\}$* . Hence, for any tangent vector field  $X$  and for a local orthonormal basis  $\{\xi_\alpha\}_{\alpha=1,\dots,p}$  ( $\xi_1 := \xi$ ) of normal vectors to  $M$ , we have the following decomposition in tangential and normal components :

$$(2.1) \quad JX = FX + u^1(X)\xi_1,$$

$$(2.2) \quad J\xi_\alpha = -U_\alpha + P\xi_\alpha, \quad \alpha = 1, \dots, p.$$

It is easily seen that  $F$  and  $P$  are skew-symmetric linear endomorphisms acting on  $T_xM$  and  $T_xM^\perp$ , respectively. Moreover, the Hermitian property of  $J$  implies

$$(2.3) \quad g(FU_\alpha, X) = -u^1(X)\bar{g}(\xi_1, P\xi_\alpha),$$

$$(2.4) \quad g(U_\alpha, U_\beta) = \delta_{\alpha\beta} - \bar{g}(P\xi_\alpha, P\xi_\beta).$$

From  $\bar{g}(JX, \xi_\alpha) = -\bar{g}(X, J\xi_\alpha)$ , we get  $g(U_\alpha, X) = u^1(X)\delta_{1\alpha}$  and hence

$$(2.5) \quad g(U_1, X) = u^1(X), \quad U_\alpha = 0, \quad \alpha = 2, \dots, p.$$

Next, applying  $J$  to (2.1), using (2.2) and (2.5) we have

$$(2.6) \quad F^2X = -X + u^1(X)U_1, \quad u^1(X)P\xi_1 = -u^1(FX)\xi_1.$$

Since  $P$  is skew-symmetric, (2.3) and the second equation of (2.6) give

$$(2.7) \quad u^1(FX) = 0, \quad P\xi_1 = 0, \quad FU_1 = 0.$$

So, (2.2) may be written in the form

$$(2.8) \quad J\xi_1 = -U_1, \quad J\xi_\alpha = P\xi_\alpha, \quad \alpha = 2, \dots, p$$

and further, we may put

$$(2.9) \quad P\xi_\alpha = \sum_{\beta=2}^p P_{\alpha\beta}\xi_\beta, \quad \alpha = 2, \dots, p,$$

where  $(P_{\alpha\beta})$  is a skew-symmetric matrix which satisfies

$$(2.10) \quad \sum_{\beta=2}^p P_{\alpha\beta}P_{\beta\gamma} = -\delta_{\alpha\gamma}, \quad \alpha, \gamma = 2, \dots, p.$$

These results imply that  $(F, U_1, u^1, g)$  defines an almost contact metric structure on  $(M, g)$  ([16]).

Now, let  $\bar{\nabla}$  and  $\nabla$  denote the Levi-Civita connection on  $\bar{M}$  and  $M$ , respectively and denote by  $D$  the normal connection induced from  $\bar{\nabla}$  in the normal bundle  $TM^\perp$  of  $M$ . Then the Gauss and Weingarten equations are given by

$$(2.11) \quad \bar{\nabla}_X Y = \nabla_X Y + h(X, Y),$$

$$(2.12) \quad \bar{\nabla}_X \xi_\alpha = -A_\alpha X + D_X \xi_\alpha, \quad \alpha = 1, \dots, p$$

for any tangent vector fields  $X$  and  $Y$  to  $M$ . Here  $h$  denotes the second fundamental form and  $A_\alpha$  is the shape operator corresponding to  $\xi_\alpha$ . They are related by  $h(X, Y) = \sum_{\alpha=1}^p g(A_\alpha X, Y)\xi_\alpha$ .

