SUBSERIES CONVERGENCE AND
SEQUENCE-EVALUATION CONVERGENCE

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Abstract. We show a series of improved subseries convergence results, e.g., in a sequentially complete locally convex space X every weakly c₀-Cauchy series on X must be c₀-convergent. Thus, if X contains no copy of c₀, then every weakly c₀-Cauchy series on X must be subseries convergent.

Let X be a locally convex space. A series $\sum x_j$ on X is said to be weakly c-convergent if for every $\{t_j\} \in c$ the series $\sum_{j=1}^{\infty} t_j x_j$ converges in $(X, weak)$, i.e., for every $\{t_j\} \in c$ there is an $x_0 \in X$ such that

$$\sum_{j=1}^{\infty} t_j f(x_j) = \lim_{n \to \infty} f(\sum_{j=1}^{n} t_j x_j) = f(x_0)$$

for each $f \in X'$, the dual of X (= the family of continuous linear functionals on X). In this case, $x_0$ is the weak sum of the series $\sum t_j x_j$ and we write $x_0 = w - \sum_{j=1}^{\infty} t_j x_j$. Similarly a series $\sum x_j$ on X is said to be c-convergent if for every $\{t_j\} \in c$ the series $\sum_{j=1}^{\infty} t_j x_j$ converges in X.

Since $c_0 \subseteq c$, if $\sum x_j$ is weakly c-convergent then $\sum x_j$ is weakly c₀-convergent and, by the Orlicz-Pettis theorem, $\sum x_j$ is c₀-convergent. Therefore we have

**Proposition 1.** If $\sum x_j$ is weakly c-convergent, then for all $f \in X'$

$$(*) \quad \sum_{j=1}^{\infty} |f(x_j)| < +\infty.$$
Proof. See [1], Theorem 2. □

Of course, if \( \sum x_j \) is (weakly) \( c_0 \)-convergent, then (*) holds and the converse is true if \( X \) is sequentially complete.

Note that with the norm \( \| \{ t_j \} \|_\infty = \sup_j |t_j| \), \( c_0 \), \( c \) and \( l^\infty \) are Banach spaces. For a locally convex space \( X \), let \( \sigma(X, X') \), \( \tau(X, X') \) and \( \beta(X, X') \) denote the weak topology, the Mackey topology and the strong topology, respectively. \( \tau(X, X') \) is just the topology of uniform convergence on weak* \((\sigma(X', X))\) compact balanced convex sets in \( X' \) and \( \beta(X, X') \) is just the topology of uniform convergence on weak* bounded sets in \( X' \). If \( (X, \| \cdot \|) \) is a Banach space, then \( \tau(X, X') = \beta(X, X') = \| \cdot \| \) by the Banach-Alaoglu theorem (see [2]).

For a locally convex space \( X \) (with the locally convex topology \( \mu \)) and an operator \( T : c \to X \) we say that \( T \) is continuous means \( T \) is \( \| \cdot \| - \mu \) continuous. But \( \mu \leq \tau(X, X') \leq \beta(X, X') \) continuity is stronger than \( \| \cdot \| - \mu \) continuity. However, by the Hellinger-Toeplitz theorem, if \( (Y, \| \cdot \|) \) is a Banach space and \( T : Y \to X \) is continuous, i.e., \( \| \cdot \| - \mu \) continuous, then \( T \) is \( \| \cdot \| - \beta(X, X') \) continuous because \( \beta(Y, Y') = \| \cdot \| \). Thus, for \( T : c \to X \), the continuity of \( T \) is equivalent to the \( \| \cdot \|_\infty - \beta(X, X') \) continuity.

It is well known that if \( \sum x_j \) is a (weakly) \( c_0 \)-convergent series on a locally convex space \( X \), then letting \( T\{ t_j \} = \sum_{j=1}^\infty t_jx_j \) for each \( \{ t_j \} \in c_0 \), \( T \) is \( \| \cdot \|_\infty - \beta(X, X') \) continuous linear operator and, hence, \( T \) is \( \| \cdot \|_\infty - \beta(X, X') \) continuous. Note that in this case the series \( \sum_{j=1}^\infty t_jx_j \) converges with respect to the original topology on \( X \) and the more strong \( \tau(X, X') \), the Mackey topology. But in the case of \( c \)-convergence, a weakly \( c \)-convergent series need not be \( c \)-convergent. The following result shows that weakly \( c \)-convergent series also gives \( \| \cdot \|_\infty - \beta(X, X') \) continuous operators.

