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# BANACH-SAKS PROPERTY ON THE DUAL OF SCHLUMPRECHT SPACE

Kyugeun Cho and Chongsung Lee

Abstract. In this paper, we show that Schlumprecht space is reflexive and the Dual of Schlumprecht space has the Banach-Saks property and study behavior of block basic sequence in Schlumprecht space.

## 1. Introduction

S. Banach and S. Saks [BS] showed that every bounded sequence in $L_p[0,1]$, $1 < p < \infty$, has a subsequence with arithmetic means converging in norm. J. Schrier [Sc] showed that $C[0,1]$ does not have this property. The above results lead us to consider the following question. What Banach space $X$ has the *Banach-Saks property* i.e., every bounded sequence in $X$ admits a subsequence whose arithmetic means converges in norm. S. Kakutani [Ka] showed that uniformly convex Banach spaces have the Banach-Saks property. T. Nishiura and D. Waterman [NW] showed that Banach spaces with the Banach-Saks property are reflexive. A. Baernstein [Ba] proved the converse by providing an example of a reflexive Banach space which does not have the Banach-Saks property. C. Seifert [Se1] showed that the dual of Baernstein space has the Banach-Saks property. In this paper, we introduce arbitrarily distortable Banach space - Schlumprecht space [Sh] and show that it is reflexive, not uniformly convex and its dual has the Banach-Saks property.

Schlumprecht [Sh] introduced a Banach space which is arbitrarily distortable. We introduce some basic definitions and construct Schlumprecht space $S$. The vector space of all real valued sequences

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$(x_n)$ whose elements are eventually zero is denoted by $c_{00}$ ; $(e_i)$ denotes the usual unit vector basis of $c_{00}$, i.e., $e_i(j) = 1$ if $i = j$ and $e_i(j) = 0$ if $i \neq j$. For $x = \sum_{i=1}^{n} \alpha_i e_i \in c_{00}$, the set $\text{supp}(x) = \{i \in \mathbb{N} : \alpha_i \neq 0\}$ is called the support of $x$. If $E$ and $F$ are two finite subsets of $\mathbb{N}$ we write $E < F$ if $\max(E) < \min(F)$, and for $x, y \in c_{00}$ we write $x < y$ if supp$(x) <$supp$(y)$. For $E \subset \mathbb{N}$ and $x = \sum_{i=1}^{\infty} x_i e_i \in c_{00}$ we put $E(x) := \sum_{i \in E} x_i e_i$.

The following lemma is essential to define the Schlumprecht space and we refer to [Sh] for the its proof. From now on, we mean $f(x)$ as $\log_2(x+1)$.

Lemma 1.1. *[Sh] Let $f(x) = \log_2(x+1)$, for $x \geq 1$. Then $f$ has the following properties :*

(1) *$f(1) = 1$ and $f(x) < x$ for all $x > 1$,*
(2) *$f$ is strictly increasing to $\infty$,*
(3) *$\lim_{x\to\infty} (f(x)/x^q) = 0$ for all $q > 0$,*
(4) *the function $g(x) = x/f(x)$, $x \geq 1$ is concave, and*
(5) *$f(x) \cdot f(y) \geq f(x \cdot y)$ for $x, y \geq 1$.*

On $c_{00}$ we define a norm $|\cdot|_k$ by induction for each $k \in \mathbb{N}$ . For $x = \sum x_n e_n \in c_{00}$ we let $|x|_0 = \max_{n \in \mathbb{N}} |x_n|$. Assuming that $|x|_k$ is defined for some $k \in \mathbb{N}$ we put

$$|x|_{k+1} = \max_{\substack{l \in \mathbb{N} \\ E_1 < E_2 < \cdots < E_l \\ E_i \subset \mathbb{N}}} \frac{1}{f(l)} \sum_{i=1}^{l} |E_i(x)|_k.$$

Since $f(1) = 1$,

$$|x|_{k+1} \geq |E(x)|_k = |x|_k,$$

where $E =$supp$(x)$. It follows that $(|x|_k)$ is increasing for any $x \in c_{00}$. Since $f(l) > 1$ for all $l \geq 2$ and

$$\frac{1}{f(l)} \sum_{k=1}^{l} |E_k(e_i)|_k \leq \frac{1}{f(l)},$$

it follows that

$$|e_i|_k = 1 \qquad \text{for any } i \in \mathbb{N} \text{ and } k \in \mathbb{N}_0.$$

We put

$$\|x\| = \max_{k \in \mathbb{N}} |x|_k, \qquad \text{for } x \in c_{00}.$$

Then $\|\cdot\|$ is a norm on $c_{00}$ and we define the Schlumprecht space $S$ as the completion of $c_{00}$ with respect to $\|\cdot\|$.

The following proposition states some easy facts about $S$.