Furthermore, putting

$$(2.13) \quad D_X \xi_\alpha = \sum_{\beta=1}^p s_{\alpha\beta}(X)\xi_\beta,$$

it follows that  $(s_{\alpha\beta})$  is the skew-symmetric matrix of connection forms of  $D$ . Next, the Gauss, Codazzi and Ricci equations are ([11]) :

$$(2.14) \quad \bar{g}(\bar{R}(X, Y)Z, W) = g(R(X, Y)Z, W) \\ + \sum_{\alpha=1}^p \{g(A_\alpha X, Z)g(A_\alpha Y, W) - g(A_\alpha Y, Z)g(A_\alpha X, W)\},$$

$$(2.15) \quad \bar{g}(\bar{R}(X, Y)Z, \xi_\alpha) = g((\nabla_X A_\alpha)Y - (\nabla_Y A_\alpha)X, Z) \\ + \sum_{\beta=1}^p \{g(A_\beta Y, Z)s_{\beta\alpha}(X) - g(A_\beta X, Z)s_{\beta\alpha}(Y)\},$$

$$(2.16) \quad \bar{g}(\bar{R}(X, Y)\xi_\alpha, \xi_\beta) = \bar{g}(R^\perp(X, Y)\xi_\alpha, \xi_\beta) + g([A_\beta, A_\alpha]X, Y)$$

for any tangent vector fields  $X, Y, Z$  and  $W$  to  $M$ .  $\bar{R}$  denotes the Riemannian curvature tensor of  $\bar{M}$  and  $R$  that of  $M$ .  $R^\perp$  is the curvature tensor of the normal connection  $D$ .

Moreover, if the ambient space  $\bar{M}$  is of constant holomorphic sectional curvature  $c$ , since

$$\begin{aligned} \bar{R}(\bar{X}, \bar{Y})\bar{Z} = \frac{c}{4} \{ & \bar{g}(\bar{Y}, \bar{Z})\bar{X} - \bar{g}(\bar{X}, \bar{Z})\bar{Y} + \bar{g}(J\bar{Y}, \bar{Z})J\bar{X} \\ & - \bar{g}(J\bar{X}, \bar{Z})J\bar{Y} - 2\bar{g}(J\bar{X}, \bar{Y})J\bar{Z} \} \end{aligned}$$

for any tangent vector fields  $\bar{X}, \bar{Y}$  and  $\bar{Z}$  to  $\bar{M}$ , the Riemannian curvature tensor  $R$  of  $M$  given by

$$(2.17) \quad \begin{aligned} R(X, Y)Z = \frac{c}{4} \{ & g(Y, Z)X - g(X, Z)Y + g(FY, Z)FX \\ & - g(FX, Z)FY - 2g(FX, Y)FZ \} \\ & + \sum_{\alpha=1}^p \{ g(A_\alpha Y, Z)A_\alpha X - g(A_\alpha X, Z)A_\alpha Y \} \end{aligned}$$

for any tangent vector fields  $X, Y$  and  $Z$  to  $M$ .

In the sequel, we consider the case of a complex space form  $\bar{M} = M^{\frac{n+p}{2}}(c)$  and  $\bar{\nabla}J = 0$ . Then by differentiating (2.1) and (2.2) covariantly and by comparing the tangential and normal parts, we have

$$(2.18) \quad (\nabla_Y F)X = u^1(X)A_1Y - g(A_1X, Y)U_1,$$

$$(2.19) \quad (\nabla_Y u^1)X = g(F A_1 Y, X),$$

$$(2.20) \quad \nabla_X U_1 = F A_1 X,$$

$$(2.21) \quad g(A_\alpha U_1, X) = - \sum_{\beta=2}^p s_{1\beta}(X) P_{\beta\alpha}, \quad \alpha = 2, \dots, p$$

for any tangent vector fields  $X$  and  $Y$  to  $M$ .

In the rest of this paper we suppose that there exists an orthonormal basis  $\{\xi_\alpha\}_{\alpha=1, \dots, p}$  of normal vectors to  $M$  each of which is parallel with respect to the normal connection  $D$ . Then from (2.13) we have

$$(2.22) \quad s_{\alpha\beta} = 0.$$

Hence, from (2.21) and (2.22) we obtain

$$(2.23) \quad A_\alpha U_1 = 0, \quad \alpha = 2, \dots, p.$$

Moreover, from (2.22) and (2.23), the Codazzi equation (2.15) becomes

$$(2.24) \quad (\nabla_X A_1)Y - (\nabla_Y A_1)X \\ = \frac{c}{4} \{g(X, U_1)FY - g(Y, U_1)FX - 2g(FX, Y)U_1\},$$

$$(2.25) \quad (\nabla_X A_\alpha)Y - (\nabla_Y A_\alpha)X = 0, \quad \alpha = 2, \dots, p$$

for any tangent vector fields  $X$  and  $Y$  to  $M$ . Also, by differentiating (2.23) covariantly and using (2.25) we get

$$(2.26) \quad (\nabla_{U_1} A_\alpha)U_1 = 0, \quad \alpha = 2, \dots, p.$$

Especially, recently Kwon and Pak [8] proved the following lemma.

