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SOME EQUATIONS ON THE SUBMANIFOLDS OF A MANIFOLD GSX_n

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Abstract. On a generalized Riemannian manifold X_n, we may impose a particular geometric structure by the basic tensor field $g_{\lambda\mu}$ by means of a particular connection $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$. For example, Einstein's manifold X_n is based on the Einstein's connection defined by the Einstein's equations. Many *recurrent* connections have been studied by many geometers, such as Datta and Singel, M. Matsumoto, and E.M. Patterson. The purpose of the present paper is to study some relations between a *generalized semisymmetric g-recurrent manifold* GSX_n and its submanifold.

All considerations in this present paper deal with the general case $n \geq 2$ and all possible classes.

1. Introduction

Let X_n be a generalized n-dimensional Riemannian manifold referred to a real coordinate system y^{ν}, with coordinate transformation $y^{\nu} \rightarrow \bar{y}^{\nu}$, for which

$$Det\left(\frac{\partial y}{\partial \bar{y}}\right) \neq 0. \tag{1.1}$$

The manifold X_n is endowed with a general real nonsymmetric tensor $g_{\lambda\mu}$, which may be split into a symmetric part $h_{\lambda\mu}$ and a skew-symmetric part $k_{\lambda\mu}$:

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu} \tag{1.2}$$

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where

$$\mathcal{G} = Det(g_{\lambda\mu}) \neq 0, \qquad \mathcal{H} = Det(h_{\lambda\mu}) \neq 0. \tag{1.3}$$

Hence, we may define a unique tensor $h^{\lambda\nu}$ by

$$h_{\lambda\mu} h^{\lambda\nu} = \delta_\mu^\nu \tag{1.4}$$

and X_n is assumed to be connected by a real nonsymmetric connection $\Gamma_\lambda{}^\nu{}_\mu$ with the following transformation rule:

$$\bar{\Gamma}_\lambda{}^\nu{}_\mu = \frac{\partial \bar{y}^\nu}{\partial y^\alpha}\left(\frac{\partial y^\beta}{\partial \bar{y}^\lambda}\frac{\partial y^\gamma}{\partial \bar{y}^\mu}\Gamma_\beta{}^\alpha{}_\gamma + \frac{\partial^2 y^\alpha}{\partial \bar{y}^\lambda \partial \bar{y}^\mu}\right). \tag{1.5}$$

This connection may also be decomposed into its symmetric part $\Lambda_\lambda{}^\nu{}_\mu$ and its skew-symmetric part $S_{\lambda\mu}{}^\nu$, called the torsion tensor of $\Gamma_\lambda{}^\nu{}_\mu$:

$$\Gamma_\lambda{}^\nu{}_\mu = \Lambda_\lambda{}^\nu{}_\mu + S_{\lambda\mu}{}^\nu \tag{1.6}$$

where

$$\Lambda_\lambda{}^\nu{}_\mu = \Gamma_{(\lambda}{}^\nu{}_{\mu)}, \qquad S_{\lambda\mu}{}^\nu = \Gamma_{[\lambda}{}^\nu{}_{\mu]}. \tag{1.7}$$

Now, we will define a manifold GSX_n.

A connection $\Gamma_\lambda{}^\nu{}_\mu$ is said to be *semisymmetric* if its torsion tensor is of the form

$$S_{\lambda\mu}{}^\nu = 2\delta^\nu_{[\lambda} X_{\mu]} \tag{1.8}$$

for an arbitrary vector $X_\mu \neq 0$.

Hereafter we assume that X_μ is a non-null vector.

A particular differential geometric structure may be imposed on X_n by the tensor field $g_{\lambda\mu}$ by means of the connection $\Gamma_\lambda{}^\nu{}_\mu$ defined by the following g-recurrent condition:

$$D_\omega g_{\lambda\mu} = -4X_\omega g_{\lambda\mu}. \tag{1.9}$$

Here, D_ω is the symbolic vector of the covariant derivative with respect to the connection $\Gamma_\lambda{}^\nu{}_\mu$.

Definition 1.1. The connection $\Gamma_\lambda{}^\nu{}_\mu$ which satisfies (1.8) is called g-recurrent connection.

Definition 1.2. A connection which is both semisymmetric and g-recurrent is called a GS-connection.

A generalized Riemannian manifold X_n on which the differential geometric structure is imposed by $g_{\lambda\mu}$ through a GS-connection is called an n-dimensional GS-manifold and will be denoted by GSX_n.

