A REMARK ON FORMALITY

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Abstract. In this paper we prove two independent theorems concerning formality of a nilmanifold and a differential graded algebra using the well-known theorem of Deligne-Griffiths-Morgan-Sullivan. We first give a rational homotopy theoretic proof to the statement that a nilmanifold is formal if and only if it is a torus. And then we study some conditions with which formality of one dga implies formality of the other in an extension of dga’s.

1. Minimal models and KS-extensions

We recall here the basic facts and notation we shall need from Sullivan’s theory of minimal models, for which the basic references are [3, 4, 8]. We assume the reader to be familiar with the basics of differential graded algebras [2] over a field $k$ of characteristic 0.

Definition. A dga $(M, d)$ is called minimal, if:

i) $M = \Lambda V$ is freely generated for some graded $k$-vector space $V$;

ii) $d$ is decomposable in the following sense: there exists an ordering in the set $\{x_a, a \in I\}$ of all free generators of $M$ such that $x_\beta < x_a \implies \deg(x_\beta) < \deg(x_a)$ and such that $dx_a \in \Lambda(V_{<a})$, $V_{<a}$ denoting the span of the $x_\beta < x_a$.

Notation. If $\{x_1, x_2, \ldots\}$ is a basis for $V$, then we write $V = \langle x_1, x_2, \ldots \rangle$ and $\Lambda V = \Lambda(x_1, x_2, \ldots)$.

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Remark. When $A = \Lambda V$ is connected, that is $A^0 = k$, ii) is equivalent to $d : V \to \Lambda^{\geq 2} V$. $\Lambda^{\geq m} V$ denotes the differential ideal of $\Lambda V$ having additive basis the monomials $x_{i_1} \cdots x_{i_k}$ with $k \geq m$.

Definition. i) A minimal model for a dga $A$ is a minimal dga $M_A$ and a dga map $\rho_A : M_A \to A$ such that the induced homomorphism on cohomology $\rho^*$ is an isomorphism.

ii) A minimal model for a space $X$ is a minimal model of the dga $A^*(X)$, the rational polynomial forms on $X$.

Example 1. i) $\Lambda(CP(n)) = \Lambda(x_2, y_{2n+1})$, $dy = x^{n+1}$.

ii) $\Lambda(T^n) = \Lambda(x_1^1, x_1^2, \ldots, x_1^n)$, $d = 0$.

The aim of the second part of this section is to describe the algebraic fibrations, which serve as models for fibrations [9]. Only augmented algebras are considered, that is, $(A, d_A)$ is always endowed with a homomorphism $\varepsilon : A \to k$ such that $\text{Ker } \varepsilon = \oplus_{k>0} A^k$.

Definition. A KS-extension is a sequence of augmentation preserving dga morphisms

$$(A, d_A) \xrightarrow{i} (A \otimes \Lambda V, d) \xrightarrow{\rho} (\Lambda V, d)$$

with the following conditions:

i) $i(a) = a \otimes 1$, $\rho = \varepsilon_A \otimes \text{id}_{\Lambda V}$, where $\varepsilon_A$ is the augmentation of $A$.

ii) there exists an ordered homogeneous basis $\{x_a : a \in I\}$ for $V$ indexed by a well ordered set $I$ such that $d(1 \otimes x_a) \in A \otimes \Lambda(V_{<a})$.

We will also call simply $(A, d_A) \xrightarrow{i} (A \otimes \Lambda V, d)$ a KS-extension.

2. Minimal model of a nilmanifold

Definition. A nilmanifold $M$ is a compact homogeneous space of the form $N/\pi$ where $N$ is a simply connected Lie group and $\pi$ is a lattice, that is, a discrete co-compact subgroup of $N$.