LEMMA 2.1. *Let  $M$  be an  $n$ -dimensional CR-submanifold of  $(n-1)$  CR-dimension immersed in a complex space form  $M^{\frac{n+p}{2}}(c)$ ,  $c \neq 0$  and let there exist an orthonormal basis  $\{\xi_\alpha\}_{\alpha=1, \dots, p}$  of normal vectors to  $M$  each of which is parallel with respect to the normal connection. If  $A_1 U_1 = \lambda U_1$  for some function  $\lambda$ , then  $\lambda$  is locally constant.*

Finally, we suppose that  $U_1$  is principal with corresponding principal curvature  $\lambda$ . Then, by Lemma 2.1,  $\lambda$  is constant on  $M$  and it satisfies

$$(2.27) \quad (\nabla_{U_1} A_1)U_1 = 0,$$

$$(2.28) \quad A_1 F A_1 = \frac{c}{4} F + \frac{1}{2} \lambda (A_1 F + F A_1)$$

by virtue of (2.20) and (2.24). Hence from (2.24) and (2.28), we get

$$(2.29) \quad \nabla_{U_1} A_1 = -\frac{1}{2} \lambda (A_1 F - F A_1).$$

### 3. Proof of Theorem 1

In this section, we are concerned with the proof of Theorem 1. Let  $M$  be an  $n$ -dimensional  $CR$ -submanifold of  $(n - 1)$   $CR$ -dimension immersed in a complex space form  $M^{\frac{n+p}{2}}(c)$ . Then  $M$  admits an almost contact metric structure  $(F, U_1, u^1, g)$ . The Lie derivative  $\mathcal{L}_{U_1}R$  of  $R$  with respect to the structure vector field  $U_1$  satisfies

$$(3.1) \quad (\mathcal{L}_{U_1}R)(X, Y, Z) = \mathcal{L}_{U_1}(R(X, Y)Z) - R(\mathcal{L}_{U_1}X, Y)Z \\ - R(X, \mathcal{L}_{U_1}Y)Z - R(X, Y)\mathcal{L}_{U_1}Z$$

for any vector fields  $X, Y$  and  $Z$  on  $M$ . From now on we shall prove the following lemma.

LEMMA 3.1. *Let  $M$  be as in Lemma 2.1. If  $\mathcal{L}_{U_1}R = 0$ , then  $A_1F = FA_1$ .*

*Proof.* Let  $T_0$  be a distribution defined by the subspace  $T_0(x) = \{u \in T_xM : g(u, U_1(x)) = 0\}$  of the tangent space  $T_xM$  of  $M$  at any point  $x$ , which is called the *holomorphic distribution*. Suppose that the structure vector field  $U_1$  is not necessarily principal. Then we can put  $A_1U_1 = \lambda U_1 + \mu V$ , where  $V$  is a unit vector field in  $T_0$ ,  $\lambda$  and  $\mu$  are smooth functions on  $M$ . From (2.17), (2.18), (2.20), (3.1) and our assumption, we have

$$(3.2) \quad 0 = \frac{c}{4}\mu[\{u^1(Y)g(Z, V) - u^1(Z)g(Y, V)\}FX \\ - \{u^1(X)g(Z, V) - u^1(Z)g(X, V)\}FY \\ - 2\{u^1(X)g(Y, V) - u^1(Y)g(X, V)\}FZ \\ + g(FY, Z)\{u^1(X)V - g(X, V)U_1\} \\ - g(FX, Z)\{u^1(Y)V - g(Y, V)U_1\} \\ - 2g(FX, Y)\{u^1(Z)V - g(Z, V)U_1\}] \\ - \frac{c}{4}\{g(FY, Z)F(A_1F - FA_1)X - g(FX, Z)F(A_1F - FA_1)Y \\ - 2g(FX, Y)F(A_1F - FA_1)Z + g((A_1F - FA_1)Y, Z)X \\ - g((A_1F - FA_1)X, Z)Y + g((A_1F^2 - F^2A_1)Y, Z)FX$$

