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ON CONFORMALLY FLAT UNIT VECTOR BUNDLES

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Abstract. We study the conformally flat unit vector bundle E_1 of constant scalar curvature for the bundle $\pi : E^{n+2} \to M^n$ over an Einstein manifold M.

1. Introduction

It is well known that there is a naturally induced metric, called the Sasaki metric, on the tangent bundle of a differentiable manifold M^n. The geometry of this metric structure has been extensively studied and many results were obtained. The question of locally symmetric tangent bundles was answered by O. Kowalski.([3]) Locally symmetric tangent sphere bundle was studied by D. Blair.([1]) He and T. Koufogiorgos have also answered concerning the case of the conformally flat tangent sphere bundle.([2])

On the other hand, the Sasaki metric on the normal bundle of a submanifold was studied by A. Borisenko and A. Yampol'skii.([4])

In this line of study, it is natural to study the geometry of the Sasaki metric on general vector bundle of a manifold. We shall study the geometry of the conformally flat unit vector bundle and prove the following theorem.

Theorem. *Let $\pi : E^{n+2} \to M^n$, $n \geq 3$ be a vector bundle over an Einstein manifold with fiber metric $g^{\perp}$ and a metric connection ∇. Suppose that the unit vector bundle E_1 is conformally flat and is of constant curvature. Then, either the connection ∇ is flat, or (M,G) admits an almost Hermitian structure.*

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2. Preliminaries

Let $\pi : E^{n+k} \to M^n$ be a vector bundle equipped with fiber metric $g^\perp$ and a metric connection ∇ where (M^n, G) is a Riemannian manifold. Let D be the Riemannian connection and $\underline{R}$ the curvature tensor of M. If $(x^1, x^2, \ldots, x^n)$ are local coordinates on M, set $q^i = x^i \circ \pi$; then, $(q^1, q^2, \ldots, q^n)$ together with the fiber coordinates $(u^1, u^2, \ldots, u^k)$ form local coordinates on E. For a vector field $X = X^i \frac{\partial}{\partial x^i}$ on M, we define

$$X^H = X^i \frac{\partial}{\partial q^i} - X^i \mu^\alpha_{\beta i} u^\beta \frac{\partial}{\partial u^\alpha}$$

where $\nabla_{\frac{\partial}{\partial x^i}} e_\beta = \mu^\alpha_{\beta i} e_\alpha$ and for a section $U = U^\alpha e_\alpha$, define

$$U^V = U^\alpha \frac{\partial}{\partial e_\alpha} .$$

The *connection map* $K : TE \to E$ is defined by

$$KX^H = 0 \text{ and } KU^V = U.$$

Then, there is a metric g on E, called the *Sasaki metric*, defined by

$$g(\tilde{X}, \tilde{Y}) = G(\pi_* \tilde{X}, \pi_* \tilde{Y}) + g^\perp(K\tilde{X}, K\tilde{Y})$$

at $(x, U) \in E$

Observe that any vector $\tilde{X}$ tangent to E, $\tilde{X} = (\pi_* \tilde{X})^H + (K\tilde{X})^V$; so it is enough to have such combinations of horizontal and vertical vectors in the following lemmas.

Now, we define the adjoint operator $\hat{R}_{UV} X$ by the equality

$$G(\hat{R}_{UV} X, Y) = g^\perp(R_{XY} U, V). \tag{1}$$

Then, we have

Lemma 2.1. *Let X and Y be tangent vector fields on M, and U and V sections of the bundle E. Then, at each point (x, W),*

$$\tilde{\nabla}_{U^V} V^V = 0 , \qquad \tilde{\nabla}_{X^H} V^V = (\nabla_X V)^V + \frac{1}{2}(\hat{R}_{WV} X)^H ,$$

$$\tilde{\nabla}_{U^V} Y^H = \frac{1}{2}(\hat{R}_{WU} Y)^H , \qquad \tilde{\nabla}_{X^H} Y^H = (D_X Y)^H - \frac{1}{2}(R_{XY} W)^V .$$

