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ON M-CONTINUITY

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Abstract. In this paper, we introduce a new class of sets, called m-sets, and the notion of m-continuity. In particular, m-sets and m-continuity are used to extend known results for α-continuity and semi-continuity and precontinuity.

1. Introduction

Let X, Y and Z be topological spaces on which no separation axioms are assumed unless explicity stated. Let S be a subset of X. The closure (resp. interior, boundary) of S will be denoted by S^- (resp. $S^0, b(S)$). A subset S of X is called semi-open set[1] (resp. preopen set[2], α-set[3]) if $S \subset S^{0-}$ (resp. $S \subset S^{-0}, S \subset S^{0-0}$). The complement of a semi-open set (resp. preopen set, α-set) is called semi-closed set (resp. preclosed set, α-closed set). The family of all semi-open sets (resp. preopen sets, α-sets) in X will be denoted by $SO(X)$ (resp. $PO(X)$, $\alpha(X)$). A function $f : X \to Y$ is called semi-continuous[1] (resp. precontinuous[2], α-continuous [4]) if $f^{-1}(V) \in SO(X)$ (resp. $f^{-1}(V) \in PO(X)$, $f^{-1}(V) \in \alpha(X)$ for each open set V of Y).

A subclass $\tau^* \subset P(X)$ is called a supratopology on X if $X \in \tau^*$ and τ^* is closed under arbitrary union. (X, τ^*) is called a supratopological space. The members of τ^* are called supraopen sets[5]. Let (X, τ) be a topological space and τ^* be a supratopology on X. We call τ^* a supratopology associated with τ if $\tau \subset \tau^*$. Let (X, τ^*) be a supratopological space and (Y, μ) be a topological space. A function $f : X \to Y$ is an S-continuous function if the inverse image of each open set in Y is a supraopen set in X[5]. Let (X, τ^*) and (Y, μ^*) be supratopological spaces. A function $f : X \to Y$ is an S^*-continuous

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function if the inverse image of each supraopen set in Y is a supraopen set in X[5].

2. m-sets induced by a supratopology

DEFINITION 2.1. Let (X, τ^*) be a supratopological space. A subset A of X is called an m-set with τ^* if $A \cap T \in \tau^*$ for all $T \in \tau^*$.

The class of all m-sets with τ^* will be denoted by τ_m.

EXAMPLE 2.2. Let $X = \{a, b, c, d\}$ and $\tau^* = \{\emptyset, X, \{a\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{b, c, d\}, \{a, b, d\}\}$. Then $\tau_m = \{\emptyset, X, \{a\}, \{b, c, d\}\}$.

REMARK. Let (X, τ) be a topological space. Since $SO(X)$ is closed with repect to arbitrary union, $SO(X)$ is a supratopology on X. For any α-set A in X, $A \cap B \in SO(X)$ for all $B \in SO(X)$. Thus A is an m-set with $SO(X)$. That is, $\alpha(X)$ is τ_m with $SO(X)$.

LEMMA 2.3. *Let (X, τ^*) be a supratopological space. Then the class τ_m of all m-sets with τ^* is contained τ^*.*

Proof. Let A be an m-subset with τ^*. And X is an element of τ^*. Now we take that $X \cap A = A$ belongs to the supratopology τ^*, by the definition of m-sets. □

THEOREM 2.4. *Let (X, τ^*) be a supratopological space. Then the class τ_m of all m-sets with τ^* is a supratopology.*

Proof. Let $\{A_\alpha\}$ be a class of members of τ_m. By definitions of the m-set and the supratopology, $(\cup A_\alpha) \cap T = \cup(A_\alpha \cap T) \in \tau^*$ for all $T \in \tau^*$. Thus the union $\cup A_\alpha$ also belongs to τ_m. □

THEOREM 2.5. *Let (X, τ^*) be a supratopological space with $\emptyset \in \tau^*$. If a subset A of X is a singleton set and $A \in \tau^*$, then A is an m-set.*

Proof. Since $A \in \tau^*$ is a singleton set, $A \cap B = \emptyset$ or A for $B \in \tau^*$. Thus A is an m-set. □

We obtain the following, by definition of m-set.

THEOREM 2.6. *Let (X, τ^*) be a supratopological space. If T is any supraopen set of τ^* in X and A is an m-set with τ^*, then $T \cap A$ is also a supraopen set.*

COLORALLY 2.7. *Let (X, τ) be a topological space and $\tau^* = PO(X)$. If $A \in \alpha(X)$ and $B \in PO(X)$, then $A \cap B \in PO(X)$.*

Proof. Since $\alpha(X) \subset SO(X) \cap PO(X)$, $\alpha(X)$ is a subclass of m-sets with $PO(X)$, and it obtained by Theorem 2.6. □

THEOREM 2.8. *Let (X, τ^*) be a supratopological space with $\emptyset \in \tau^*$. Then the class τ_m of all m-subsets of X is a topology on X.*

Proof. Since $\emptyset \cap T = \emptyset \in \tau^*$ and $X \cap T = T \in \tau^*$ for all $T \in \tau^*$, $\emptyset$ and $X \in \tau_m$.

