Kangweon-Kyungki Math. Jour. **6** (1998), No. 2, pp. 259–279

BACKWARD SELF-SIMILAR STOCHASTIC PROCESSES IN STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. For the forward-backward semimartingale, we can define the backward semimartingale flow which is generated by the backward canonical stochastic differential equation. Therefore, we define the backward self-similar stochastic processes, and we study the backward self-similar stochastic flows through the canonical stochastic differential equations.

0. Introduction

In the previous work [5], for the C-valued forward-backward semimartingale, Kunita defined the inverse flow which is a backward semimartingale flow generated by the canonical backward stochastic differential equation(SDE). On the other hand, in [3] and [4], he studied the self-similar stochastic flows generated by the canonical SDE on the manifolds. Therefore, for the forward-backward semimartingale, we can define the forward and the backward stochastic flows by the canonical SDE. Thus, the purpose of this paper is to define the backward self-similar processes and the backward self-similar stochastic flows, and study them through the canonical SDE on $\mathbb{R}^d$.

To define the backward self-similar process, it is convenient to use the (inverse) dilation which is also an invertible linear transformation. Therefore, we define the backward self-similar semimartingale with respect to a dilation and study the backward self-similarity for the flows which are generated by the backward SDE. Thus, first, we think the relation of self-similarities between the backward driving processes and

Received June 30, 1998.
1991 Mathematics Subject Classification: 60G18,60H07,60H10.
Key words and phrases: SDE, Backward stochastic flows, Self-similar process.

the backward stochastic flows through the canonical SDE. Further, for the forward-backward self-similar processes, we also think the two-sided self-similar stochastic flows through the canonical SDE.

Section I is the preliminary part. In this section, we define the canonical SDE and the backward SDE. Further, we also define the backward self-similar processes. Section II is the main part of this results. In this section, we study the backward self-similar stochastic processes and the backward self-similar stochastic flows through the canonical SDE. In section III, we will deal with the density of the self-similar stochastic flow which is a solution of the canonical SDE.

I. Preliminaries

For a non-negative integer m, we denote by $C^m := C^m(\mathbb{R}^d;\mathbb{R}^d)$ the set of all maps from $\mathbb{R}^d$ into itself which are m -times continuously differentiable. In case $m = 0$, we denote it $C := C(\mathbb{R}^d;\mathbb{R}^d)$ which is the space of continuous maps from $\mathbb{R}^d$ into itself equipped with the compact uniform topology. Let $0 < \delta \leq 1$. We denote by $C_b^{m+\delta} := C_b^{m+\delta}(\mathbb{R}^d;\mathbb{R}^d)$ the set of all $v \in C^m$ such that derivatives $D^\alpha v$ are bounded and uniformly $\delta - H\ddot{o}lder$ continuous for any $\alpha$with $|\alpha| \leq m$. Let $\tilde{C} := \tilde{C}(\mathbb{R}^d \times \mathbb{R}^d;\mathbb{S}_+)$, where $\mathbb{S}_+$ is the space of $d \times d$ - matrices. We define the subspace $\tilde{C}_b^{m+\delta} = \tilde{C}_b^{m+\delta}(\mathbb{R}^d \times \mathbb{R}^d;\mathbb{S}_+)$ of $\tilde{C}$ similarly.

Let $(\Omega, \mathcal{F}, P)$ be a probability space where the filtration $\mathcal{F}_t; t \in [0,\infty)$ of sub-σ-field of $\mathcal{F}$ is defined. Let $X(x,t), t \geqq 0$ be a family of $\mathbb{R}^d$-valued stochastic process with spatial parameter $x \in \mathbb{R}^d$ defined on $(\Omega, \mathcal{F}, P)$. If $X(x,t)$ is continuous in x for each t a.s., we can regard it as a C -valued process. We denote it sometimes by $X(t) = X(x,t), t \geq 0$.

Let $X(x,t)$ be a cadlag semimartingale with values in C. We define the point process $N(t,E)$ over $[0,\infty) \times C$ associated with $X(t)$ by

$$N((s,t],E) = \sum_{s<r\leq t} \mathcal{X}_E(\Delta X(r)), \Delta X(s) = X(s) - X(s-),$$

where E is a Borel subset of C excluding 0. Then there exists a unique predictable process $\hat{N}(t,E)$ which is called the compensator such that

$$\tilde{N}(t,E) = N(t,E) - \hat{N}(t,E)$$

is a localmartingale. For a bounded Borel subset U of C, consider a C-valued semimartingale $X(x,t)$ which is represented as;

$$\begin{aligned} X(x,t) &= X_c(x,t) + X_d(x,t) \\ &= M^c(x,t) + B^c(x,t) + \int_U v(x)\tilde{N}(t,dv) + \int_{U^c} v(x)N(t,dv), \end{aligned}$$

where $M^c(x,t)$ is a continuous localmartingale for any x, $B^c(x,t)$ is a continuous predictable process of bounded variation for any x, and the integral form

$$\int_U v(x)\tilde{N}(t,dv)$$

is a discontinuous localmartingale part of $X(x,t)$ for any x.

Let $A_t, t \in [0,\infty)$ be a continuous increasing process adapted to the filtration $\mathcal{F}_t$ such that $A_0 = 0$ a.s. Then there exist predictable processes $a^{ij}(x,y,t)$ and $b^i(x,t)$, and for the compensator $\hat{N}(t,E)$, there exists a predictable measure-valued process $\nu_t(E)$ satisfying

$$\langle M^{c,i}(x,t), M^{c,j}(y,t)\rangle = \int_0^t a^{ij}(x,y,s)dA_s,$$

$$B^{c,i}(x,t) = \int_0^t b^i(x,s)dA_s,$$

and

$$\hat{N}(t,E) = \int_0^t \nu_s(E)dA_s.$$

The system (a,b,ν) is called the characteristic of semimartingale $X(x,t)$ with respect to A_t.

Let $X(x,t), t \geq 0$ be a C-valued cadlag semimartingale equipped with the characteristic (a,b,ν). We introduce a condition;