LEMMA 2.2. *Let X and Y be tangent vector fields on M, and U and V sections of the bundle E. Then, at each point (x, W),*

$$\begin{aligned}\tilde{R}_{X^H Y^H} Z^H &= [\,\underline{R}_{XY} Z + \frac{1}{4}\hat{R}_{W R_{ZY} W} X + \frac{1}{4}\hat{R}_{W R_{XZ} W} Y \\ &\quad + \frac{1}{2}\hat{R}_{W R_{XY} W} Z\,]^H + \frac{1}{2}[\,(\nabla_Z R)_{XY} W\,]^V \\ \tilde{R}_{X^H U^V} V^V &= -[\,\frac{1}{2}\hat{R}_{UV} X + \frac{1}{4}\hat{R}_{WU}\hat{R}_{WV} X\,]^H \\ &\quad etc.\end{aligned}$$

The proofs of these lemmas are very routine computations and we omit them here. Also, in Lemma 2.2, the cases mentioned are all we need for the rest of this paper.

Now, we consider a hypersurface E_1 of E defined by

$$E_1 = \{U \in E : |U| = 1\}$$

called the unit vector bundle. The metric on E_1 induced from the Sasaki metric on E is denoted by g', the Riemannian connection of g' by ∇', and its Riemannian curvature tensor by $R'_{\tilde{X}\tilde{Y}}\tilde{Z}$.

Notice that the vector field $W = U^\alpha (e_\alpha)^V$ is a unit normal and the position vector of a point W in E_1. Then, we consider the Weingarten map A, defined by $A\tilde{X} = -\tilde{\nabla}_{\tilde{X}} W$, of the immersion $\iota : E_1 \to E$.

For a vertical vector field V tangent to E_1, we have using Lemma 2.1

$$\tilde{\nabla}_{\iota_* V} W = (\iota_* V u^\alpha)(e_\alpha)^V + u^\alpha \tilde{\nabla}_{\iota_* V}(e_\alpha)^V = \iota_* V,$$

and for $X^H = (X^i, X^{n+\alpha})$ tangent to E_1,

$$\begin{aligned}\tilde{\nabla}_{X^H} W &= \tilde{\nabla}_{X^H} u^\alpha (e_\alpha)^V \\ &= (X^H u^\alpha)\frac{\partial}{\partial u^\alpha} + u^\alpha \tilde{\nabla}_{X^H}(e_\alpha)^V \\ &= -\,\mu^\alpha_{\beta i} u^\beta X^i \frac{\partial}{\partial u^\alpha} + u^\alpha((\nabla_X e_\alpha)^V + \frac{1}{2}(R_{W e_\alpha} X^H) \\ &= -\,\mu^\alpha_{\beta i} u^\beta X^i \frac{\partial}{\partial u^\alpha} + u^\alpha X^i (0 + \mu^\beta_{\alpha i})\frac{\partial}{\partial u^\beta} + \frac{1}{2}(R_{WW} X)^H \\ &= 0\end{aligned}$$

Hence, $A = -Id$ on vertical vectors and $A = 0$ on horizontal vectors. From this and the well-known identity for the second fundamental form σ

$$g(\sigma(\tilde{X}, \tilde{Y}), \tilde{V}) = g(A_{\tilde{V}} \tilde{X}, \tilde{Y}),$$

we have that

$$\sigma(\tilde{X}, \tilde{Y}) = 0 \tag{2}$$

if at least one of $\tilde{X}$ and $\tilde{Y}$ is horizontal.

In this paper, we consider the vector bundle $\pi : E^{n+k} \to M^n$ only with $k = 2$. Since each fiber has dimension 2, we can choose orthonormal sections $\{U, V\}$. Then, we can write

$$\nabla_X U = k(X)V \text{ and } \nabla_X V = -k(X)U,$$

where k is a 1-form. Thus,

$$\begin{aligned} R_{XY}U &= \nabla_X k(X)V - \nabla_Y k(X)Y - k([X,Y])V \\ &= 2dk(X,Y)V. \end{aligned}$$

We define a linear operator L by

$$G(LX, Y) = 2dk(X, Y).$$

Then, we have

$$G(LX, Y) = 2dk(X, Y) = g^{\perp}(R_{XY}U, V) = G(\hat{R}_{UV}X, Y) \tag{3}$$

and

$$\begin{aligned} G(L^2X, Y) &= G(\hat{R}_{UV}\hat{R}_{UV}X, Y) \\ &= -G(\hat{R}_{UV}X, \hat{R}_{UV}Y) = -G(LX, LY). \end{aligned} \tag{4}$$

Thus, from (3), we can write $L = \hat{R}_{UV}$.