$$\begin{aligned}
& -g((A_1F^2 - F^2A_1)X, Z)FY - 2g((A_1F^2 - F^2A_1)X, Y)FZ\} \\
& + g((\nabla_{U_1}A_1)Y, Z)A_1X - g((\nabla_{U_1}A_1)X, Z)A_1Y \\
& + g(A_1Y, Z)\{(\nabla_{U_1}A_1)X + (A_1F - FA_1)A_1X\} \\
& - g(A_1X, Z)\{(\nabla_{U_1}A_1)Y + (A_1F - FA_1)A_1Y\} \\
& - \sum_{\alpha=2}^p [\{g(A_1FA_\alpha Y, Z) - g((\nabla_{U_1}A_\alpha)Y, Z) - g(A_\alpha FA_1Y, Z)\}A_\alpha X \\
& - g(A_\alpha Y, Z)\{(\nabla_{U_1}A_\alpha)X - FA_1A_\alpha X + A_\alpha FA_1X\}] \\
& + \sum_{\alpha=2}^p [\{g(A_1FA_\alpha X, Z) - g((\nabla_{U_1}A_\alpha)X, Z) - g(A_\alpha FA_1X, Z)\}A_\alpha Y \\
& - g(A_\alpha X, Z)\{(\nabla_{U_1}A_\alpha)Y - FA_1A_\alpha Y + A_\alpha FA_1Y\}]
\end{aligned}$$

for any vector fields  $X$ ,  $Y$  and  $Z$  on  $T_xM$ . Putting  $Z = U_1$  and taking  $X$  and  $Y$  in the holomorphic distribution  $T_0$  in (3.2) and using (2.23) and (2.26), we have

$$\begin{aligned}
(3.3) \quad 0 &= \frac{c}{4}\mu\{g(Y, FV)X - g(X, FV)Y\} + g((\nabla_{U_1}A_1)U_1, Y)A_1X \\
& - g((\nabla_{U_1}A_1)U_1, X)A_1Y \\
& + \mu[g(Y, V)\{(\nabla_{U_1}A_1)X + (A_1F - FA_1)A_1X\} \\
& - g(X, V)\{(\nabla_{U_1}A_1)Y + (A_1F - FA_1)A_1Y\}] \\
& + \sum_{\alpha=2}^p \mu\{g(A_\alpha Y, FV)A_\alpha X - g(A_\alpha X, FV)A_\alpha Y\}.
\end{aligned}$$

Again, putting  $Y = Z = U_1$  and taking  $X$  in the holomorphic distribution  $T_0$  in (3.2) and using (2.23), we have

$$\begin{aligned}
(3.4) \quad \lambda(\nabla_{U_1}A_1)X &= \mu g(X, V)(\nabla_{U_1}A_1)U_1 - d\lambda(U_1)A_1X \\
& + g((\nabla_{U_1}A_1)U_1, X)A_1U_1 \\
& + \frac{c}{4}\mu g(X, FV)U_1 + \mu^2 g(X, V)(A_1F - FA_1)V \\
& - \lambda\mu^2 g(X, V)FV - \lambda(A_1F - FA_1)A_1X.
\end{aligned}$$

Eliminating  $(\nabla_{U_1}A_1)X$  and  $(\nabla_{U_1}A_1)Y$  in (3.3) and (3.4) and using