Condition (A). For a positive predictable process $K_t, t \geq 0$ satisfying

$$\int_0^T K_t dA_t < \infty \text{ a.s. for any } T > 0,$$

(i) $a(x,y,t)$ is a continuous $\tilde{C}_b^{1+1}$-valued process satisfying

$$\|a(t)\|\tilde{}_{1+1} \leq K_t \text{ a.s.}$$

(ii) $b(x,t)$ is a continuous C_b^{0+1}-valued process satisfying

$$\|b(t)\|_{0+1} \leq K_t \text{ a.s.}$$

(iii) The measure $\nu_t(\cdot)$ is supported by C_b^{1+1}. Further, there exists a Borel set $U \subset C_b^{1+1}$ such that for some constant $c > 0$, $\|\nu\|_{1+1} \leq c$ for all $v \in U$, and

$$\nu_t(U^c) \leq K_t, \text{ and } \int_U \|v\|_{1+1}^2 \nu_t(dv) \leq K_t.$$

Let $\{\xi_t, t \geq 0\}$ be an $\mathbb{R}^d$-valued cadlag process satisfying Condition (A) adapted to $(\mathcal{F}_t)$. Then we can define the *Itô* integrals and the *Stratonovich* integrals, respectively;

$$\int_s^t X(\xi_{r-}, dr), \text{ and } \int_s^t X(\xi_r, \circ dr).$$

Let $v(x)$ be a *Lipschitz* continuous vector field. Then by Condition (A)-(iii), the possible infinite sum

$$\sum_{s \leq t} [exp(\Delta X(s))(x) - x - \Delta X(x,s)]$$

is absolutely convergent a.s.. Therefore, we can define the canonical integral of a cadlag semimartingale ξ_t based on the vector field-valued semimartingale $X(t)$ as following;

$$\begin{aligned}\int_s^t X(\xi_r, \diamond dr) &= \int_s^t X_c(\xi_r, \circ dr) + \int_s^t X_d(\xi_{r-}, dr) \\ &\quad + \sum_{s \leq r \leq t} [exp(\Delta X(r))(\xi_{r-}) - \xi_{r-} - \Delta X(\xi_{r-}, r)],\end{aligned}$$

where the first part and the second part of the right hand side are *Stratonovich* integral and *Itô* integral, respectively.

Let $X(x,t), t \geq 0$ be a C-valued semimartingale whose characteristic satisfy Condition (A). Consider a canonical SDE which is represented by

$$\text{(I-1)} \qquad \xi_t(x) = x + \int_0^t X(\xi_s(x), \diamond ds),$$

where $0 \leq s \leq t$. The process ξ_t satisfying (I-1) is called a solution of the canonical SDE (I-1) driven by the vector field-valued semimartingale $X(t)$.

PROPOSITION I-1. *Assume that the characteristics of the C-valued semimartingale $X(t)$ satisfy Condition (A). Then the canonical SDE (I-1) has a unique solution $\xi_{s,t}(x), t \geq s$ for any s, x. Further, a certain version $\xi_{s,t}(x)$ of the solution admits the following properties;*

(i) $\xi_{s,u}(x) = \xi_{t,u}(\xi_{s,t}(x))$ holds for all $x \in \mathbb{R}^d$ and $s < t < u$, a.s.

(ii) The map $\xi_{s,t} : \mathbb{R}^d \to \mathbb{R}^d$ is an onto homeomorphism for all $s < t$ a.s.

(iii) $\xi_{s,t}$ is a C-valued cadlag processes in both s and t.

The above $\xi_{s,t}$ is called the stochastic flow of homeomorphisms generated by X_t.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{\mathcal{F}_{s,t}; 0 \leq s \leq t \leq T\}$ be a two parameter family of sub-σ-field of $\mathcal{F}$ which contains all null sets and satisfy

$$\mathcal{F}_{s,t} \subset \mathcal{F}_{s',t'}, \text{ if } s' \leq s \leq t \leq t',$$

and

$$\cap_{\epsilon>0}\mathcal{F}_{s,t+\epsilon} = \mathcal{F}_{s,t}, \text{ and } \cap_{\epsilon>0} \mathcal{F}_{s-\epsilon,t} = \mathcal{F}_{s,t}$$

for any $s < t$. A C-valued cadlag process $\{X_t, t \geq 0\}$ is called a forward-backward semimartingale if $X_t - X_s, t \in [s, T]$ is a forward semimartingale adapted to the filtration $(\mathcal{F}_{s,t})_{t\in[s,T]}$ for any s and also $X_t - X_s, s \in [0, t]$ is a backward semimartingale adapted to the filtration $(\mathcal{F}_{s,t})_{s\in[0,t]}$ for any t.

Let $\{\xi_s, 0 \leq s \leq t\}$ (t is fixed) be a process adapted to the filtration $(\mathcal{F}_{s,t})_{0\leq s\leq t<\infty}$. The backward *Itô* integral of ξ_s based on a forward-backward semimartingale $X(x, t)$ is defined by

$$\int_s^t X(\xi_{r-}, \hat{d}r) = lim_{|\delta|\to 0} \sum_{k=1}^{m} [X(\xi_{t_k}, t_k) - X(\xi_{t_k}, t_{k-1})]$$

This integral is also a backward cadlag semimartingale with respect to s. The backward *Stratonovich* integral is defined similarly.

The canonical backward integral of a cadlag semimartingale ξ_t based on the forward-backward semimartingale $X(x, t)$ can be defined similarly;

$$\int_s^t X(\xi_r, \diamond\hat{d}r) = \int_s^t X_c(\xi_r, \circ\hat{d}r) + \int_s^t X_d(\xi_{r-}, \hat{d}r) + \sum_{s\leq r\leq t} [exp(\Delta X(r))(\xi_{r-}) - \xi_{r-} - \Delta X(\xi_{r-}, r)],$$

where the first term of the right hand side is the Stratonovich integral.

PROPOSITION I-2. *Let $X(t)$ be the C-valued semimartingale of Proposition I-1. Assume that $X(t)$ is a forward-backward semimartingale. Then the inverse flow $\xi_{s,t}^{-1}$ is a cadlag C-valued process both in s and t. Further, it is a backward semimartingale and satisfies the following Itô backward SDE;*

$$\xi_{s,t}^{-1}(y) = y + \int_s^t \hat{X}(\xi_{r,t}^{-1}(y), \hat{d}r), \tag{I-2}$$

where

$$\hat{X}(x,t) = -X(x,t) + \int_s^t c(x,s)dA_s + \sum_{s\le t}[e^{-\Delta X(s)}(x) - x - \Delta X(x,s)]$$

Thus $\xi_{s,t}^{-1}$ is represented as a solution of a canonical backward SDE driven by $-X$;

$$\xi_{s,t}^{-1}(y) = y + \int_s^t (-X)(\xi_{r,t}^{-1}(y), \diamond\hat{d}r), \tag{I-3}$$

Now, we consider a forward-backward semimartingale $X(t)$ having the characteristic (a, b, ν) with respect to A_t associated with U. It is known that under the following Condition (A^*), we can define the forward flow $\xi_{s,t}(x)$ and the backward flow $\xi_{s,t}^{-1}(y)$, respectively.