3. Proof of the main theorem

We now prove our main theorem:

We take an orthonormal basis $\{X_i^H, V\}, i = 1, \cdots, n$, tangent to E_1 so that $\{X_i\}$ form an orthonormal basis of M. Then, using the Gauss equation for E_1 in E and (2), we have

$$\begin{aligned} g'(Q'X^H, Y^H) &= \sum_{i=1}^{n} g'(R'_{X^H X_i^H} X_i^H, Y^H) + g'(R'_{X^H V} V, Y^H) \\ &= \sum_{i=1}^{n} \big(g(\tilde{R}_{X^H X_i^H} X_i^H, Y^H) \\ &\quad + g(\sigma(X^H, Y^H), \sigma(X_i^H, X_i^H)) \\ &\quad - g(\sigma(X_i^H, Y^H), \sigma(X_i^H, X^H)) \big) + g(\tilde{R}_{X^H V} V, Y^H) \\ &\quad + g(\sigma(X^H, Y^H), \sigma(V, V)) - g(\sigma(V, Y^H), \sigma(V, X^H)) \\ &= \sum_{i=1}^{n} g(\tilde{R}_{X^H X_i^H} X_i^H, Y^H) + g(\tilde{R}_{X^H V} V, Y^H) \end{aligned}$$

Continuing this computation using Lemma 2.2, we have at U

$$\begin{aligned} g'(Q'X^H, Y^H) &= \sum_{i=1}^{n} g([R_{X X_i} X_i + \frac{3}{4} \hat{R}_{U R_{X X_i} U} X_i]^H, Y^H) \\ &\quad + g(-\frac{1}{4}[\hat{R}_{UV} \hat{R}_{UV} X]^H, Y^H) \\ &= G(\underline{Q}X, Y) \\ &\quad - \frac{3}{4} \sum_{i=1}^{n} g^{\perp}(R_{X X_i} U, R_{Y X_i} U) + \frac{1}{4} G(\hat{R}_{UV} X, \hat{R}_{UV} Y) \qquad (5) \end{aligned}$$

and

$$g'(Q'V,V) = \sum_{i=1}^{n} g'(R'_{VX_i^H}X_i^H, V)$$
$$= \sum_{i=1}^{n} g(\tilde{R}_{X_i^H V}V, X_i^H)$$
$$= \sum_{i=1}^{n} g(-\frac{1}{4}[\hat{R}_{UV}\hat{R}_{UV}X_i]^H, X_i^H)$$
$$(6) \qquad = \frac{1}{4}\sum_{i=1}^{n} G(\hat{R}_{UV}X_i, \hat{R}_{UV}X_i)$$

Using (5) and (6), we also have

$$R' = \sum_{i=1}^{n} g'(Q'X_i^H, X_i^H) + g'(Q'V,V)$$
$$= \sum_{i=1}^{n} G(\underline{Q}X_i, X_i) - \frac{3}{4}\sum_{i,j=1}^{n} g^{\perp}(R_{X_iX_j}U, R_{X_iX_j}U)$$
$$+ \frac{1}{2}\sum_{i=1}^{n} G(\hat{R}_{UV}X_i, \hat{R}_{UV}X_i)$$
$$(7) \qquad = \underline{R} + \frac{1}{2}\sum_{i=1}^{n} |\hat{R}_{UV}X_i|^2 - \frac{3}{4}\sum_{i,j=1}^{n} |R_{X_iX_j}U|^2$$

But, due to (4), we also have

$$trL^2 = \sum_{i=1}^{n} G(L^2X_i, X_i)$$
$$(8) \qquad = -\sum_{i=1}^{n} G(\hat{R}_{UV}X_i, \hat{R}_{UV}X_i)$$

Thus, from (7) and (8), we get

$$(9) \qquad R' = \underline{R} - \frac{1}{2}trL^2 - \frac{3}{4}\sum_{i,j=1}^{n} |R_{X_iX_j}U|^2$$