Proposition 1.2. *[Sh]*

(1) *The sequence of unit vectors $(e_i)$ is a 1-subsymmetric and 1-unconditional basis of the Schlumprecht space $S$ ; i.e., for any $x = \sum_{i=1}^{\infty} x_i e_i \in S$, any strictly increasing sequence $(n_i) \subset \mathbb{N}$ and any $(\epsilon_i)_{i \in \mathbb{N}} \in \{-1, 1\}^{\mathbb{N}}$ it follows that*

$$\|x\| = \left\| \sum_{i=1}^{\infty} x_i e_i \right\| = \left\| \sum_{i=1}^{\infty} \epsilon_i x_i e_{n_i} \right\|.$$

(2) *For $x \in S$,*

$$\|x\| = \max \left\{ |x|_0, \sup_{\substack{l \geq 2 \\ E_1 < E_2 < \cdots < E_l \\ E_i \subset \mathbb{N}}} \frac{1}{f(l)} \sum_{i=1}^{l} \|E_i(x)\| \right\}$$

(3) *For $n \in \mathbb{N}$ we have that*

$$\left\| \sum_{i=1}^{n} e_i \right\| = \frac{n}{f(n)}.$$

The following is the main result of [Sh].

Theorem 1.3. *The Schlumprecht space $S$ is arbitrarily distortable and does not contain an isomorphic copy of $l_1$.*

## 2. Banach-Saks Property on the Dual of Schlumprecht space

In this chapter, we show that $S$ is reflexive, not uniformly convex and $S^*$ has the Banach-Saks Property. Finally, we carefully examine the rather special behavior of block basic sequence in Schlumprecht space.

THEOREM 2.1. *$S$ is a reflexive space.*

*Proof.* It suffices to show that $S$ does not contain $c_0$, by Theorem 1.3 and Proposition 1.2.(1). Suppose $c_0$ is isomorphic to a subspace of $S$. Then there exists a sequence $\{y_n\}$ of $S$ which is equivalent to the unit vectors $\{e_n\}$ of $c_0$, that is, there exists $m,\ M > 0$ such that

$$m\|\sum a_n e_n\|_{c_0} \leq \|\sum a_n y_n\| \leq M\|\sum a_n e_n\|_{c_0}.$$

Since the unit vector $\{e_n\}$ of $c_0$ is convergent weakly to zero and bounded away from zero in norm, so is $\{y_n\}$. By the Bessaga-Pelcynski selection principle, there exists a subsequence $\{y'_n\}$ of $\{y_n\}$ which is equivalent to a normalized block basis $\{u_j\}$ of unit vectors $\{e_n\}$ of $S$. Then we have

$$\left\|\sum_{k=1}^{n} y'_k\right\| \leq M$$

and

$$\left\|\sum_{k=1}^{n} u_k\right\| \geq \frac{1}{f(n)}\sum_{k=1}^{n}\|E_k(\sum_{k=1}^{n} u_k)\| = \frac{n}{f(n)},$$

where $E_k$=supp $(u_k)$.

Since $\frac{n}{f(n)} \to \infty$ as $n \to \infty$ by Lemma 1.1.(3), we get the contradiction to the fact that $\{u_j\}$ is equivalent to $\{y'_k\}$. □

We show that $S$ is not uniformly convex. For this, we need following easy lemma.

LEMMA 2.2. *Let* $x = \left(\frac{f(2)}{2} \pm \epsilon\right) e_1 + \left(\frac{f(2)}{2} \mp \epsilon\right) e_2 \in S$ *and* $\frac{f(2)}{2} \leq \frac{f(2)}{2} + \epsilon \leq 1$. *Then* $\|x\| = 1$.

*Proof.* Since the number of supp$(x)$ is 2, by Proposition 1.2.(2),

$$\begin{aligned}\|x\| &= \max\left\{|x|_0, \frac{1}{f(2)}\left(\|E_1(x)\| + \|E_2(x)\|\right)\right\} \\ &= \max\left\{\frac{f(2)}{2} + \epsilon, 1\right\} = 1.\end{aligned}$$

□

Using Lemma 2.2, we get the following proposition.

PROPOSITION 2.3. *The Schlumprecht space* $S$ *is not uniformly convex.*

*Proof.* . Let $\epsilon > 0$, $\frac{f(2)}{2} + \epsilon \leq 1$ and

$$\begin{aligned}x &= \left(\frac{f(2)}{2} + \epsilon\right) e_1 + \left(\frac{f(2)}{2} - \epsilon\right) e_2 \\ y &= \left(\frac{f(2)}{2} - \epsilon\right) e_1 + \left(\frac{f(2)}{2} + \epsilon\right) e_2.\end{aligned}$$

Then $\|x\| = \|y\| = 1$ and

$$\begin{aligned}\|x + y\| &= f(2)\|e_1 + e_2\| \\ &= f(2)\frac{2}{f(2)} \qquad \text{by Proposition 1.2.(3)} \\ &= 2.\end{aligned}$$

□

By Proposition 2.3 and Theorem 2.1, we can ask a natural question : does $S$ or $S^*$ has the Banach-Saks property ? The following Lemma is the criterion for testing for the Banach-Saks property.

LEMMA 2.4. *[Se2] Suppose $X$ is a reflexive Banach space whose basis is $\{x_n\}$. Then $X$ has the Banach-Saks property if and only if every bounded block basic sequence with respect to $\{x_n\}$ admits a subsequence whose arithmetic means converges to zero in norm.*

Now we are ready to get our main theorem which is focused throughout this paper.