Condition (A^*). For a positive predictable process $K_t, t \ge 0$ satisfying

$$\int_0^T K_t dA_t < \infty \text{ a.s. for any } T > 0,$$

(i) $a(x,y,t)$ is a continuous $\tilde{C}_b^{2+1}$-valued process satisfying

$$\|a(t)\|_{2+1}^{\sim} \le K_t \text{ a.s.}$$

(ii) $b(x,t)$ is a continuous C_b^{1+1}-valued process satisfying

$$\|b(t)\|_{1+1} \le K_t \text{ a.s.}$$

(iii) The measure $\nu_t(\cdot)$ is supported by C_b^{2+1}. Further, there exists a Borel set $U \subset C_b^{2+1}$ such that for some constant $c > 0$, $\|\nu\|_{2+1} \le c$ for all $v \in U$, and

$$\nu_t(U^c) \le K_t, \text{ and } \int_U \|v\|_{2+1}^2 \nu_t(dv) \le K_t.$$

PROPOSITION I-3. *(c.f.[5]) Let X_t be a C-valued semimartingale satisfying Condition (A^*). Further, assume that X_t is a forward-backward semimartingale. Let $\{\xi_{s,t}; 0 \leq s \leq t \leq T\}$ be a stochastic flow determined by the SDE;*

$$\xi_{s,t}(x) = x + \int_s^t X(\xi_{s,r-}(x), \diamond ds).$$

Then the inverse $\xi_{s,t}^{-1}(y)$ is a backward semimartingale and satisfies the canonical backward SDE;

$$\xi_{s,t}^{-1}(y) = y + \int_s^t (-X)(\xi_{r,t}^{-1}(y), \diamond \hat{d}r).$$

Let $\{\gamma_r\}_{r>0}$ be a family of diffeomorphisms of manifold M satisfying the following (a)-(d).

(a) $\gamma_r(p)$ is differentiable with respect to $(r, p) \in (0, \infty) \times M$.

(b) $\gamma_r \circ \gamma_s = \gamma_{rs}$ holds for all $r, s > 0$.

(c) There exists a point $p_0 \in M$ such that $\gamma_r(p_0) = p_0$ holds for all $r > 0$.

(d) $\lim_{r\to 0} \gamma_r(p) = p_0$ holds uniformly on the compact sets of M.

Then we call it a dilation over M. Now, we define the dilation on $\mathbb{R}^d$, and recall an operator self-similarity with exponent Q for $\mathbb{R}^d$-valued processes. Let Q be an $d \times d$-matrix such that real parts of its eigenvalues are all positive. Consider an invertible linear transformation γ_r from $\mathbb{R}^d$ to itself of the form;

$$\begin{aligned} \gamma_r &:= exp(log\ r)Q, \text{ for } r > 0 \\ &:= r^Q. \end{aligned}$$

Then, because of $r^Q s^Q = (rs)^Q$, the linear transformations $\{\gamma_r\}_{r>0}$ satisfy $\gamma_r\gamma_s = \gamma_{rs}$ for all $s, t > 0$, and also we can define that;

$$\gamma_r(x) \to 0 \text{ as } r \to 0,$$

for any $x \in \mathbb{R}^d$. We call this one-parameter group $\{\gamma_r\}_{r>0}$ of automorphisms as a *dilation* with exponent Q on $\mathbb{R}^d$.

Let $\{X_t; t \in [0,T]\}$ be a forward-backward semimartingale. An $\mathbb{R}^d$-valued forward process $\{X_t, t \geq 0\}$ is called self-similar with respect to a dilation $\{\gamma_r\}_{r>0}$ if the law of the stochastic process $\{\gamma_r X_t, t \geq 0\}$ is equal to that of $\{X_{rt}, t \geq 0\}$ for any $r > 0$. Let $\{\hat{X}_t, 0 \leq t \leq T\}$ be a backward semimartingale on the same probability space. An $\mathbb{R}^d$-valued backward process $\hat{X}_t$ is backward self-similar with respect to a dilation $\{\delta_r\}_{r>0}$ if the laws of the stochastic processes $\{\delta_r \hat{X}_t, t \in [0,T]\}$ and $\{\hat{X}_{t/r}, t \in [0,T]\}$ are same for any $r > 0$. Since a dilation is an invertible linear transformation, we can think an inverse linear transformations $\{\delta_r^{-1}\}_{r>0}$ as an inverse dilation of $\{\delta_r\}_{r>0}$. Thus, if we assume that $\hat{X}_t$ is backward self-similar with respect to a dilation $\{\delta_r\}_{r>0}$, then we can get the following re! lati on; by the law,

$$\delta_r^{-1}\hat{X}_t = \hat{X}_{rt}, \text{ for all } r > 0,$$

because of

$$\hat{X}_t = \delta_r^{-1} \circ \delta_r \hat{X}_t = \delta_r^{-1}\hat{X}_{t/r}.$$

We think an inverse linear transformations $\{\gamma_r^{-1}\}_{r>0}$ of the (forward) dilation $\{\gamma_r\}_{r>0}$, and assume the following relation; by the law,

$$\gamma_r^{-1}\hat{X}_t = \hat{X}_{t/r}, \text{ for all } r > 0.$$

If Q, the exponent of dilation $\gamma_r = r^Q$, is semisimple, then we get $\gamma_r^{-1} = r^{-Q}$, where $-Q$ is the inverse matrix of Q. Thus, we get $\gamma_r^{-1} = \gamma_{1/r}$ for all $r > 0$, and

$$\gamma_r^{-1}\hat{X}_t = \gamma_{1/r}\hat{X}_t = \hat{X}_{t/r}.$$

II. Backward self-similar stochastic flows

Consider a canonical SDE of the form;

$$d\xi_t(x) = \sum_{j=1}^{m} v_j(\xi_t(x)) \diamond dZ_t^j \tag{II-1}$$

with initial condition $\xi_0(x) = x$, which is driven by a vector field-valued semimartingale $X_t(x) = \sum_{j=1}^{m} v_j(x) Z_t^j$, where $Z_t = (Z_t^1, Z_t^2, \cdots, Z_t^m)$

is an $\mathbb{R}^m$-valued semimartingale and $v_1, v_2, \cdots, v_m$ are the smooth complete vector fields on $\mathbb{R}^d$. Let $\mathcal{L}$ be an algebra generated by the vector fields $v_1, v_2, \cdots, v_m$. Then the linear combination $\sum_{j=1}^m v_j Z_t^j$ can be an element of $\mathcal{L}$.

By the solution of (II-1), we can define an $\mathbb{R}^d$-valued semimartingale flow $\{\xi_{s,t}(x); 0 \le s \le t \le T\}$ adapted to $\mathcal{F}_{s,t} = \sigma(Z_{s,t}; 0 \le s \le t \le T)$ satisfying;

(II-2)

$$\begin{aligned}\xi_{s,t}(x) &= x + \sum_{j=1}^m \int_s^t v_j(\xi_{s,r-}(x)) \diamond dZ_r^j \\ &= x + \sum_{j=1}^m \int_s^t v_j(\xi_{s,r}(x)) \circ dZ_r^{c,j} + \sum_{j=1}^m \int_s^t v_j(\xi_{s,r-}(x)) dZ_r^{d,j} \\ &+ \sum_{s \le r \le t} [exp(\sum_{j=1}^m \Delta Z_r^j v_j)(\xi_{s,r-}(x)) \\ &- \xi_{s,r-}(x) - \sum_{j=1}^m v_j(\xi_{s,r-}(x)) \Delta Z_r^j].\end{aligned}$$

We assume that;

(A.1) $dim(\mathcal{L}) < \infty$,

(A.2) $dim(\mathcal{L}(x)) = d$ hold for all $x \in \mathbb{R}^d$, where $\mathcal{L}(x) = \{v_x; v \in \mathcal{L}\}$ and v_x is the projection of v to the point $x \in \mathbb{R}^d$.