The following theorems have been proved ([3])[1].

Theorem 1.3. *If the system (1.8) admits a solution $\Gamma_\lambda{}^\nu{}_\mu$, it must be of the form*

$$\Gamma_\lambda{}^\nu{}_\mu = \Lambda_\lambda{}^\nu{}_\mu + 2\delta^\nu_{[\lambda} X_{\mu]}. \tag{1.10}$$

Theorem 1.4. *If the system (1.9) admits a solution $\Gamma_\lambda{}^\nu{}_\mu$, it must be of the form*

$$\Gamma_\lambda{}^\nu{}_\mu = \{{}_\lambda{}^\nu{}_\mu\} - V^\nu{}_{\lambda\mu} - 2S^\nu{}_{(\lambda\mu)} + S_{\lambda\mu}{}^\nu \tag{1.11}$$

where

$$V^\nu{}_{\lambda\mu} = 2X^\nu h_{\lambda\mu} - 4X_{(\lambda}\delta^\nu_{\mu)}. \tag{1.12}$$

Theorem 1.5. *If the system (1.9) admits a solution $\Gamma_\lambda{}^\nu{}_\mu$ with its semi-symmetric torsion tensor, it must be of the form*

$$\Gamma_\lambda{}^\nu{}_\mu = \{{}_\lambda{}^\nu{}_\mu\} + 2\delta^\nu_\lambda X_\mu. \tag{1.13}$$

2. Preliminaries

[1]Numbers in brackets refer to the references at the end of the paper.

This section is a brief collection of basic concepts, results, and notations needed in the present paper[1].

Let X_m be a submanifold of X_n defined by a system of sufficiently differentiable equations

$$y^\nu = y^\nu(x^1, ..., x^m) \tag{2.1}$$

where the matrix of derivatives

$$B_i^\nu = \frac{\partial y^\nu}{\partial x^i}$$

is of rank m. Hence at each point of X_m, there exists *the first set* $\{B_i^\nu, \underset{x}{N}^\nu\}$ *of n linearly independent nonnull vectors.*

The m vectors B_i^ν are tangential to X_m and the $n-m$ vectors $\underset{x}{N}^\nu$ are normal to X_m and mutually orthogonal. That is

$$h_{\alpha\beta} B_i^\alpha \underset{x}{N}^\beta = 0, \qquad h_{\alpha\beta} \underset{x}{N}^\alpha \underset{y}{N}^\beta = 0 \qquad \text{for } x \neq y. \tag{2.2}$$

The process of determining the set $\{\underset{x}{N}^\nu\}$ is not unique unless $m = n-1$.

However, we may choose their magnitudes such that

$$h_{\alpha\beta} \underset{x}{N}^\alpha \underset{x}{N}^\beta = \varepsilon_x \tag{2.3}$$

where $\varepsilon_x = \pm 1$ according as the left-hand side of (2.3) is positive or negative.

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[1] In our further considerations in the present paper, we use the following types of indices ($m < n$): (1) Lower Greek indices α, β, γ,..., running from 1 to n and used for the holonomic components of tensors in X_n. (2) Capital Latin indices A,B,C,..., running from 1 to n and used for the C-nonholonomic components of tensors in X_n at points of X_m. (3) Lower Latin indices i, j, k,..., with the exception of x, y, and z, running from 1 to m. (4) Lower Latin indices x, y, z, running from $m+1$ to n. The summation convention is operative with respect to each set of the above indices within their range, with exception of x, y, z.

3. The induced connection on X_m of GSX_n $(m < n)$

If $\Gamma_\lambda{}^\nu{}_\mu$ is a connection on X_n, the connection $\Gamma_i{}^k{}_j$ defined by

$$(3.1)\qquad \Gamma_i{}^k{}_j = B^k_\gamma(B^\gamma_{ij} + \Gamma_\alpha{}^\gamma{}_\beta B^\alpha_i B^\beta_j), \qquad B^\gamma_{ij} = \frac{\partial B^\gamma_i}{\partial x^j} = \frac{\partial^2 y^\gamma}{\partial x^i \partial x^j}$$

is called the *induced connection* of $\Gamma_\lambda{}^\nu{}_\mu$ on X_m of X_n.