**Theorem 2.** Let \( X \) be a locally convex space and \( \sum x_j \) a weakly \( c \)-convergent series on \( X \). Define \( T : c \to X \) by \( T\{ t_j \} = w - \sum_{j=1}^\infty t_jx_j \), \( \{ t_j \} \in c \). Then \( T \) is a continuous linear operator and, hence, \( T \) is \( \| \cdot \|_\infty - \beta(X, X') \) continuous.
Proof. If \( \{t_j\} \in c \), then
\[
\sum_{j=1}^{\infty} t_j f(x_j) = \lim_n \sum_{j=1}^{n} t_j f(x_j) = \lim_n f\left(\sum_{j=1}^{n} t_j x_j\right) = f\left(w - \sum_{j=1}^{\infty} t_j x_j\right)
\]
for all \( f \in X' \). Suppose that \( \lim_\alpha \{t_{\alpha j}\} = \{t_j\} \) in \((c, \text{weak})\). It is well known that \( f \in c' \) if and only if there exists a \( \gamma \in \mathbb{C} \) and a
\[
\{\gamma_j\} \in l^1 = \{\{\delta_j\} : \sum_{j=1}^{\infty} |\delta_j| < +\infty\}
\]
such that
\[
f\{s_j\} = \gamma \lim_j s_j + \sum_{j=1}^{\infty} \gamma_j s_j
\]
for \( \{s_j\} \in c \). Therefore,
\[
\lim_\alpha [\gamma \lim_j t_{\alpha j}] + \lim_\alpha \sum_{j=1}^{\infty} t_{\alpha j} \gamma_j = \gamma \lim_j t_j + \sum_{j=1}^{\infty} t_j \gamma_j
\]
for every \( \gamma \in \mathbb{C} \) and \( \{\gamma_j\} \in c \). Letting \( \gamma = 0 \), we then have
\[
\lim_\alpha \sum_{j=1}^{\infty} t_{\alpha j} \gamma_j = \sum_{j=1}^{\infty} t_j \gamma_j \text{ for all } \{\gamma_j\} \in l^1.
\]
Now let \( f \in X' \) be arbitrary. By Proposition 1, \( \{f(x_j)\} \in l^1 \). Therefore,
\[
\lim_\alpha f(T\{t_{\alpha j}\}) = \lim_\alpha f\left(w - \sum_{j=1}^{\infty} t_{\alpha j} x_j\right) = \lim_\alpha \sum_{j=1}^{\infty} t_{\alpha j} f(x_j) = \sum_{j=1}^{\infty} t_j f(x_j)
\]
\[
= f\left(w - \sum_{j=1}^{\infty} t_j x_j\right) = f(T\{t_j\}).
\]
This shows that \( T \) is weak-weak continuous. By the Hellinger-Toeplitz theorem ([2], P. 169, Corollary. 6), \( T \) is \( \beta(c, c') - \beta(X, X') \) continuous. But \( \beta(c, c') = \| \cdot \|_{\infty} \) so \( T \) is \( \| \cdot \|_{\infty} - \beta(X, X') \) continuous. \( \square \)
A series $\sum x_j$ on a locally convex space $X$ is said to be weakly $c$-Cauchy if for every $\{t_j\} \in c$, $\{\sum_{j=1}^{n} t_j x_j\}_{n=1}^{\infty}$ is a Cauchy sequence in $(X, \text{weak})$, i.e., for each $f \in X'$,

$$\{\sum_{j=1}^{n} t_j f(x_j)\}_{n=1}^{\infty} = \{f(\sum_{j=1}^{n} t_j x_j)\}_{n=1}^{\infty}$$

is a Cauchy sequence in $\mathbb{C}$. Clearly, $\sum x_j$ is weakly $c$-Cauchy if and only if for every $\{t_j\} \in c$ and $f \in X'$ the series $\sum_{j=1}^{\infty} t_j f(x_j)$ converges.

The following result shows that a weakly $c$-Cauchy series on a sequentially complete locally convex space must be $c_0$-convergent. Note that Banach spaces are sequentially complete locally convex spaces.