(A.3) The semimartingale $\{Z_t\}$ is nondegenerate.

Then it is known that, for any $x \in \mathbb{R}^d$, the equation (II-2) has a global unique solution $\{\xi_{s,t}(x); 0 \le s \le t \le T\}$ which is called a stochastic flow generated by SDE (II-2).

A two-parameters stochastic flow $\{\xi_{s,t}(x); 0 \le s \le t \le T\}$ generated by the SDE (II-2) is said forward self-similar with respect to the dilation $\{\psi_r\}_{r>0}$ if the laws of $\{\psi_r \circ \xi_{s,t} \circ \psi_r^{-1}(x); 0 \le s \le t \le T\}$ and $\{\xi_{s,rt}(x); 0 \le s \le t \le T\}$ are same for any $r > 0$. Thus, we get the followings;

PROPOSITION II-1. *(c.f.[3] Theorem 2.2) Suppose that the stochastic flow $\{\xi_{s,t}(x); 0 \le s \le t \le T\}$ driven by $\{Z_t\}$ through SDE (II-2) is self-similar with respect to a certain dilation $\{\psi_r\}_{r>0}$. Then the $\mathbb{R}^d$-valued driving process $\{Z_t; t \ge 0\}$ is also self-similar with respect to a* *dilation* $\{\gamma_r\}_{r>0}$ *such that* $d\psi_r = \gamma_r$.

Proposition II-2. *(c.f.[3] Theorem 2.4) Let* $\{\xi_{s,t}(x); 0 \le s \le t \le T\}$ *be a stochastic flow on* $\mathbb{R}^d$ *driven by* $\mathbb{R}^d$*-valued self-similar semimartingale* $\{Z_t, t \ge 0\}$ *with respect to dilation* $\gamma_r = r^Q$ *through SDE (II-2). Suppose that the exponent* Q *of dilation* $\{\gamma_r\}_{r>0}$ *admits a linear extension* $\tilde{Q}$ *such that* $\tilde{\gamma}_r = r^{\tilde{Q}}$ *on the space* $\mathcal{L}$*. Then the stochastic flow* $\{\xi_{s,t}(x); 0 \le s \le t \le T\}$ *is also self-similar with respect to a certain dilation* $\{\psi_r\}_{r>0}$ *such that* $d\psi_r = \tilde{\gamma}_r$.

For a backward vector field-valued semimartingale process $\hat{X}_t(x) = \sum_{j=1}^m v_j(x)\hat{Z}_t^j$, consider a backward SDE of the form;

$$d\xi_s^{-1}(y) = \sum_{j=1}^m (-v_j)(\xi_s^{-1}(y)) \diamond \hat{d}Z_s^j. \tag{II-3}$$

Then we can define the inverse flow $\{\xi_{s,t}^{-1}; 0 \le s \le t \le T\}$ by the solution of following backward SDE;

$$\xi_{s,t}^{-1}(y) = y + \sum_{j=1}^m \int_s^t (-v_j)(\xi_{u,t}^{-1}(y)) \diamond \hat{d}Z_u^j. \tag{II-4}$$

similarly as SDE (II-2).

To study the backward self-similar stochastic flow, we define it. A two-parameters backward stochastic flow $\{\hat{\xi}_{s,t}(y); 0 \le s \le t \le T\}$ is backward self-similar with respect to a dilation $\{\theta_r\}_{r>0}$ if, for fixed t, the laws of $\{\theta_r \circ \hat{\xi}_{s,t} \circ \theta_r^{-1}(y)\}$ and $\{\hat{\xi}_{s/r,t}(y)\}$ are same for any $r > 0$.

Since an inverse flow is a backward flow, if the inverse flow $\{\xi_{s,t}^{-1}(y); 0 \le s \le t \le T\}$ generated by the backward SDE (II-4) is backward self-similar with respect to the dilation $\{\theta_r\}_{r>0}$, then we get that, for the inverse dilation $\{\theta_r^{-1}\}_{r>0}$ of $\{\theta_r\}_{r>0}$, the laws of $\{\theta_r^{-1} \circ \xi_{s,t}^{-1} \circ \theta_r(y)\}$ and $\{\xi_{rs,t}^{-1}(y)\}$ are same for any $r > 0$. Further, if we think an inverse linear transformations $\{\psi_r^{-1}\}_{r>0}$ of the (forward) dilation $\{\psi_r\}_{r>0}$ and assume that $\{\xi_{s,t}^{-1}(y); 0 \le s \le t \le T\}$ is backward self-similar with respect to $\{\psi_r^{-1}\}_{r>0}$, then we get the relation; by the law,

$$\psi_r^{-1} \circ \xi_{s,t}^{-1} \circ \psi_r(y) = \xi_{s/r,t}^{-1}(y) \text{ for all } r > 0.$$

Thus we can get the following;

Theorem II-1. *Let* $\{\xi_{s,t}^{-1}(y); 0 \leq s \leq t \leq T\}$ *be an* $\mathbb{R}^d$*-valued backward self-similar stochastic flow with respect to dilation* $\{\theta_r\}_{r>0}$ *generated by the backward SDE (II-4). Then the driving process* $\{\hat{X}_t; 0 \leq t \leq T\}$ *is also backward self-similar with respect to a dilation* $\{d\theta_r\}_{r>0}$*, which is of the form*

$$d\theta_r = \delta_r, r > 0.$$

Proof. For a fixed $r > 0$, we put

$$\tilde{\xi}_{s,t}^{-1}(y) := \theta_r \circ \xi_{s,t}^{-1} \circ \theta_r^{-1}(y),\ 0 \leq s \leq t \leq T.$$

Then, since $\{\xi_{s,t}^{-1}\}$ satisfies (II-4), we get that $\{\tilde{\xi}_{s,t}^{-1}\}$ satisfies; for $s \leq u \leq t$,

(II-5)

$$\begin{aligned}\tilde{\xi}_{s,t}^{-1}(y) &= y + \sum_{j=1}^{m} \int_s^t (-v_j)(\tilde{\xi}_{u,t}^{-1}(y)) \diamond \hat{d}Z_u^j \\ &= y + \sum_{j=1}^{m} \int_s^t (-v_j)(\tilde{\xi}_{u,t}^{-1}(y)) \circ \hat{d}Z_u^{c,j} + \sum_{j=1}^{m} \int_s^t (-v_j)(\tilde{\xi}_{u,t}^{-1}(y)) \circ \hat{d}Z_u^{d,j} \\ &+ \sum_{s\leq u\leq t} [exp(\sum_{j=1}^{m} \Delta\hat{Z}_u^j(-v_j))(\tilde{\xi}_{u,t}^{-1}(y)) - \tilde{\xi}_{u,t}^{-1}(y) \\ &- \sum_{j=1}^{m}(-v_j)(\tilde{\xi}_{u,t}^{-1}(y))\Delta\hat{Z}_u^j].\end{aligned}$$