The following statements have been already proved([3]):

(a) The torsion tensor $S_{ij}{}^k$ of the induced connection $\Gamma_i{}^k{}_j$ is the induced tensor of the torsion tensor $S_{\lambda\mu}{}^\nu$ of the connection $\Gamma_\lambda{}^\nu{}_\mu$. That is

$$(3.2)\qquad S_{ij}{}^k = S_{\alpha\beta}{}^\gamma B^\alpha_i B^\beta_j B^k_\gamma.$$

(b) The induced connection $\{{}_i{}^k{}_j\}$ of $\{{}_\lambda{}^\nu{}_\mu\}$ is the Christoffel symbol defined by h_{ij}. That is

$$(3.3)\qquad \{{}_i{}^k{}_j\} = \frac{1}{2} h^{kp}(\partial_i h_{jp} + \partial_j h_{ip} - \partial_p h_{ij}).$$

(c) On an X_m of GSX_n, the induced connection $\Gamma_i{}^k{}_j$ is of the form

$$(3.4)\qquad \Gamma_i{}^k{}_j = \{{}_i{}^k{}_j\} + 2\delta^k_i X_j.$$

Here $\{{}_i{}^k{}_j\}$ are the induced Christoffel symbols defined by (3.3) and X_j is the induced vector on X_m of a vector $X_\mu \neq 0$ determining $\Gamma_\lambda{}^\nu{}_\mu$.

(d) On an X_m of GSX_n, a necessary and sufficient condition for the induced connection $\Gamma_i{}^k{}_j$ to be g-recurrent is

$$(3.5)\qquad \sum_x k_{x[i} \overset{x}{\Lambda}_{j]k} = 0, \qquad \text{where} \qquad \overset{x}{\Lambda}_{ij} = (\nabla_\beta \overset{x}{N}_\alpha) B^\alpha_i B^\beta_j.$$

Let $\overset{o}{D}_j$ be the symbolic vector of the generalized covariant derivative with respect to the $x's$. That is

$$(3.6)\qquad \overset{o}{D}_j B^\alpha_i = B^\alpha_{ij} + \Gamma_\beta{}^\alpha{}_\gamma B^\beta_i B^\gamma_j - \Gamma_i{}^k{}_j B^\alpha_k.$$

Then the vector $\overset{o}{D}_j B_i^\alpha$ in X_n is normal to X_m and is given by

$$(3.7)\qquad \overset{o}{D}_j B_i^\alpha = -\sum_x \overset{x}{\Omega}_{ij} \underset{x}{N}^\alpha$$

where

$$(3.8)\qquad \overset{x}{\Omega}_{ij} = -(\overset{o}{D}_j B_i^\alpha)\overset{x}{N}_\alpha.$$

And we know that the tensors $\overset{x}{\Omega}_{ij}$ are the induced tensors on X_m of the tensor $D_\beta \overset{x}{N}_\alpha$ in X_n. That is

$$(3.9)\qquad \overset{x}{\Omega}_{ij} = (D_\beta \overset{x}{N}_\alpha) B_i^\alpha B_j^\beta.$$

The tensor $\overset{x}{\Omega}_{ij}$ will be called the *generalized coefficients of the second fundamental form* of X_m.

4. The generalized fundamental equations for X_m of GSX_n

On an X_m of GSX_n, the following identities hold ([2]):

$$(4.1)\qquad \overset{o}{D}_j B_i^\alpha = -\sum_x \overset{x}{\Lambda}_{ij} \underset{x}{N}^\alpha \qquad \text{where} \qquad \overset{x}{\Lambda}_{ij} = (\bigtriangledown_\beta \overset{x}{N}_\alpha) B_i^\alpha B_j^\beta$$

(Generalized Gauss formulas for an X_m of GSX_n)

$$(4.2)\qquad \overset{o}{D}_j \underset{x}{N}^\alpha = (\varepsilon_x h^{im} \overset{x}{\Lambda}_{mj}) B_i^\alpha + \sum_y (\varepsilon_y \underset{x}{\overset{y}{H}}_\gamma B_j^\gamma + 2\delta_x^y X_j) \underset{y}{N}^\alpha.$$

(Generalized Weingarten equations on an X_m of GSX_n)

In order to derive the generalized *Gauss-Codazzi equations,* we need the following curvature tensors of GSX_n and X_m:

$$(4.3)\qquad R_{\omega\mu\lambda}{}^\nu = 2(\partial_{[\mu} \Gamma_{|\lambda|}{}^\nu{}_{\omega]} + \Gamma_\lambda{}^\alpha{}_{[\omega} \Gamma_{|\alpha|}{}^\nu{}_{\mu]})$$

$$(4.4)\qquad R_{ijk}{}^h = 2(\partial_{[j} \Gamma_{|k|}{}^h{}_{i]} + \Gamma_k{}^p{}_{[i} \Gamma_{|p|}{}^h{}_{j]})$$