**Theorem 3.** Let $X$ be a sequentially complete locally convex space. If a series $\sum x_j$ on $X$ is weakly $c$-Cauchy, then $\sum x_j$ is $c_0$-convergent, i.e., for each $\{t_j\} \in c_0$ the series $\sum_{j=1}^{n} t_j x_j$ converges.

**Proof.** Suppose $\sum_{j=1}^{\infty} |f(x_j)| = +\infty$ for some $f \in X'$. There is an integer $n_1 > 1$ such that $\sum_{j=1}^{n_1} |f(x_j)| > 1$. There is an integer $n_2 > n_1$ such that $\sum_{j=1}^{n_2} |f(x_j)| > \sum_{j=1}^{n_1} |f(x_j)| + 2$. There is an $n_3 > n_2$ such that $\sum_{j=1}^{n_3} |f(x_j)| > \sum_{j=1}^{n_2} |f(x_j)| + 3$. Continuing this construction we have an integer sequence $1 = n_0 < n_1 < n_2 < n_3 < \cdots$ such that

$$\sum_{j=n_k+1}^{n_{k+1}} |f(x_j)| > k + 1, \quad k = 0, 1, 2, 3, \cdots$$

Let $t_1 = 0$, $t_j = \frac{1}{k+1} \sgn f(x_j)$, $n_k < j \leq n_{k+1}$, $k = 0, 1, 2, 3, \cdots$. Then $t_j \to 0$ so $\{t_j\} \in c_0 \subseteq c$. But

$$\sum_{j=1}^{N} t_j f(x_j) = \sum_{j=2}^{\infty} t_j f(x_j) = \sum_{k=0}^{N} \sum_{j=n_k+1}^{n_{k+1}} \frac{1}{k+1} (\sgn f(x_j)) f(x_j)$$

$$= \sum_{k=0}^{N} \frac{1}{k+1} \sum_{j=n_k+1}^{n_{k+1}} |f(x_j)| > \sum_{k=0}^{N} 1 = N + 1,$$
for all \( N \in \mathbb{N} \), i.e., \( \sum_{j=1}^{\infty} t_j f(x_j) \) diverges. This contradicts that \( \sum x_j \) is weakly \( c \)-Cauchy. So \( \sum_{j=1}^{\infty} |f(x_j)| < +\infty \), foa all \( f \in X' \). Let

\[
A = \left\{ \sum_{j=1}^{n} \alpha_j x_j : n \in \mathbb{N}, |\alpha| \leq 1, 1 \leq j \leq n \right\}.
\]

For every \( f \in X' \),

\[
|f(\sum_{j=1}^{n} \alpha_j x_j)| = \left| \sum_{j=1}^{n} \alpha_j f(x_j) \right| \leq \sum_{j=1}^{n} |\alpha_j||f(x_j)| \\
\leq \sum_{j=1}^{n} |f(x_j)| \leq \sum_{j=1}^{\infty} |f(x_j)| < +\infty,
\]

for all \( \sum_{j=1}^{n} \alpha_j x_j \in A \). This shows that \( A \) is weakly bounded and, hence, bounded by the Mackey theorem ([2], p.114, Theorem 1).

Now suppose that \( \{t_j\} \in c_0 \), i.e., \( t_j \to 0 \). Without loss of generality, we assume that for all \( j_0 \) there exists \( j > j_0 \) such that \( t_j \neq 0 \). Let \( U \) be a neighborhood of \( 0 \in X \). Letting \( \alpha_k = \sup_{j \geq k}|t_j| \), \( \alpha_k \to 0 \). Since \( A \) is bounded, there is a \( \delta > 0 \) such that \( \alpha A \subseteq U \) for all \( |\alpha| \leq \delta \). Since \( \alpha_k \to 0 \), there is a \( k_0 \in \mathbb{N} \) such that if \( k \geq k_0 \), then \( |\alpha_k| \leq \delta \). Therefore, if \( m > k \geq k_0 \), then

\[
\sum_{j=k}^{m} t_j x_j = \alpha_k \sum_{j=k}^{m} \frac{t_j}{\alpha_k} x_j \\
= \alpha_k \left( 0x_1 + 0x_2 + \cdots + 0x_{k-1} + \sum_{j=k}^{m} \frac{t_j}{\alpha_k} x_j \right) \\
\in \alpha_k A \subseteq U.
\]

This shows that \( \{\sum_{j=1}^{n} t_j x_j\}_{n=1}^{\infty} \) is Cauchy and, hence, the series \( \sum_{j=1}^{\infty} t_j x_j \) converges because \( X \) is sequentially complete. \( \square \)
THEOREM 4. Let $X$ be a sequentially complete locally convex space. For a series $\sum x_j$ on $X$, the following conditions are equivalent.