(2.26), we obtain

$$\begin{aligned}
 (3.5) \quad 0 = & \frac{c}{4} \mu [\lambda \{g(Y, FV)X - g(X, FV)Y\} + \mu \{g(X, FV)g(Y, V) \\
 & - g(X, V)g(Y, FV)\}U_1] \\
 & + \lambda \{g((\nabla_{U_1} A_1)U_1, Y)A_1 X - g((\nabla_{U_1} A_1)U_1, X)A_1 Y\} \\
 & + \mu [g(Y, V)\{g((\nabla_{U_1} A_1)U_1, X)A_1 U_1 - d\lambda(U_1)A_1 X\} \\
 & - g(X, V)\{g((\nabla_{U_1} A_1)U_1, Y)A_1 U_1 - d\lambda(U_1)A_1 Y\}] \\
 & + \sum_{\alpha=2}^p \mu \{g(A_\alpha Y, FV)A_\alpha X - g(A_\alpha X, FV)A_\alpha Y\}
 \end{aligned}$$

for any vector fields  $X$  and  $Y$  in  $T_0$ . Now, putting  $X = V$  and  $Y = FV$  in (3.5) and using (2.26), we get

$$\begin{aligned}
 (3.6) \quad 0 = & \frac{c}{4} \mu (\lambda V - \mu U_1) + \lambda \{g((\nabla_{U_1} A_1)U_1, FV)A_1 V \\
 & - g((\nabla_{U_1} A_1)U_1, V)A_1 FV\} \\
 & - \mu \{g((\nabla_{U_1} A_1)U_1, FV)A_1 U_1 - d\lambda(U_1)A_1 FV\} \\
 & + \sum_{\alpha=2}^p \mu \{g(A_\alpha FV, FV)A_\alpha V - g(A_\alpha V, FV)A_\alpha FV\}.
 \end{aligned}$$

Taking the inner product of (3.6) with  $U_1$  and using (2.23), we obtain  $\mu = 0$ , that is, the structure vector field  $U_1$  is principal. Hence, by Lemma 2.1,  $\lambda$  is constant. If  $\lambda = 0$ , then putting  $X = U_1$  in (3.2) and using (2.23), (2.26) and  $c \neq 0$ , we get  $A_1 F - F A_1 = 0$ . Next, suppose that  $\lambda \neq 0$ . Then from (3.4) and (2.27) we have

$$(3.7) \quad (\nabla_{U_1} A_1)X + A_1 F A_1 X - F A_1^2 X = 0$$

for any vector field  $X$  in  $T_0$ .

Furthermore, putting  $Y = Z = U_1$  in (3.2) and using (2.23) and (2.27), we see that (3.7) holds for any vector field  $X$ . This implies that

$$(3.8) \quad F(A_1^2 - \lambda A_1 - \frac{c}{4}I)X = 0$$

for any vector field  $X$ , where  $I$  denotes the identity transformation and we have used (2.28) and (2.29). (3.8) is equivalent to

$$A_1^2 - \lambda A_1 - \frac{c}{4}(I - u^1 \otimes U_1) = 0,$$

from which it follows that  $A_1$  satisfies  $(A_1 F - F A_1)^2 = 0$ , where we have used that (2.28) and  $A_1 F^2 = F^2 A_1 = -A_1 + \lambda u^1 \otimes U_1$ . Hence, we have  $A_1 F - F A_1 = 0$ .  $\square$

*Proof of Theorem 1.* Combining Lemma 3.1 and Theorem A, we have Theorem 1.  $\square$

#### 4. Proof of Theorem 2

In this section, we are concerned with the proof of Theorem 2. Let  $M$  be an  $n$ -dimensional  $CR$ -submanifold of  $(n-1)$   $CR$ -dimension immersed in a complex space form  $M^{\frac{n+p}{2}}(c)$ . Then  $M$  admits an almost contact metric structure  $(F, U_1, u^1, g)$ . The covariant derivative  $\nabla_{U_1} R$  of  $R$  with respect to the structure vector field  $U_1$  satisfies

$$(4.1) \quad (\nabla_{U_1} R)(X, Y, Z) = \nabla_{U_1}(R(X, Y)Z) - R(\nabla_{U_1} X, Y)Z \\ - R(X, \nabla_{U_1} Y)Z - R(X, Y)\nabla_{U_1} Z$$

for any vector fields  $X, Y$  and  $Z$  on  $M$ . From now on we shall prove the following lemma.