The following notation will be used in further considerations:

$$(4.5)\qquad \underset{x}{\overset{y}{H}}_\gamma = \varepsilon_y (\bigtriangledown_\gamma \underset{x}{N}^\alpha) \overset{y}{N}_\alpha$$

THEOREM 4.1. *On an X_m of GSX_n, the curvature tensors defined by (4.3) and (4.4) satisfy the following identities:*

(4.6)
$$R_{ijk}{}^{h} = R_{\beta\gamma\epsilon}{}^{\alpha} B_i^\beta B_j^\gamma B_k^\epsilon B_\alpha^h + 2\sum_x \overset{x}{\Lambda}_{k[i}(\overset{x}{\Lambda}_{j]m}\varepsilon_x h^{hm} - \delta_{j]}^h X_x + k_{j]}{}^h X_x + k_{j]x} X^h)$$

(The generalized Gauss equations for an X_m of GSX_n)

(4.7)
$$2\overset{o}{D}_{[k}\overset{x}{\Lambda}_{j]i} = R_{\beta\gamma\epsilon}{}^{\alpha} B_k^\beta B_j^\gamma B_i^\epsilon \overset{x}{N}_\alpha + 6\overset{x}{\Lambda}_{i[k} X_{j]} + 2\sum_y \overset{y}{\Lambda}_{i[k}(B_{j]}^\gamma \varepsilon_x \underset{y}{\overset{x}{H}}{}_\gamma + X_{j]} k_y{}^x + k_{j]}{}^x X_y)$$

(The generalized Codazzi equations for an X_m of GSX_n)

Proof. In virtue of (3.1),(3.6),(4.3) and (4.4), we have

(4.8)
$$\begin{aligned}2\overset{o}{D}_{[k}\overset{o}{D}_{j]}B_i^\alpha &= 2[\partial_{[k}(\overset{o}{D}_{j]}B_i^\alpha) - \Gamma_{[j}{}^m{}_{k]}(\overset{o}{D}_m B_i^\alpha) \\ &\quad - \Gamma_i^m{}_{[k}(\overset{o}{D}_{j]}B_m^\alpha) + \Gamma_\beta{}^\alpha{}_\gamma(\overset{o}{D}_{[j}B_{|i|}^\beta)B_{k]}^\gamma] \\ &= -R_{\epsilon\gamma\beta}{}^\alpha B_i^\beta B_j^\gamma B_k^\epsilon + R_{kji}{}^m B_m^\alpha + 4\sum_x \overset{x}{\Lambda}_{i[j}X_{k]}\underset{x}{N}^\alpha\end{aligned}$$

On the other hand, the equations (4.1) and (4.2) give

(4.9)
$$\begin{aligned}2\overset{o}{D}_{[k}\overset{o}{D}_{j]}B_i^\alpha &= -2\sum_x \overset{o}{D}_{[k}(\overset{x}{\Lambda}_{j]i}\underset{x}{N}^\alpha) \\ &= 2\sum_x(\overset{o}{D}_{[j}\overset{x}{\Lambda}_{k]i})\underset{x}{N}^\alpha + 2\sum_x \overset{x}{\Lambda}_{i[k}\overset{o}{D}_{j]})\underset{x}{N}^\alpha \\ &= 2\sum_x(\overset{o}{D}_{[j}\overset{x}{\Lambda}_{k]i} + \overset{x}{\Lambda}_{i[k}X_{j]})\underset{x}{N}^\alpha \\ &\quad + 2\sum_{x,y}\overset{x}{\Lambda}_{i[k}(B_{j]}^\gamma \varepsilon_x \underset{x}{\overset{y}{H}}{}_\gamma + X_{j]}k_x{}^y + k_{j]}{}^y X_x)\underset{y}{N}^\alpha \\ &\quad + 2\sum_x \overset{x}{\Lambda}_{i[k}(\overset{x}{\Lambda}_{j]m}\varepsilon_x h^{pm} - \delta_{j]}^p X_x + k_{j]x}X_x + k_{j]}{}^p X_x)B_p^\alpha\end{aligned}$$