It is well known that $N$ is diffeomorphic to some $\mathbb{R}^n$ and therefore, $M$ is $K(\pi, 1)$. Furthermore, this entails the fact that $\pi$ is a finitely generated torsion free nilpotent group.
The general theory of nilmanifolds is contained in [6]. We only need a minimal model of a nilmanifold. Following [7] we decompose $M = K(\pi, 1)$ into a tower $S^1$-bundles

$$S^1 \to M_{i-1} \xrightarrow{\tau_i} \mathbb{C}P(\infty), \quad i = 2, \ldots, n$$

which is, in fact, the Postnikov decomposition of $M$ with $k$-invariants the $\tau_i$. Note that $[M_{i-1}, \mathbb{C}P(\infty)] = [M_{i-1}, K(\mathbb{Z}, 2)] = H^2(M_{i-1}; \mathbb{Z})$.

**Lemma 1.** [7] The minimal model of a nilmanifold $M^n$ of dimension $n$ has the form

$$\Lambda(M^n) = (\Lambda(x_1, \ldots, x_n), d), \deg(x_i) = 1$$

with $dx_i = \tau_i$, where $\tau_i$ is a cocycle representing the class $\tau_i \in H^2(M_{i-1}; \mathbb{Z})$.

### 3. Formality of a dga and the theorem of Deligne-Griffiths-Morgan-Sullivan

The basic reference for this section is [1]. Let $M$ be a minimal dga and $H^*(M)$ the cohomology of $M$ viewed as a dga with the differential 0.

**Definition.** i) $M$ is **formal** if there is a dga map $\Psi : M \to H^*(M)$ inducing the identity on cohomology.

ii) A dga $(A, d_A)$ is a **formal consequence** of its cohomology algebra if its minimal model is formal.

iii) A smooth manifold $M$ is **formal** if the de Rham algebra $\Omega^*(M)$ is a formal consequence of its cohomology algebra.

**Example 2.** Consider the 3-dimensional Heisenberg group $U_3(\mathbb{R})$ and mod out by $U_3(\mathbb{Z})$. The resulting manifold $M$ is a 3-manifold obtained as a principle bundle,

$$S^1 \to M \to T^2.$$ 

The minimal model of $M$ is given by

$$\Lambda(M) = \Lambda(x, y, z), \deg(x) = \deg(y) = \deg(z) = 1$$
with \( dx = 0 = dy \) and \( dz = xy \). Thus \( xz \), for example, is closed but not exact. But since \( x \cdot H^1(M) = 0 \), there can be no map of \( M \to H^*(M) \) inducing the identity in cohomology. Hence \( M \) is not formal.

We will use the following criterion for formality.

**Lemma 2.** (Deligne-Griffiths-Morgan-Sullivan) [1] A minimal dga \((\Lambda V, d)\) is formal if and only if \( V \) decomposes as a direct sum \( V = C \oplus N \) with \( d(C) = 0 \) and \( d \) injective on \( N \) such that every closed element in the ideal generated by \( N \) is exact.

Nonexact cocycles in the ideal \((N)\) are called **Massey products**.

### 4. Main theorems

We now present our main theorems.

**Theorem 1.** A nilmanifold \( M^n \) is formal if and only if it is a torus.

**Proof.** Since the minimal model of a torus is given by the dga \((\Lambda(x_1, \ldots, x_n), 0)\) where each \( x_i \) has degree one and the differential is 0 (See Example 1), it is clearly formal. Conversely, let \( M \) have a minimal model of the form \((\Lambda(x_1, \ldots, x_n), d)\) where \( \deg(x_i) = 1, i = 1, \ldots, n \) and \( d \neq 0 \). Then there exists \( k \) such that \( dx_1 = \cdots = dx_{k-1} = 0 \), \( dx_k \neq 0 \). By the minimality condition \( dx_k \) can be written as \( dx_k = \sum_{i<j<k} x_i x_j \). Let \( \{x_{i_1}, x_{i_2}, \ldots, x_{i_l}\}, i_1 < i_2 < \cdots < i_l, \) be the set of different \( x_i \)'s appearing in the summation of \( dx_k \). We may assume that \( l > 2 \) (See Example 2). Consider the element \( a = \Sigma x_{i_1} \cdots \hat{x}_{i_s} \cdots x_{i_l} \) ranging all the permutations of \( \{i_1, \ldots, i_l\} \). Suppose that \( \Lambda V \) is formal. Then \( V \) has a decomposition \( V = C \oplus N \) as in Lemma 2. Then clearly \( ax_k \in (N) \). Note that \( d(ax_k) = (da)x_k \pm adx_k = \pm(\Sigma x_{i_1} \cdots \hat{x}_{i_s} \cdots x_{i_l})(\Sigma x_i x_j) = 0 \) since each term reduces to 0. But it is not hard to see that \( ax_k \) is not a coboundary, which is a contradiction. Hence \( d = 0 \), and \( M \) has the rational homotopy type of a torus. We now follow the argument in [7] P.204 to conclude that \( M \) has the homotopy type of a torus. \( \square \)