1. $\sum x_j$ is a weakly unconditional Cauchy series, i.e., for all $f \in X'$, $\sum_{j=1}^{\infty} |f(x_j)| < +\infty$.
2. For every $\{t_j\} \in l^\infty$, $\{\sum_{j \in \Delta} t_j x_j : \Delta \subseteq \mathbb{N} \text{ finite}\}$ is bounded.
3. $\sum x_j$ is $c_0$-convergent, i.e., for every $\{t_j\} \in c_0$, the series $\sum_{j=1}^{\infty} t_j x_j$ converges.
4. $\sum x_j$ is weakly $c_0$-Cauchy, i.e., the series $\sum_{j=1}^{\infty} t_j f(x_j)$ converges for every $\{t_j\} \in c_0$ and $f \in X'$.
5. $\sum x_j$ is weakly $c$-Cauchy, i.e., the series $\sum_{j=1}^{\infty} t_j f(x_j)$ converges for every $\{t_j\} \in c$ and $f \in X'$.
6. $\{\sum_{j=1}^{n} t_j x_j : n \in \mathbb{N}, |t_j| \leq 1, 1 \leq j \leq n\}$ is bounded.

Proof. By Theorem 2 of [1], (1)=$(2)=$(3) since $X$ is sequentially complete. Since $c_0 \subseteq c$, (5)⇒(4). As in the proof of Theorem 3, (4) ⇒ (1) ⇒ (6) ⇒ (3) ⇒ (4). So (1)=$(2)=$(3)=$(4)=$(6) and (5)⇒(4). Suppose (4) holds. Then (1) holds because (1)=$(4)$, i.e., $\sum_{j=1}^{\infty} |f(x_j)| < +\infty$, for all $f \in X'$. Since $\{t_j\} \in c \Rightarrow \{t_j\}$ is bounded,

$$\sum_{j=1}^{\infty} |t_j f(x_j)| = \sum_{j=1}^{\infty} |t_j| |f(x_j)| \leq \sup_j |t_j| \sum_{j=1}^{\infty} |f(x_j)| < +\infty.$$

This shows that $\sum_{j=1}^{\infty} t_j f(x_j)$ converges for all $\{t_j\} \in c$. □

COROLLARY 5. If $X$ is a sequentially complete locally convex space, then (1)=$(2)=$(3)=$(4)=$(5)=$(6)=$(7)=$(8)=$(9)=$(10).

7. $\sum_{j=1}^{\infty} |t_j f(x_j)| < +\infty$, for all $\{t_j\} \in c_0$, $f \in X'$.
8. $\sum_{j=1}^{\infty} |t_j f(x_j)| < +\infty$, for all $\{t_j\} \in c$, $f \in X'$.
9. $\sum_{j=1}^{\infty} |t_j f(x_j)| < +\infty$, for all $\{t_j\} \in l^\infty$, $f \in X'$.
10. $\sum_{j=1}^{\infty} t_j f(x_j)$ converges for every $\{t_j\} \in l^\infty$, and $f \in X'$.

Proof. $\{t_j\} \in l^\infty \Rightarrow \{t_j \text{ sgn } f(x_j)\} \in l^\infty$, so (9)=$(10)$.

$(1) \Rightarrow (9) \Rightarrow (8) \Rightarrow (7) \Rightarrow (4) \Rightarrow (1)$. □

Now we give the main result of this paper.
THEOREM 6. Let $X$ be a sequentially complete locally convex space. The following conditions are equivalent.

(a) $X$ contains no copy of $c_0$.

(b) Each weakly $c_0$-Cauchy series on $X$ is $c$-convergent, i.e., if $\sum_{j=1}^{\infty} t_j f(x_j)$ converges for every $\{t_j\} \in c_0$ and $f \in X'$, then $\sum_{j=1}^{\infty} t_j x_j$ converges for each $\{t_j\} \in c$.