Suppose $A, B \in \tau_m$. By definition of m-set, we obtain $B \cap T \in \tau^*$ and $A \cap (B \cap T) \in \tau^*$ for all $T \in \tau^*$. Thus $(A \cap B) \in \tau_m$.

And by Theorem 2.4., the proof is completed. □

Now the class τ_m is called an m-topology with τ^* and the members of τ_m are called m-open sets. A subset B of X is called an m-closed set if the complement of B is an m-open set. Thus the intersection of any family of m-closed sets is a m-closed set and the union of finitely many m-closed sets is an m-closed set.

In case τ_m is an m-topology with τ^* on X, the topological space (X, τ_m) with τ^* will be denoted by (X, τ_m, τ^*).

REMARK. In a space (X, τ), if τ^* is an associated supratopology with τ, an m-set need not be an open set, and vice versa.

EXAMPLE 2.9.

Let $X = \{a, b, c, d\}$. Consider $\tau = \{\emptyset, X, \{a, b\}\}$ and $\tau^* = \{\emptyset, X, \{a, b\}, \{b, d\}, \{a, b, d\}\}$. Then τ^* is a supratopoogy associated with τ and $\{a, b, d\}$ is an m-set but it is not an open set. And $\{a, b\}$ is an open set but it is not an m-set.

Definition 2.10. Let (X, τ_m, τ^*) be an m-topological space.

(1) The m-interior of A is defined as the union of all m-open sets contained in A. The m-interior of A is denoted by $mintA$.
(2) The m-closure of A is defined as the intersection of all m-closed sets containing A. The m-closure of A is denoted by $mclA$.

By the above definitions, we obtain the following properties.

Theorem 2.11. *Let (X, τ_m, τ^*) be an m-topological space and A be a subset of X.*

(1) *A is m-open if and only if $A = mintA$.*
(2) *A is m-closed if and only if $A = mclA$.*
(3) *$mcl(mclA) = mclA$ and $mint(mintA) = mintA$.*
(4) *$A \subset B$ implies $mclA \subset mclB$.*
(5) *$mclA \cup mclB = mcl(A \cup B)$.*

3. m-continuity

Definition 3.1. Let (X, τ_m, τ^*) be an m-topological space and (Y, μ) be a topological space. A mapping $f : X \to Y$ is called an m-continuous if the inverse image of each open set of Y is an m-open set in X.

Remark. In general, there is no relation between the continuity and the m-continuity.

Example 3.2. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a, b\}\}$ and $\mu = \{\emptyset, X, \{a, b, d\}\}$. Now we take a supratopology $\tau^* = \{\emptyset, X, \{a, b\}, \{b, d\}, \{a, b, d\}\}$ for τ. Then $\tau_m = \{\emptyset, X, \{a, b, d\}\}$. Let $f : (X, \tau, \tau^*) \to (X, \tau)$ be the identity function. Then f is continuous but it is not m-continuous. And if $f : (X, \tau, \tau^*) \to (X, \mu)$ be the identity function. Then f is m-continuous but it is not continuous.

Theorem 3.3. *Let (X, τ_m, τ^*) be an m-topological spaces and (Y, μ) be a topological spaces. If $f : (X, \tau_m, \tau^*) \to (Y, \mu)$ is a mapping, then the following statements are equivalent:*

(1) *f is an m-continuous.*

(2) *The inverse image of each closed set in* Y *is* m*-closed.*
(3) *For each* $x \in X$*, and each open set* $V \subset Y$ *containing* $f(x)$*, there exists* $W \in \tau_m$ *such that* $x \in W$*,* $f(W) \subset V$*.*
(4) $f(mclA) \subset clf(A)$ *for every* $A \subset X$*.*
(5) $mcl(f^{-1}(B)) \subset f^{-1}(cl(B))$ *for every* $B \subset Y$*.*

Proof. (1)⇒(2). Let B be closed in Y. Since $Y - B$ is open in Y and $X - f^{-1}(B)$ is m-open, thus $f^{-1}(B)$ is m-closed.

(2)⇒(1). Let V be open in Y. Since $Y - V$ is closed in Y and $X - f^{-1}(V)$ is m-closed, $f^{-1}(V)$ is m-open.

(1)⇒(3). For each $x \in X$, and each open set V containing $f(x)$. Set $W = f^{-1}(V)$. Then W is m-open, $x \in W$, and $f(W) \subset V$.

(3)⇒(4). We will show that for each $b \in mclA$, $f(b) \in cl(f(A))$. Let V be an open neighborhood of $f(b)$, then there exists $W \in \tau_m$ such that $b \in W$ and $f(W) \subset V$. Since $b \in mclA$, $W \cap A \neq \emptyset$. $f(W \cap A) \neq \emptyset$ and $f(W) \cap f(A) \neq \emptyset$. Thus $V(f(b)) \cap f(A) \neq \emptyset$ and $f(b) \in cl(f(A))$.