Therefore, we get, for $0 \leq s \leq u \leq t \leq T$,

(II-6)

$$\begin{aligned}\tilde{\xi}_{s,t}^{-1}(y) = y &+ \sum_{j=1}^{m} \int_s^t (-v_j)(\theta_r \circ \theta_r^{-1} \circ \tilde{\xi}_{u,t}^{-1}(y)) \circ \hat{d}Z_u^{c,j} \\ &+ \sum_{j=1}^{m} \int_s^t (-v_j)(\theta_r \circ \theta_r^{-1} \circ \tilde{\xi}_{u,t}^{-1}(y)) \circ \hat{d}Z_u^{d,j} \\ &+ \sum_{s\leq u\leq t} [\theta_r exp(\sum_{j=1}^{m} \Delta\hat{Z}_u^j(-v_j))(\theta_r^{-1} \circ \tilde{\xi}_{u,t}^{-1}(y)) - \tilde{\xi}_{u,t}^{-1}(y) \\ &- \sum_{j=1}^{m}(-v_j)(\theta_r \circ \theta_r^{-1} \circ \tilde{\xi}_{u,t}^{-1}(x))\Delta\hat{Z}_u^j].\end{aligned}$$

But since

$$(-v_j)(\theta_r \circ \theta_r^{-1} \circ \xi_{u,t}^{-1}(y)) = d\theta_r \circ (-v_j) \circ \tilde{\xi}_{u,t}^{-1}(y)),$$

and

$$\theta_r(exp(\sum_{j=1}^{m} \Delta \hat{Z}_u^j(-v_j))(\theta_r^{-1} \circ \tilde{\xi}_{u,t}^{-1}(y)) = exp(d\theta_r \sum_{j=1}^{m} Z_u^j(-v_j))(\tilde{\xi}_{u,t}^{-1}(y)),$$

we see that $\{\tilde{\xi}_{s,t}^{-1}\}$ is driven by $\{d\theta_r \sum_{j=1}^{m}(-v_j)\hat{Z}_s^j\}$.

On the other hand, $\{\tilde{\xi}_{s/r,T}^{-1}(y)\}$ is driven by $\{\sum_{j=1}^{m}(-v_j)\hat{Z}_{s/r}^j(y)\}$. Since the law of $\{\tilde{\xi}_{s,t}^{-1}(y)\}$ coincides with the law of $\{\sum_{j=1}^{m}(-v_j)\hat{Z}_{s/r}^j(y)\}$, we get that the law of $\{\sum_{j=1}^{m} d\theta_r(-v_j)\hat{Z}_s^j\}$ coincides with the law of $\{\sum_{j=1}^{m}(-v_j)\hat{Z}_{s/r}^j(y)\}$ for any $r > 0$. This implies that $d\theta_j(v_j) \in \mathcal{L}$ for any j. Thus $d\theta_r$ maps $\mathcal{L}$ into itself. Let Q be an exponent of the dilation $d\theta_r = \delta_r$. Then the law of the process $\{\delta_r \hat{X}_t\}$ coincides with the law of the process $\{\hat{X}_{t/r}\}$ for any $r > 0$. This show that the driving process $\{\hat{X}_t; t \in [0, T]\}$ is backward self-similar with respect to dilation $d\theta_r = \delta_r$. □

Theorem II-2. *Let $\{\xi_{s,t}^{-1}(y); 0 \le s \le t \le T\}$ be an inverse flow on $\mathbb{R}^d$ driven by a backward self-similar semimartingale $\{\hat{Z}_t, t \ge 0\}$ with respect to the dilation $\delta_r = r^Q$ through SDE (II-4). Suppose that Q admit a linear extension $\tilde{Q}$ such that $\tilde{\delta}_r = r^{\tilde{Q}}$ on the space $\mathcal{L}$. Then the inverse flow $\{\xi_{s,t}^{-1}(y); 0 \le s \le t \le T\}$ is also backward self-similar with respect to a certain dilation $\{\theta_r\}_{r>0}$ such that $d\theta_r = \tilde{\delta}_r$.*

Proof. It is need to construct the dilation $\{\theta_r\}_{r>0}$ which makes the backward self-similar flow $\xi_{s,t}^{-1}(y)$. For the purpose, for a given automorphism $\tilde{\delta}_r$ of $\mathcal{L}$, we have to construct a diffeomorphism $\{\theta_r\}_{r>0}$ of $\mathbb{R}^d$ such that $d\theta_r = \tilde{\delta}_r$.

On the other hand, let $\tilde{\delta}_r$ be an automorphism of $\mathcal{L}$. Then, by the theory of [3], we know that there exists a unique diffeomorphism θ_r of $\mathbb{R}^d$ such that, for any $x \in \mathbb{R}^d$, $\theta_r(x) = x$ and $d\theta_r = \tilde{\delta}_r$. Therefore, for the inverse linear transformation $\tilde{\delta}_r^{-1}$ of δ_r, we can get the inverse diffeomorphism θ_r^{-1} of θ_r such that $d\theta_r^{-1} = \tilde{\delta}_r^{-1}$. This dilation $\{\theta_r^{-1}\}_{r>0}$ makes the backward self-similar flow $\xi_{s,t}^{-1}(y)$. Indeed, let $\tilde{Q}$ be the exponent of inverse dilation of $\tilde{\delta}_r = r^{\tilde{Q}}$. Then $\{\tilde{\delta}_r^{-1}\}_{r>0}$ such that $\tilde{\delta}_r^{-1} = r^{\tilde{Q}^*}$ is an inverse dilation on $\mathcal{L}$. Then, by the same theory as above (c.f. [2]), there exist an one-parameter group of diffeomorphisms $\{\theta_r^{-1}\}_{r>0}$ such that $\theta_r^{-1}(y) = y$ and $d\theta_r^{-1} = \tilde{\delta}!_r^{-1}$ hold for any $r > 0$. It is immediate that this inverse dilation $\{\theta_r^{-1}\}_{r>0}$ is a dilation which we want to find.

Finally, we shall prove that the inverse flow $\{\xi_{s,t}^{-1}(y)\}$ is backward self-similar with respect to this dilation $\{\theta_r^{-1}\}_{r>0}$. Set

$$\tilde{\xi}_{s,t}^{-1}(y) := \theta_r^{-1} \circ \xi_{s,t}^{-1} \circ \theta_r(y),\ 0 \le s \le t \le T,$$

and $\tilde{Z}_t := d\theta_r \hat{Z}_t$. Then, from the equation (II-6), we get;
(II-7)

$$\begin{aligned}
\tilde{\xi}_{s,t}^{-1}(y) &= y + \sum_{j=1}^{m} \int_s^t (-v_j)(\tilde{\xi}_{u,t}^{-1}(y)) \diamond \hat{d}\tilde{Z}_u^j \\
&= y + \sum_{j=1}^{m} \int_s^t (-v_j)(\tilde{\xi}_{u,t}^{-1}(y)) \circ \hat{d}\tilde{Z}_u^{c,j} + \sum_{j=1}^{m} \int_s^t (-v_j)(\tilde{\xi}_{u,t}^{-1}(y)) \circ \hat{d}\tilde{Z}_u^{d,j} \\
&+ \sum_{s \le u \le t} [exp(\sum_{j=1}^{m} \Delta\tilde{Z}_u^j(-v_j))(\tilde{\xi}_{u,t}^{-1}(y) - \tilde{\xi}_{u,t}^{-1}(y) \\
&- \sum_{j=1}^{m}(-v_j)(\tilde{\xi}_{u,t}^{-1}(x))\Delta\tilde{Z}_u^j].
\end{aligned}$$