LEMMA 4.1. *Let  $M$  be as in Lemma 2.1 with  $n \geq 3$ . If  $\nabla_{U_1} R = 0$ , then  $\nabla_{U_1} A_1 = 0$ .*

*Proof.* From (2.17), (2.18), (4.1) and our assumption, we have

$$(4.2) \quad 0 = \frac{c}{4} [\{u^1(Y)g(A_1 U_1, Z) - u^1(Z)g(A_1 U_1, Y)\}FX \\ - \{u^1(X)g(A_1 U_1, Z) - u^1(Z)g(A_1 U_1, X)\}FY \\ - 2\{u^1(X)g(A_1 U_1, Y) - u^1(Y)g(A_1 U_1, X)\}FZ \\ + g(FY, Z)\{u^1(X)A_1 U_1 - g(A_1 U_1, X)U_1\} \\ - g(FX, Z)\{u^1(Y)A_1 U_1 - g(A_1 U_1, Y)U_1\}]$$

$$\begin{aligned}
& -2g(FX, Y)\{u^1(Z)A_1U_1 - g(A_1U_1, Z)U_1\}] \\
& + g((\nabla_{U_1}A_1)Y, Z)A_1X - g((\nabla_{U_1}A_1)X, Z)A_1Y \\
& + g(A_1Y, Z)(\nabla_{U_1}A_1)X - g(A_1X, Z)(\nabla_{U_1}A_1)Y \\
& + \sum_{\alpha=2}^p \{g((\nabla_{U_1}A_\alpha)Y, Z)A_\alpha X - g((\nabla_{U_1}A_\alpha)X, Z)A_\alpha Y \\
& + g(A_\alpha Y, Z)(\nabla_{U_1}A_\alpha)X - g(A_\alpha X, Z)(\nabla_{U_1}A_\alpha)Y\}
\end{aligned}$$

for any vector fields  $X$ ,  $Y$  and  $Z$  on  $T_xM$ . Suppose that the structure vector field  $U_1$  is not necessarily principal. Then we can put  $A_1U_1 = \lambda U_1 + \mu V$ , where  $V$  is a unit vector field in  $T_0$ , and  $\lambda$  and  $\mu$  are smooth functions on  $M$ . Let  $M_0$  be the non-empty open subset of  $M$  consisting of points  $x$  at which  $\mu(x) \neq 0$ . Hereafter unless otherwise stated, we shall discuss on the subset  $M_0$  of  $M$ . By the form  $A_1U_1 = \lambda U_1 + \mu V$ , (4.2) is reformed as

$$\begin{aligned}
(4.3) \quad 0 &= \frac{c}{4} \mu \{u^1(Y)g(Z, V) - u^1(Z)g(Y, V)\}FX \\
& - \{u^1(X)g(Z, V) - u^1(Z)g(X, V)\}FY \\
& - 2\{u^1(X)g(Y, V) - u^1(Y)g(X, V)\}FZ \\
& + g(FY, Z)\{u^1(X)V - g(X, V)U_1\} \\
& - g(FX, Z)\{u^1(Y)V - g(Y, V)U_1\} \\
& - 2g(FX, Y)\{u^1(Z)V - g(Z, V)U_1\}] \\
& + g((\nabla_{U_1}A_1)Y, Z)A_1X - g((\nabla_{U_1}A_1)X, Z)A_1Y \\
& + g(A_1Y, Z)(\nabla_{U_1}A_1)X - g(A_1X, Z)(\nabla_{U_1}A_1)Y \\
& + \sum_{\alpha=2}^p \{g((\nabla_{U_1}A_\alpha)Y, Z)A_\alpha X - g((\nabla_{U_1}A_\alpha)X, Z)A_\alpha Y \\
& + g(A_\alpha Y, Z)(\nabla_{U_1}A_\alpha)X - g(A_\alpha X, Z)(\nabla_{U_1}A_\alpha)Y\}
\end{aligned}$$

for any vector fields  $X$ ,  $Y$  and  $Z$  on  $T_xM$ . Putting  $Z = U_1$  and taking  $X$  and  $Y$  in the holomorphic distribution  $T_0$  in (4.3) and using (2.23) and (2.26), we have

$$\begin{aligned}
(4.4) \quad 0 &= \frac{c}{4} \mu \{-g(Y, V)FX + g(X, V)FY - 2g(FX, Y)V\} \\
& + g((\nabla_{U_1}A_1)U_1, Y)A_1X - g((\nabla_{U_1}A_1)U_1, X)A_1Y \\
& + \mu \{g(Y, V)(\nabla_{U_1}A_1)X - g(X, V)(\nabla_{U_1}A_1)Y\}.
\end{aligned}$$