(4)⇒(5). Let $A = f^{-1}(B)$ for $B \subset Y$. Then $f(mcl(A)) \subset cl(f(A)) \subset cl(B)$, and $mcl(f^{-1}(B)) \subset f^{-1}(cl(B))$.

(5)⇒(2). Let $B \subset Y$ be closed. Then $mcl(f^{-1}(B)) \subset f^{-1}(cl(B)) = f^{-1}(B)$, and $f^{-1}(B)$ is an m-closed set. □

Remark. If $f : (X, \tau_m, \tau^*) \to (Y, \mu)$ is an m-continuous function and $g : (Y, \mu) \to (Z, \nu)$ is a continuous function, then $g \circ f$ is m-continuous.

Lemma 3.4. *Let* $f : (X, \tau) \to (Y, \mu)$ *be an* α*-continuous function. Then*

(1) *For each subset* A *of* X*,* $f(cl_\alpha(A)) \subset (f(A))^-$ *if and only if* $f(A^{-0-}) \subset (f(A))^-$*.*
(2) *For each subset* B *of* Y*,* $cl_\alpha(f^{-1}(B)) \subset f^{-1}(B^-)$ *if and only if* $(f^{-1}(B))^{-0-} \subset f^{-1}(B^-)$*.*

Proof. Since $cl_\alpha(A) = A \cup cl(int(cl(A)))$, the properties are proved obviously. □

By Theorem 3.3 and Lemma 3.4, easily we get the following properties.

Corollary 3.5. *Let $f : (X, \tau_m, SO(X)) \to (Y, \mu)$ is a function, the followings are equivalent:*

(1) *f is α-continuous.*
(2) *The inverse image of each closed set in Y is m-closed set.*
(3) *For each $x \in X$, and each open set $V \subset Y$ containing $f(x)$, there exists $W \in \tau_m$ such that $x \in W$, $f(W) \subset V$.*
(4) *$f(A^{-0-}) \subset cl(f(A))$ for every $A \subset X$.*
(5) *$(f^{-1}(B))^{-0-} \subset f^{-1}(cl(B))$ for every $B \subset Y$.*

Definition 3.6. A function $f : (X, \tau_m, \tau^*) \to (Y, \mu_m, \mu^*)$ is an mS-continuous function if the inverse image of each m-set in Y is a supraopen set in X.

The following theorem is a straightforward result of Mashhour(Theorem 2.1.[5]).

Theorem 3.7. *Let $f : (X, \tau, \tau^*) \to (Y, \mu, \mu^*)$ be a function. Then the followings are equivalent :*

(1) *f is an mS-continuous.*
(2) *The inverse image of each m-closed set in Y is a supraclosed set.*
(3) *$(f^{-1}(V))^{sc} \subset f^{-1}(mcl(V))$, for every $V \subset Y$.*
(4) *$f(U^{sc}) \subset mcl(f(U))$, for every $U \subset X$.*
(5) *For any point $x \in X$ and any m-open set V of Y containing $f(x)$, there exists $U \in \tau^*$ such that $x \in U$ and $f(U) \subset V$.*

Remark. Let $f : (X, \tau, \tau^*) \to (Y, \mu, \mu^*)$ be a function. Then we can get the following diagrams :

(1) m-continuity $\Longrightarrow$ S-continuity
(2) S^*-continuity $\Longrightarrow$ mS-continuity
(3) In $\tau \subset \tau_m$,

continuity $\Longrightarrow$ m-continuity $\Longrightarrow$ S-continuity

(4) In $\tau \subset \tau_m$ and $\mu \subset \mu_m$,

m-continuity $\Longrightarrow$ S-continuity $\Longleftarrow$ mS-continuity $\Longleftarrow$ S^*-continuity

(5) In $\tau^* = SO(X)$,

continuity $\Longrightarrow$ m-continuity(=α-continuity) $\Longrightarrow$ semi-continuity

(6) In $\tau^* = PO(X)$,

continuity $\Longrightarrow$ α-continuity $\Longrightarrow$ m-continuity $\Longrightarrow$ precontinuity

References

1. N. Levine, *Semi-open sets and semi- continuity in topological spaces*, Am. Math. Monthly **70** (1963), 36-41.
2. A.S.Mashhour, M.E.Abd El-Monsef and S.N.El.Deeb, *On precontinuous and weak precontinuous mappings*, Proc. Math. Phys. Soc. Egypt **51** (1981a), 47–53.
3. O. Njastad, *On some classes of nearly open sets*, pacific journal of mathematics **15** (1964), 961–970.
4. A.S.Mashhour. I.A.Hasanein and S.N.El-deeb, *α-cotinuous and α-open mappings*, Acta Math. Hung. **41** (1983), 213-218.
5. A.S.Mashhour, A.A.Allam, F.S.Mahmoud and F.H.Khedr, *On supratopological spaces*, Indian J. Pure Appl. Math **14** (1983), 502-510.

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