Therefore, SDE (II-7) shows that $\{\tilde{\xi}_{u,t}^{-1}(y)\}$ is driven by $\{\sum_{j=1}^{m}(-v_j)\tilde{Z}_u^j\}$. Since the backward process $\{\hat{X}_u\}$ such that $\hat{X}_u = \sum_j(-v_j)\hat{Z}_u^j$ is backward self-similar with respect to dilation $\{\tilde{\delta}_r^{-1}\}$ and $d\theta_r^{-1} = \tilde{\delta}_r^{-1}$ holds, the law of semimartingale $\{\sum_{j=1}^{m}(-v_j)\tilde{Z}_u^j\}$ coincides with the law of the semimartingale $\{\sum_{j=1}^{m}(-v_j)\tilde{Z}_{rt}\}$. This implies that the law of the flow $\{\tilde{\xi}_{s,t}^{-1}\}$ coincides with the law of the flow $\{\xi_{rs,t}^{-1}\}$ for any $r > 0$. Thus $\{\xi_{s,t}^{-1}\}$ is backward self-similar with respect to the dilation $\{\theta_r^{-1}\}_{r>0}$.□

THEOREM II-3. *Let $X_t(x) = \sum_{j=1}^m v_j(x)Z_t^j$ be a forward-backward vector field -valued semimartingale. Let $\{\xi_{s,t}(x); 0 \leq s \leq t \leq T\}$ be a forward self-similar stochastic flow with respect to a dilation $\{\psi_r\}_{r>0}$ generated by SDE (II-2). If the inverse flow $\xi_{s,t}^{-1}(y)$ is generated by the canonical backward SDE (II-4), then it is backward self-similar with respect to the inverse dilation $\{\psi_r^{-1}\}_{r>0}$ of $\{\psi_r\}_{r>0}$.*

Proof. Let $\xi_{s,t}(x)$ be a self-similar stochastic flow generated by SDE (II-2). Then from the Proposition I-3, we see that $\xi_{s,t}^{-1}(y)$ is a inverse flow of $\xi_{s,t}(x)$ and satisfies the backward SDE (II-4).

If $\xi_{s,t}(x)$ is a self-similar stochastic flow with respect to dilation $\{\psi_r\}_{r>0}$, then from the Proposition II-1, the driving process $X_t(x)$ is also self-similar with respect to dilation $\{d\psi_r\}_{r>0}$. Since $\{X_t; 0 \leq s \leq t \leq T\}$ is a forward-backward semimartingale, the backward process $\{\hat{X}_t; 0 \leq s \leq t \leq T\}$ is also backward self-similar with respect to the inverse dilation $\{d\psi_r^{-1}\}_{r>0}$. Therefore, from Theorem II-2, there is a dilation $\{\psi_r^{-1}\}_{r>0}$ such that the inverse flow $\xi_{s,t}^{-1}(y)$ is backward self-similar with respect to the inverse dilation $\{\psi_r^{-1}\}_{r>0}$. □

Now, we will introduce the definition of two-sided self-similar stochastic flow. Because the stochastic flow $\{\xi_{s,t}; 0 \leq s \leq t \leq T\}$ driven by the forward-backward semimartingale $\{X_t; t \in [0,T]\}$ is also a two-parameters forward-backward semimartingale flow, we can define as following; A two-parameters stochastic flow $\{\xi_{s,t}; 0 \leq s \leq t \leq T\}$ is two-sided self-similar with respect to backward dilation $\{\theta_r\}_{r>0}$ and to forward dilation $\{\psi_r\}_{r>0}$, where $\{\theta_r\}_{r>0}$ play a role to the backward flow and $\{\psi_r\}_{r>0}$ play a role to the forward flow, if the laws of

$$\{\theta_r \circ (\psi_r \circ \xi_{s,t} \circ \psi_r^{-1}) \circ \theta_r^{-1}; 0 \leq s \leq t \leq T\}$$

(or $\{\psi_r \circ (\theta_r \circ \xi_{s,t} \circ \theta_r^{-1}) \circ \psi_r^{-1}; 0 \leq s \leq t \leq T\}$) and $\{\xi_{s/r,rt}\}$ are same for all $r > 0$. Therefore, if $\{\theta_r\}_{r>0}$ is an identity matrix, then the two-sided self-similar flow $\{\xi_{s,t}; 0 \leq s \leq t \leq T\}$ is only forward self-similar, and if $\{\psi_r\}_{r>0}$ is an identity matrix, then it becomes only backward self-similar.

THEOREM II-4. *Let $\{\xi_{s,t}; 0 \leq s \leq t \leq T\}$ be an $\mathbb{R}^d$-valued two-sided self-similar semimartingale flow such that the forward flow $\{\xi_{s,t}\}$*

is generated by the forward SDE (II-2) and the backward flow $\{\xi_{s,t}^{-1}\}$ *is generated by the backward SDE (II-4). Then there exist a forward-backward semimartingale* $\{X_t; t \in [0,T]\}$ *such that the forward semimartingale* X_t *is a driving process of the forward flow* $\xi_{s,t}(x)$ *and is forward self-similar, and the backward semimartingale* $\hat{X}_t$ *is a driving process of the inverse flow* $\xi_{s,t}^{-1}(y)$ *and is backward self-similar.*

Proof. Let $\{\theta_r\}_{r>0}$ be an identity matrix. For the forward self-similar semimartingale flow $\xi_{s,t}(x)$ with respect to $\{\psi_r\}_{r>0}$, from the Proposition II-1, we get the forward semimartingale $\{X_t, t \in [0,T]\}$ as a driving process such that X_t is forward self-similar with respect to a dilation $\{d\psi_r\}_{r>0}$.