(c) Each weakly $c$-Cauchy series on $X$ is $c$-convergent, i.e., if $\sum_{j=1}^{\infty} t_j f(x_j)$ converges for every $\{t_j\} \in c$ and $f \in X'$, then $\sum_{j=1}^{\infty} t_j x_j$ converges for each $\{t_j\} \in c$.

Proof. (a)$\Rightarrow$(b). Suppose $\sum_{j=1}^{\infty} \alpha_j f(x_j)$ converges for every $\{\alpha_j\} \in c_0$ and $f \in X'$. Let $\{t_j\} \in c$. Then $\alpha_j t_j \to 0$ for each $\{\alpha_j\} \in c_0$ so $\sum_{j=1}^{\infty} \alpha_j f(t_j x_j)$ converges for every $\{\alpha_j\} \in c_0$ and $f \in X'$. By theorem 4 ((3)$\Rightarrow$(4)), $\sum_{j=1}^{\infty} \alpha_j t_j x_j$ converges for each $\{\alpha_j\} \in c_0$, i.e., $\{t_j x_j\} \in CMC(X)$ (see [3]). Since $X$ contains no copy of $c_0$, by Theorem 4 of [3], $\sum_{j=1}^{\infty} t_j x_j$ converges, i.e., (b) holds.

(b)$\Rightarrow$(c) : $c_0 \subseteq c$.

(c)$\Rightarrow$(a). Suppose $X$ contains a copy of $c_0$. Say that $c_0 \subseteq X$. Let $e_j$ denotes the sequence that has 1 at the $j$-th spot and 0 elsewhere, i.e., $e_j = (0, \cdots, 0, 1, 0, 0, \cdots)$. For every $\{t_j\} \in c$ and $f = \{\alpha_j\} \in l^1 = c'_0$,

$$\sum_{j=1}^{n} |t_j f(e_j)| = \sum_{j=1}^{n} |f(t_j e_j)| = |f(\sum_{j=1}^{n} t_j e_j)| = |f(t_1, t_2, \cdots, t_n, 0, 0, \cdots)| = \left| \sum_{j=1}^{n} t_j \alpha_j \right| \leq \sum_{j=1}^{n} |t_j| |\alpha_j| \leq sup_j |t_j| \sum_{j=1}^{n} |\alpha_j| \leq sup_j |t_j| \sum_{j=1}^{n} |\alpha_j| < +\infty,$$

for all $n \in \mathbb{N}$, i.e., for every $\{t_j\} \in c$ and $f \in c'_0$, $\sum_{j=1}^{\infty} t_j f(e_j)$ converges.

However, letting $t_j = 1$ for all $j$, $\{t_j\} = \{1\} \in c$ but the series $\sum_{j=1}^{\infty} e_j$
diverges in $c_0$:

$$\| \sum_{j=m}^{n} e_j \|_{\infty} = \|(0, \cdots, 0, 1, 1, \cdots, 1, 0, 0, \cdots)\|_{\infty} = 1$$

for all $1 \leq m < n < +\infty$. If $\lim_{n} \sum_{j=1}^{n} e_j = x \in X \setminus c_0$, then

$$\lim_{m, n \to \infty} \| \sum_{j=m}^{n} e_j \|_{\infty} = 0.$$ 

So $\sum_{j=1}^{\infty} e_j$ diverges in $X$. This contradicts (c). \hfill \Box

**Corollary 7.** If a sequentially complete locally convex space $X$ contains no copy of $c_0$, then every weakly $c$-convergent series on $X$ is $c$-convergent.

By Theorem 4 of [3], we have

**Theorem 8.** Let $X$ be a sequentially complete locally convex space. The followings are equivalent.

1. $X$ contains no copy of $c_0$.
2. Each weakly $c_0$-Cauchy series on $X$ is bounded multiplier convergent, i.e., if $\sum_{j=1}^{\infty} t_j f(x_j)$ converges for every $\{t_j\} \in c_0$ and $f \in X'$, then $\sum_{j=1}^{\infty} t_j x_j$ converges for each $\{t_j\} \in l^\infty$, the family of bounded number sequences.

**Proof.** (1) $\Rightarrow$ (b). So if $\sum_{j=1}^{\infty} t_j f(x_j)$ converges for every $\{t_j\} \in c_0$ and $f \in X'$, then $\{x_k\} \in CMC(X)$ but (1) $\Rightarrow$ $CMC(X) = BMC(X)$ by Theorem 4 of [3]. \hfill \Box

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