Now, since E_1 is conformally flat and since $\dim E_1 = n+1$ is at least 3, we have, in view of the famous Weyl conformal curvature tensor,

$$\begin{aligned} g'(R'_{\tilde{X}\tilde{Y}}\tilde{Z},\tilde{W}) = {} & \frac{1}{n-1}\big((g'(\tilde{Y},\tilde{Z})g'(Q'\tilde{X},\tilde{W}) - g'(\tilde{Z},\tilde{X})g'(Q'\tilde{Y},\tilde{W}) \\ & + g'(Q'\tilde{Y},\tilde{Z})g'(\tilde{X},\tilde{W}) - g'(Q'\tilde{X},\tilde{W})g'(\tilde{Y},\tilde{W})\big) \\ & - \frac{R'}{n(n-1)}\big(g'(\tilde{Y},\tilde{Z})g'(\tilde{X},\tilde{W}) - g'(\tilde{X},\tilde{Z})g'(\tilde{Y},\tilde{W})\big) \end{aligned}$$

From this together with (5), we have at U, for X and Y orthogonal,

$$\begin{aligned} g'(R'_{X^H V}V, Y^H) &= \frac{1}{n-1}g'(Q'X^H, Y^H) \\ &= \frac{1}{n-1}\big(G(\underline{Q}X, Y) \\ &\quad - \frac{3}{4}\sum_{i=1}^{n} g^{\perp}(R_{XX_i}U, R_{YX_i}U) + \frac{1}{4}G(LX, LY)\big) \end{aligned} \tag{10}$$

On the other hand, again using the Gauss equation and Lemma 2.2 successively, we have

$$\begin{aligned} g'(R'_{X^H V}V, Y^H) &= g(\tilde{R}_{X^H V}V, Y^H) \\ &= -\frac{1}{4}G(\hat{R}_{UV}\hat{R}_{UV}X, Y) \\ &= \frac{1}{4}G(LX, LY) \end{aligned} \tag{11}$$

So, comparing (10) and (11), we get

$$\begin{aligned} \frac{1}{4}(1 - \frac{1}{n-1})G(LX, LY) &= \frac{1}{n-1}\big(G(\underline{Q}X, Y) \\ &\quad - \frac{3}{4}\sum_{i=1}^{n} g^{\perp}(R_{XX_i}U, R_{YX_i}U)\big) \end{aligned} \tag{12}$$

But, from (3), we see that

$$\begin{aligned} \sum_{i=1}^{n} g^{\perp}(R_{XX_i}U, R_{YX_i}U) &= 4\sum_{i=1}^{n} g^{\perp}(dk(X, X_i)V, dk(X, X_i)V) \\ &= \sum_{i=1}^{n} G(LX, X_i)G(LY, X_i) \\ &= G(LX, LY) \end{aligned} \tag{13}$$

where we have the last equality since $\{X_i\}$ is an orthonormal basis. Therefore, we have, using (4) and (12)

$$G(L^2X,Y)=-\frac{4}{n+1}G(\underline{Q}X,Y)$$

and hence,

$$L^2+\frac{4}{n+1}\underline{Q}=\alpha I \tag{14}$$

where α is a function and I is the identity transformation.

Now, from (4) and (13), we have

$$\sum_{i,j=1}^{n}|R_{X_iX_j}U|^2=-trL^2$$

Therefore, (9) gives

$$R'=\underline{R}+\frac{1}{4}trL^2.$$

Now, the trace of (14) yields

$$n\alpha=trL^2+\frac{4}{n+1}\underline{R}=4R'-\frac{4n}{n+1}\underline{R}\,.$$

Thus, since R' is a constant and M is Einstein with $\dim M\geq 3$, α is a constant.

Again, since M is Einstein, i.e., $\underline{Q}=\frac{\underline{R}}{n}I$, we have, from (14),

$$L^2=-\beta I \tag{15}$$

where $\beta=\frac{4\underline{R}}{n(n+1)}-\alpha$ is a constant. Now, taking $X=Y$ in (4), we easily see that $\beta\geq 0$.

Case 1: $\beta=0$

In this case, $L^2=0$. Taking $X=Y$ in (4) again, we have that $|LX|^2=0$ for any X, that is, $L=0$. Hence, by the definition (1) of $\hat{R}$, we have

$$R_{XY}W=0$$

for any $X, Y,$ and W, i.e., the connection is flat.

Case 2: $\beta > 0$

We define a tensor field J by $J = \frac{1}{\sqrt{\beta}}L$. From the definition of J, it is clear that J is an almost complex structure on M. Moreover, using (4) and (15),

$$\begin{aligned} G(LX, LY) =& \frac{1}{\beta}G(LX, LY) \\ =& -\frac{1}{\beta}G(L^2X, Y) \\ =& G(X, Y) \end{aligned}$$

This completes the proof. □

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