On the other hand, if $\{\psi_r\}_{r>0}$ is an identity matrix, then the backward flow $\{\hat{\xi}_{s,t}(y)\}$ generated by (II-4) is backward self-similar with respect to the backward dilation $\{\theta_r\}_{r>0}$, and there exists a driving process $\{\hat{X}_s; s \in [0,T]\}$ such that $\hat{X}_s$ is backward self-similar with respect to dilation $\{d\theta_r\}_{r>0}$. Thus, if we think $\{\hat{\xi}_{s,t}(y)\}$ as an inverse flow, we get the backward semimartingale $\{\hat{X}_s; s \in [0,T]\}$ as a driving process such that $\hat{X}_s$ is backward self-similar with respect to dilation $d\theta_r$. □

THEOREM II-5. *For the vector field-valued forward-backward semimartingale* $\{X_t; t \in [0,T]\}$*, if the forward process* X_t *is forward self-similar and the backward process* $\hat{X}_t$ *is backward self-similar, then there exists a two-sided self-similar semimartingale flow* $\{\xi_{s,t}; 0 \le s \le t \le T\}$ *such that the forward flow* $\xi_{s,t}(x)$ *is generated by the forward SDE (II-2), and the backward flow* $\xi_{s,t}^{-1}(y)$ *is generated by the backward SDE (II-4).*

Proof. For the forward-backward semimartingale $\{X_t; t \in [0,T]\}$, if X_t is forward self-similar, then from Proposition II-2, there exists $\xi_{s,t}(x)$ generated by SDE (II-2) such that $\xi_{s,t}$ is self-similar with respect to dilation $\{\psi_r\}_{r>0}$. Thus we get that the laws of $\{\psi_r \circ \xi_{s,t} \circ \psi_r^{-1}\}$ and $\{\xi_{s,rt}\}$ are same for any $r > 0$.

For the flow $\{\xi_{s,rt}\}$, if we fixed t for a while, we can think the backward flow $\{\xi_{s,rt}^{-1}\}$ generated by SDE (II-4) whose driving process

$\{\hat{X}_s; s \in [0, rt]\}$ is backward semimartingale. From the assumption, the backward process $\{\hat{X}_s; s \in [0, rt]\}$ is self-similar with respect to a backward dilation $\{\delta_r\}_{r>0}$. If we apply Theorem II-2, there exists a backward self-similar flow $\{\xi_{s,rt}^{-1}\}$ with respect to the backward dilation $\{\theta_r\}_{r>0}$ such that $d\theta_r = \delta_r$. Therefore, from the definition of the backward self-similar flow, if we apply that, by the law,

$$\psi_r \circ \xi_{s,t} \circ \psi_r^{-1} = \xi_{s,rt},$$

then we get that, by the law,

$$\theta_r \circ (\psi_r \circ \xi_{s,t} \circ \psi_r^{-1}) \circ \theta_r^{-1} = \xi_{s/r,rt}. \qquad \square$$

III. Density of self-similar flows

For a canonical SDE (II-1), consider the stochastic flow $\{\xi_{s,t}(x); 0 \leq s \leq t \leq T\}$ generated by SDE (II-2). If we use a Levy process $Z_t = (Z_t^1, Z_t^2, \cdots, Z_t^m)$ of the form;

$$\text{(III-1)} \qquad Z_t^j = W_t^j + b^j t + \int_E z^j \tilde{N}_p((0,t], dz), j = 1, 2, \cdots, m,$$

where $W_t = (W_t^1, W_t^2, \cdots, W_t^m)$ is a Brownian motion, and the compensator $\hat{N}_p((0,t], dz)$ of Poisson point process N_p is of the form

$$\hat{N}_p(ds, dz) = G(dz)ds,$$

where $G(dz)$ is a Lebesgue measure. Then the solution of canonical SDE (II-2) can be represented as;

(III-2)

$$\begin{aligned} \xi_t(x) = x &+ \sum_{j=1}^m \int_0^t v_j(\xi_s(x)) dW_s^j + \int_0^t \mathcal{L}(\xi_{s-}(x)) ds \\ &+ \int_0^t \int_E [exp(\sum_{j=1}^m z^j v_j)(\xi_{s-}(x)) - \xi_{s-}(x)] \tilde{N}_p(ds, dz), \end{aligned}$$

where

$$\mathcal{L}(x) = \mathcal{A}(x) + \int_E [exp(\sum_{j=1}^m z^j v_j)(x) - x - \sum_{j=1}^m z^j v_j(x)]G(dz),$$

$$\mathcal{A}(x) = (1/2)\sum_{j=1}^m v_j^2(x) + v_0(x).$$

Form the equation (III-2), we put as

$$\mathbf{c}(x,z) = exp(\sum_{j=1}^m z^j v_j)(x) - x,$$

and

$$\tilde{\mathbf{c}}(x,z) = \mathbf{c}(x,z) + x.$$

Then we know that $D_x\tilde{\mathbf{c}}(x,z)$ is invertible.

On the other hand, if we put

$$\text{(III-3)} \qquad \mathbb{B}(x) = (a^{ik}(x))_{d\times d},$$

where

$$(a^{ik}(x))_{d\times d} = \sigma_{d\times m}(x)(\sigma_{d\times m}(x))^t$$

and

$$\sigma_{d\times m}(x) = \begin{pmatrix} v_1^1(x) & v_2^1(x) & \cdots & v_m^1(x) \\ v_1^2(x) & v_2^2(x) & \cdots & v_m^2(x) \\ \cdots & & & \\ v_1^d(x) & v_2^d(x) & \cdots & v_m^d(x) \end{pmatrix}_{d\times m},$$

then we see that

$$\mathbb{B}(\tilde{\mathbf{c}}(x,z)) = (D_z\mathbf{c}(x,z))(D_z\mathbf{c}(x,z))^t.$$

Thus we put as following;

$$\text{(III-4)} \qquad \mathbb{C}(x,z) = (D_x\tilde{\mathbf{c}}(x,z))^{-1}\mathbb{B}(\tilde{\mathbf{c}}(x,z))[(D_x\tilde{\mathbf{c}}(x,z)^{-1}]^t.$$

We make two assumptions;

Assumption ($\mathbb{A}$). There exist two constants $\zeta, \theta > 0$ such that

$$|\tilde{\mathbf{c}}(x,z)| \le \zeta(1+|x|^\theta)$$

for all $x \in \mathbb{R}^d$ and $z \in E$.

Assumption ($\mathbb{B}$). There is a Borel subset $\Gamma = \{(x,z)\} \subset \mathbb{R}^d \times E$ such that for any $y \in \mathbb{R}^d$ and for the x-section $\Gamma_x \subset \Gamma$, if $G(\Gamma_x) = \infty$,

$$(\cup_{z\in\Gamma_z}\{y|\mathbb{C}(x,z)y=0\}) \cap \{y|\mathbb{B}(x)y=0\} = \{0\},$$

if $G(\Gamma_x) < \infty$,

$$\mathbb{R}^d \cap \{y|\mathbb{B}(x)y=0\} = \{0\}.$$

Then we get the existence theorem of density.

Proposition III-1. *(c.f.[1]) Under* ($\mathbb{A}$) *and* ($\mathbb{B}$), *the solution* $\xi_t(x)$ *of (III-2) has a density* $y \to p_t(x,y)$ *for all* $x \in \mathbb{R}^d$ *and* $t \in (0,T]$.

Remark. In some sense, this theorem is general. If $Rank\mathbb{B}(x) = d$, then we can get the same result. See Corollary. Even though $Rank\mathbb{B}(x) < d$, $\xi_t(x)$ of (III-2) can have the density. In this case, to get the density, it must be $Rank\mathbb{B}(x)$ (or $Rank\mathbb{C}(x,z)$)$= d/2$, because of $Rank\mathbb{B}(x) = Rank\mathbb{C}(x,z)$.