By means of (4.8) and (4.9), we have

$$(4.10)\quad R_{kji}{}^{m}B^{\alpha}_{m} = R_{\epsilon\gamma\beta}{}^{\alpha}B^{\epsilon}_{k}B^{\beta}_{i}B^{\gamma}_{j} + 2\sum_{x}(\overset{o}{D}_{[j}\overset{x}{\Lambda}_{k]i} + 3\overset{x}{\Lambda}_{i[k}X_{j]})\underset{x}{N}^{\alpha}$$
$$+ 2\sum_{x,y}\overset{x}{\Lambda}_{i[k}(B^{\gamma}_{j]}\varepsilon_{x}\underset{x}{\overset{y}{H}}_{\gamma} + X_{j]}k_{x}{}^{y} + k_{j]}{}^{y}X_{x})\underset{y}{N}^{\alpha}$$
$$+ 2\sum_{x}\overset{x}{\Lambda}_{i[k}(\overset{x}{\Lambda}_{j]m}\varepsilon_{x}h^{pm} - \delta^{p}_{j]}X_{x} + k_{j]x}X_{x} + k_{j]}{}^{p}X_{x})B^{\alpha}_{p}$$

Multiplying both sides of (4.10) by B^{h}_{α}, we have(4.6). Similarly, the identity (4.7) follows by multiplying $\overset{z}{N}_{\alpha}$ into both sides of (4.10). □

5. Parallelism. Paths

In this section we investigate parallelism and paths in X_n and GSX_n. Let C be any curve in X_n, given by

$$(5.1)\qquad y^{\nu} = y^{\nu}(t).$$

DEFINITION 5.1. A vector field V^{ν} is said to be parallel along C with respect to a connection $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ if it satisfies the following condition:

$$(5.2a)\qquad \frac{dy^{\alpha}}{dt}V^{[\lambda}D_{\alpha}V^{\nu]} = 0, \qquad V^{\nu} \neq \rho\frac{dy^{\alpha}}{dt}D_{\alpha}V^{\nu}, \qquad \rho \neq 0$$

or equivalently,

(5.2b)

$$V^{[\lambda}(\frac{dV^{\nu]}}{dt} + \Gamma_{\beta}{}^{\nu]}{}_{\alpha}V^{\beta}\frac{dy^{\alpha}}{dt}) = 0, \qquad V^{\nu} \neq \rho\frac{dy^{\alpha}}{dt}D_{\alpha}V^{\nu}, \qquad \rho \neq 0.$$

In particular, the curves whose tangents are parallel along themselves are called the *paths* in X_n with respect to $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$. A path with respect to $\{{}_{\lambda}{}^{\nu}{}_{\mu}\}$ is called a *geodesic* of X_n.

Therefore, a curve C in X_n, given by (5.1), is a path if it satisfies (5.3).

$$(5.3)\qquad \frac{dy^{[\lambda}}{dt}(\frac{d^2y^{\nu]}}{dt^2} + \Gamma_{\alpha}{}^{\nu]}{}_{\beta}\frac{dy^{\alpha}}{dt}\frac{dy^{\beta}}{dt}) = 0$$

As a consequence of (5.3), we have the following result:

THEOREM 5.2. *Every path C in GSX_n is a geodesic.*

THEOREM 5.3. *A necessary and sufficient condition that parallelism be the same along every curve in X_n with respect to two connections one of which is a GS connection is that other connection $\bar{\Gamma}_{\lambda}{}^{\nu}{}_{\mu}$ be given by*

$$\bar{\Gamma}_{\lambda}{}^{\nu}{}_{\mu} = \{{}_{\lambda}{}^{\nu}{}_{\mu}\} + 2\delta_{\lambda}^{\nu} A_{\mu} \qquad \textit{for an arbitrary vector } A_{\mu}. \tag{5.4}$$

Proof. Suppose that parallelism is the same along every curve with respect to two connections $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ and $\bar{\Gamma}_{\lambda}{}^{\nu}{}_{\mu}$. Then $\bar{\Gamma}_{\lambda}{}^{\nu}{}_{\mu}$ is given by ([3])

$$\bar{\Gamma}_{\lambda}{}^{\nu}{}_{\mu} = \Gamma_{\lambda}{}^{\nu}{}_{\mu} + 2\delta_{\lambda}^{\nu} P_{\mu} \quad \textit{for an arbitrary vector } P_{\mu}. \tag{5.5}$$

By means of (1.12) and (5.5) , we have (5.4). □

REMARK 5.4. As an immediate consequence of Theorem 5.3, we know that if parallelism is preserved along every curve in X_n with respect to a GS connection $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$, then the other connection $\bar{\Gamma}_{\lambda}{}^{\nu}{}_{\mu}$ is also a GS connection.

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