Next, putting  $Y = Z = U_1$  and taking  $X$  in the holomorphic distribution  $T_0$  in (4.3) and using (2.20), (2.23) and (2.26) we have

$$(4.5) \quad \lambda(\nabla_{U_1} A_1)X = g(\nabla_{U_1} A_1)U_1, X)A_1U_1 \\ + \mu g(X, V)(\nabla_{U_1} A_1)U_1 - d\lambda(U_1)A_1X.$$

Combining (4.4) and (4.5) and using (2.26), we obtain

$$(4.6) \quad 0 = \frac{c}{4}\lambda\mu\{-g(Y, V)FX + g(X, V)FY - 2g(FX, Y)V\} \\ + \mu\{g(Y, V)g((\nabla_{U_1} A_1)U_1, X) \\ - g(X, V)g((\nabla_{U_1} A_1)U_1, Y)\}A_1U_1 \\ + \{\lambda g((\nabla_{U_1} A_1)U_1, Y) - \mu d\lambda(U_1)g(Y, V)\}A_1X \\ - \{\lambda g((\nabla_{U_1} A_1)U_1, X) - \mu d\lambda(U_1)g(X, V)\}A_1Y$$

for any vector fields  $X$  and  $Y$  in  $T_0$ . Let  $L(U_1, V, FV)$  be a distribution defined by the subspace  $L_x(U_1, V, FV)$  in the tangent space  $T_xM$  spanned by the vectors  $U_1(x)$ ,  $V(x)$  and  $FV(x)$  at any point  $x$  in  $M$ , and let  $T_1$  be the orthogonal complement in the tangent bundle  $TM$  of the distribution  $L(U_1, V, FV)$ . Then  $T_1$  is not empty because of  $n \geq 3$ . For any unit vector field  $X$  in  $T_1$ , putting  $Y = FX$  in (4.6) and using (2.26), we have

$$(4.7) \quad \frac{c}{2}\mu V = g((\nabla_{U_1} A_1)U_1, FX)A_1X - g((\nabla_{U_1} A_1)U_1, X)A_1FX,$$

provided  $\lambda \neq 0$ . Suppose that there is a unit vector field  $X_0$  in  $T_1$  at which  $g((\nabla_{U_1} A_1)U_1, X_0) = 0$ . Then from (4.7) we obtain

$$(4.8) \quad \frac{c}{2}\mu V = g((\nabla_{U_1} A_1)U_1, FX_0)A_1X_0 \neq 0.$$

Accordingly we can put  $A_1X_0 = \omega(X_0)V$ , where  $\omega$  is a 1-form on  $M_0$ . Putting  $X = X_0$ ,  $Y = V$  in (4.6), we have

$$(4.9) \quad \frac{c}{4}\lambda\mu FX_0 - \omega(X_0)\{\lambda g((\nabla_{U_1} A_1)U_1, V) - \mu d\lambda(U_1)\}V = 0.$$

Thus, since  $FX_0$  and  $V$  are orthonormal vector fields and  $\lambda \neq 0$ , (4.9) implies  $\mu = 0$ , a contradiction. Accordingly we get

$$(4.10) \quad g((\nabla_{U_1} A_1)U_1, X) \neq 0$$

for any non-zero vector field  $X$  in  $T_1$ . On the other hand, putting  $Y = FV$  in (4.6) and using (2.26), we have

$$(4.11) \quad g((\nabla_{U_1} A_1)U_1, FV)A_1X - g((\nabla_{U_1} A_1)U_1, X)A_1FV = 0$$

for any  $X$  in  $T_1$  under the assumption  $\lambda \neq 0$ . If  $g((\nabla_{U_1} A_1)U_1, FV) = 0$ , then from (4.10) and (4.11) we obtain  $A_1FV = 0$ . Now, we suppose that  $g((\nabla_{U_1} A_1)U_1, FV) \neq 0$ . From (4.10) and (4.11) we have

$$(4.12) \quad A_1X = \theta(X)A_1FV, \quad \theta(X) \neq 0$$

for any non-zero vector field  $X$  in  $T_1$ , where  $\theta$  is a 1-form on  $M_0$ . Hence, for any non-zero vector fields  $X$  and  $Y$  in  $T_1$ , we obtain

$$(4.13) \quad A_1\{\theta(Y)X - \theta(X)Y\} = 0, \quad \theta(X) \neq 0, \quad \theta(Y) \neq 0.$$