**Theorem 2.** Let \( i : (\Lambda V, d) \to (\Lambda V \otimes \Lambda W, D) \) be a KS-extension. Then we have the followings:
i) if \((\Lambda V, d)\) is formal and \(i^*\) is an epimorphism, then \((\Lambda V \otimes \Lambda W, D)\) is formal and,

ii) if \((\Lambda V \otimes \Lambda W, D)\) is formal and \(i^*\) is a monomorphism, then \((\Lambda V, d)\) is formal.

Proof. 1) Since \((\Lambda V, d)\) is formal, there exists a dga map \(\Phi : \Lambda V \to H^*(AV)\) such that \(\Phi^* = id\). We proceed by induction on the number \(n\) of generators of \(W\). When \(n = 1\), that is \(W = \langle y \rangle\), define a map \(\Psi : \Lambda V \otimes \Lambda(y) \to H^*(\Lambda V \otimes \Lambda(y))\) by \(\Psi|_{\Lambda V} = i^*\Phi\) and \(\Phi(y) = 0\). \(\Psi\) is indeed a dga map since \(dy \in Z(\Lambda V)\), the cocycles in \(\Lambda V\). Since \(i^*\) is onto, each element in \(H^*(\Lambda V \otimes \Lambda(y))\) has a preimage which maps identically into itself by \(\Phi^* = id\). Hence, \(\Phi = id\). Now assume that the statement is true when \(n = k-1\) and \(i^*_1 : H^*(\Lambda V) \to H^*(\Lambda V \otimes \Lambda(y_1, \ldots, y_{k-1}, y_k))\) is an epimorphism. Note that \(i^*_2 : H^*(\Lambda V \otimes \Lambda(y_1, \ldots, y_{k-1})) \to H^*(\Lambda V \otimes \Lambda(y_1, \ldots, y_k))\) is also an epimorphism and \(dy_k \in \Lambda V \otimes \Lambda(y_1, \ldots, y_{k-1})\). Repeating the above argument we conclude that \(\Lambda V \otimes \Lambda(y_1, \ldots, y_k)\) is formal.

2) Suppose that \((\Lambda V \otimes \Lambda W, D) = (\Lambda(V \oplus W), D)\) is formal. By Lemma 2 there exists a decomposition \(V \oplus W = C \oplus N\) with \(D(C) = 0\) and \(D\) is injective on \(N\) such that every closed element in \(N\), the ideal generated by \(N\) in \(\Lambda V \otimes \Lambda W\), is exact. By taking \(C' = C \cap V\) and \(N' = N \cap V\) we have \(d(C) = 0\) and \(d\) is injective on \(N'\) since \(D|_V = d\). Let \(a \in (N')\), the ideal generated by \(N'\) in \(\Lambda V\), and \(da = 0\). Since \(a \in (N') \subset (N)\), \(a = Db\) for some \(b \in \Lambda V \otimes \Lambda W\). Since \(i^*([a]) = [Db] = 0\) and \(i^*\) is a monomorphism we have \([a] = 0\). Hence \(a = da'\) for some \(a' \in \Lambda V\), which completes the proof.

Remark. For any non-formal dga \((\Lambda V, d)\) we may continuously add generators to kill the Massey products producing an extension \((\Lambda V \otimes \Lambda W, D)\) which is formal. Clearly \(i^* : H^*(\Lambda V, d) \to H^*(\Lambda V \otimes \Lambda W, D)\) is not a monomorphism.

References


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