Corollary. *If* $Rank\mathbb{B}(x) = d$ *or* $Rank\mathbb{C}(x,z) = d$ *for all* $z \in E$, *then the solution* $\xi_t(x)$ *of (III-2) has a density* $y \to p_t(x,y)$ *for all* $x \in \mathbb{R}^d$ *and* $t \in (0,T]$.

From the above Proposition III-1, we can define a density of stochastic flow which is generated by some SDE (III-2). Therefore, we can think the densities of the distribution of self-similar stochastic flow generated by (III-2). We denote by $P_{s,t}(x,A)$ the distribution of $\xi_{s,t}(x)$. If the stochastic flow $\{\xi_{s,t}(x); 0 \le s \le t \le T\}$ is self-similar, then we have

$$P_{s,t}(x,A) = P_{s,rt}(\psi_r(x), \psi_r(A)).$$

Let $p_{s,t}(x,y)$ be the density of distribution $P_{s,t}(x,A)$ of $\xi_{s,t}(x)$. For the simplicity, we assume that $p_t(x,y) := p_{0,t}(x,y)$. Then we get;

PROPOSITION III-2. *(c.f.[4] Theorem 4.1) Let the stochastic flow $\{\xi_{s,t}(x); 0 \leq s \leq t \leq T\}$ generated by SDE (III-2) be self-similar with respect to dilation $\{\psi_r\}_{r>0}$. If $p_t(x,y)$ is a density of the probability distribution of $\xi_t(x)$ with respect to Lebesgue measure $G(\cdot)$, then we get*

$$p_t(x,y) = (1/det(\psi_t))p_1(\psi_t^{-1}(x), \psi_t^{-1}(y)).$$

Sketch of Proof. For a fixed $s = 0$, we put $\xi_{s,t}(x) := \xi_t(x)$. Then

$$\mathbb{P}(\xi_t(x) \in A) = \int_A p_t(x,y)G(dy),$$

where $G(\cdot)$ is a Lebesgue measure, and

$$\begin{aligned}\mathbb{P}(\psi_t \circ \xi_1 \circ \psi_t^{-1}(x) \in A) &= \mathbb{P}(\xi_1(\psi_t^{-1}(x)) \in \psi_t^{-1}(A)) \\ &= \int_{\psi_t^{-1}(A)} p_1(\psi_t^{-1}(x), y)G(dy) \\ &= \int_A p_1(\psi_t^{-1}(x), \psi_t^{-1}(y))G(d\psi_t^{-1}(y)) \\ &= \int_A p_1(\psi_t^{-1}(x), \psi_t^{-1}(y))(1/det(\psi_t))G(dy).\end{aligned}$$

Therefore, we get

$$p_t(x,y) = (1/det(\psi_t))p_1(\psi_t^{-1}(x), \psi_t^{-1}(y)). \qquad \square$$

For the forward-backward vector field-valued semimartingale $X_t(x)$, let us think the backward SDE (II-3). If we use a backward Levy process $\hat{Z}_t = (\hat{Z}_t^1, \hat{Z}_t^2, \cdots, \hat{Z}_t^m)$ for the Z_t of (III-1), we can define the inverse flow $\xi_{s,t}^{-1}(y)$ by the solution of the backward SDE;
(III-5)

$$\begin{aligned}\xi_t^{-1}(y) = y &+ \sum_{j=1}^m \int_0^t (-v_j)(\xi_s^{-1}(y))\hat{d}W_s^j + \int_0^t \hat{\mathcal{L}}(\xi_s^{-1}(y))\hat{d}s \\ &+ \int_0^t \int_E [exp(\sum_{j=1}^m z^j(-v_j))(\xi_s^{-1}(y)) - \xi_s^{-1}(y)]\tilde{N}_p(\hat{d}s, \hat{d}z),\end{aligned}$$

where

$$\hat{\mathcal{L}}(y) = \hat{\mathcal{A}}(y) + \int_E [exp(\sum_{j=1}^m z^j(-v_j))(y) - y - \sum_{j=1}^m z^j(-v_j)(y)]G(\hat{d}z),$$

$$\hat{\mathcal{A}}(y) = (1/2)\sum_{j=1}^m (-v_j)^2(y) + (-v_0)(y).$$

Let $P_{s,t}^{-1}(y, A)$ be the distribution of $\xi_{s,t}^{-1}(y)$. Then, from the Proposition III-1, we can define a density of the distribution of the inverse flow $\xi_{s,t}^{-1}(y)$. If the inverse flow $\{\xi_{s,t}^{-1}(y); 0 \le s \le t \le T\}$ is backward self-similar, we have

$$P_{s,t}^{-1}(y, A) = P_{s/r,t}^{-1}(\psi_r^{-1}(y), \psi_r^{-1}(A)).$$

Let $p_{s,t}^{-1}(x, y)$ be the density of distribution $P_{s,t}^{-1}(y, A)$. For the simplicity, we assume that $p_s^{-1}(y, x) := p_{s,T}^{-1}(y, x)$. Then we get;

Theorem III-1. *Let $\{\xi_{s,t}^{-1}(y); 0 \le s \le t \le T\}$ be an inverse self-similar stochastic flow with respect to inverse dilation $\{\psi_r^{-1}\}_{r>0}$ generated by SDE (III-5). If $p_s^{-1}(y, x)$ is a density of the probability distribution of $\xi_s^{-1}(y)$, then we get;*

$$p_s^{-1}(y, x) = (1/det(\psi_s^{-1}))p_1^{-1}(\psi_s(y), \psi_s(x)).$$

Proof. For a fixed $t = T$, we put $\xi_{s,T}^{-1}(y) := \xi_s^{-1}(y)$. Then

$$\mathbb{P}(\xi_s^{-1}(y) \in A) = \int_A p_s^{-1}(y, x)G(\hat{d}x),$$

and

$$\begin{aligned}\mathbb{P}(\psi_s^{-1} \circ \hat{\xi}_1 \circ \psi_s(y) \in A) &= \mathbb{P}(\hat{\xi}_1(\psi_s(y)) \in \psi_s(A)) \\ &= \int_{\psi_s(A)} p_1^{-1}(\psi_s(y), x)G(\hat{d}x) \\ &= \int_A p_1^{-1}(\psi_s(y), \psi_s(x))G(\hat{d}\psi_t(x)) \\ &= \int_A p_1^{-1}(\psi_s(y), \psi_s(x))(1/det(\psi_s^{-1}))G(\hat{d}x).\end{aligned}$$

Therefore, we get the result. □

References

1. K. Bichteler, J-B. Gravereaux, J. Jacod, *Malliavin Calculus for Processes with Jumps*, Gordon and Breach Science Publishers, 1989.
2. Z.J. Jurek, J.D. Mason, *Operator-limit Distributions in Probability Theory*, John Wiley and Sons N.Y., 1993.
3. H. Kunita, *Stochastic flows with self-similar properties*, (preprint).
4. H. Kunita, *Asymptotically self-similar stochastic flows and their short time asymptotics*, (preprint).
5. H. Kunita, *Kunit's note*, (preprint).

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