If we put  $Z_1 = \theta(Y_1)X_1 - \theta(X_1)Y_1$  for given linearly independent vector fields  $X_1$  and  $Y_1$  in  $T_1$ , then from (4.13)  $A_1Z_1 = 0$  and hence  $A_1FV = 0$  by virtue of (4.12).

Next, putting  $X = V$  and  $Y = FV$  in (4.6), and using  $A_1FV = 0$ , we have

$$(4.14) \quad \frac{3c}{4}\lambda\mu V - g((\nabla_{U_1} A_1)U_1, FV)(\lambda A_1V - \mu A_1U_1) = 0.$$

Consequently, we get  $g((\nabla_{U_1} A_1)U_1, FV) \neq 0$  which together with (4.11) we have  $A_1X = 0$  for any vector field  $X$  in  $T_1$ . Hence, putting  $X = U_1$  and taking  $Y, Z \in T_1$  in (4.3), and using (2.23) and (2.26), we obtain

$$(4.15) \quad \lambda g((\nabla_{U_1} A_1)Y, Z)U_1 + \mu \left\{ \frac{c}{4}g(FY, Z) + g((\nabla_{U_1} A_1)Y, Z) \right\} V = 0.$$

Accordingly it turns out to be  $\mu = 0$  on  $M_0$  provided  $\lambda \neq 0$ , a contradiction. This means that  $U_1$  is principal on  $M'$ , where  $M'$  denotes the open subset of  $M$  consisting of points  $x$  at which  $\lambda(x) \neq 0$ . Thus, putting  $Y = Z = U_1$  in (4.3) and using (2.23), (2.26) and (2.27), we have  $\nabla_{U_1} A_1 = 0$ .

Now, let us denote by  $Int(M - M')$  the interior of the subset  $(M - M')$ . Then  $\lambda = 0$  on  $Int(M - M')$ . Suppose that  $U_1$  is not principal

on  $\text{Int}(M - M')$ . Then the subset  $M_1$  of  $\text{Int}(M - M')$  consisting of points  $x$  at which  $\mu(x) \neq 0$  is non-empty open set. Hence, from (4.5) we have

$$(4.16) \quad g((\nabla_{U_1} A_1)U_1, X)V + g(X, V)(\nabla_{U_1} A_1)U_1 = 0$$

on  $M_1$  for any vector field  $X$  in  $T_0$ . Accordingly  $g((\nabla_{U_1} A_1)U_1, Y) = 0$  for any vector field  $Y$  in  $T_0$  orthogonal to  $V$ . Taking the inner product of (4.16) with  $X$  in  $T_0$ , we have  $g((\nabla_{U_1} A_1)U_1, X)g(X, V) = 0$ . Putting  $X = V$  in this equation, we get  $g((\nabla_{U_1} A_1)U_1, V) = 0$ . Hence, putting  $X = V$  in (4.16), we obtain  $(\nabla_{U_1} A_1)U_1 = 0$  on  $M_1$ . Taking  $X$  and  $Y$  in  $T_0$  orthogonal to  $V$  in (4.4) and using  $(\nabla_{U_1} A_1)U_1 = 0$ , we obtain  $g(FX, Y) = 0$  on  $M_1$ , a contradiction. This means that  $U_1$  is principal with corresponding principal curvature  $\lambda = 0$ . Accordingly we have  $\nabla_{U_1} A_1 = 0$  on  $\text{Int}(M - M')$  by virtue of (2.29). This completes the proof by the continuity of  $\nabla_{U_1} A_1 = 0$ .  $\square$

*Proof of Theorem 2.* Combining (2.29), Lemma 4.1 and Theorem A, we have Theorem 2.  $\square$

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