THEOREM 2.5. *$S^*$ has the Banach-Saks property, where $S^*$ is the dual space of the Schlumprecht space.*

*Proof.* We note that $\{e_n\}$ is shrinking and the biothogonal functionals $\{e^*\}$ form a Schauder basis of $S^*$, since $S$ is reflexive. Let $\{x_n^*\}$ be a bounded block basic sequence with respect to $\{e_n^*\}$, where $x_n^* = \sum_{j \in F_n} \alpha_j e_j^*$, $F_1 < F_2 < \cdots F_n < \cdots$ . Let $x = \sum_{j=1}^{\infty} x_j e_j \in S$, $\|x\| = 1$. Then

$$\begin{aligned}
\left| \left\langle \sum_{m=1}^{n} x_m^*, x \right\rangle \right| &= \left| \sum_{m=1}^{n} \left\langle \sum_{j \in F_m} \alpha_j e_j^*, \sum_{j=1}^{\infty} x_j e_j \right\rangle \right| \\
&\le \sum_{m=1}^{n} \left| \sum_{j \in F_m} \alpha_j x_j \right| \\
&= \sum_{m=1}^{n} |\langle x_m^*, x_{F_m} \rangle| \\
&\le \sum_{m=1}^{n} M \|x_{F_m}\|, \qquad \text{where } M = \sup_m \|x_m^*\| \\
&\le M f(n) \|x\|, \qquad \text{by Proposition 1.2 (2)} \\
&= M f(n).
\end{aligned}$$

Hence

$$\frac{1}{n} \left\| \sum_{m=1}^{n} x_m^* \right\| \le \frac{M f(n)}{n} \to 0 \qquad \text{as } n \to \infty, \text{ by Lemma 1.1 (3)}$$

This completes the proof. □

Finally, we carefully examine the rather special behavior of block basic sequence in Schlumprecht space.

THEOREM 2.6. *Let* $y_n = \sum_{i=p_n+1}^{p_{n+1}} a_i e_i$, $(n = 1, 2, \cdots)$ *be a normalized block basic sequence of scalars* $\{e_n\}_{n=1}^{\infty}$*. Then for every sequence of scalars* $\{b_n\}_{n=1}^{\infty}$*,*

$$\left\| \sum_n b_n e_n \right\| \leq \left\| \sum_n b_n y_n \right\|.$$

*Proof.* We show for every choice of $(b_n)$,

$$\left| \sum_n b_n e_n \right|_m \leq \left\| \sum_n b_n y_n \right\| \qquad \text{for every } m.$$

For $m = 0$,

$$\begin{aligned} \left| \sum_n b_n e_n \right|_0 &= \sup_n |b_n| \\ &= \sup_n \|b_n y_n\| \\ &\leq \sup_n \left\| \sum_n b_n y_n \right\|, \end{aligned}$$

since $(e_n)$ is 1-unconditional. Suppose our result up to some positive integer $m$. Let $x = \sum b_n e_n,\ y = \sum b_n y_n$. Then for $E_1 < \cdots < E_l$,

$$\begin{aligned} \frac{1}{f(l)} \sum_{j=1}^{l} |E_j(x)|_m &\leq \frac{1}{f(l)} \sum_{j=1}^{l} \left\| \sum_{n \in E_j} b_n y_n \right\| \\ &\qquad \text{by the induction hypothesis} \\ &= \frac{1}{f(l)} \sum_{j=1}^{l} \|F_j(y)\| \\ &\qquad \text{where } F_j = \cup_{n \in E_j} \text{supp} y_n \\ &\leq \|y\|. \end{aligned}$$

Thus, $|x|_{m+1} \leq \|y\|$. This completes the proof. □

## References


[Ba] Baernstein, A., II, *On reflexivity and summability*, Studia Math. **42** (1972), 91-94.

[BS] Banach, S. and Sak, S., *Sur la convergence forte dans les champs $L^p$*, Studia Math. **2** (1930), 51-57.

[Ka] Kakutani, S., *Weak convergence in uniformly convex spaces*, Tohoku Math. **45** (1938), 188-193.

[NW] Nishiura, T. and Waterman, *Reflexivity and summability*, Studia Math. **23** (1963), 53-57.

[Pe] Pettis, B. J., *A proof that every uniformly convex space is reflexive*, Duke Math. J. **5** (1939), 249-253.

[Sc] Schreier, J., *Ein Gegenbeispiel zur Theorie der schwachen Konvergenz*, Studia Math. **2** (1930), 58-62.

[Se1] Seifert, C. J., *The dual of Baernstein's space and the Banach-Saks property*, Bull. Acad. Polon. Sci. **26** (1978), 237-239.

[Se2] Seifert, C. J., *Averaging in Banach space, (dissertation)*, Kent State University (1977).

[Sh] Schlumprecht, Th., *An arbitrarily distortable Banach space*, Isr. J. Math. **76** (1991), 81-95.



Institute For Basic Science
Inha University
Inchon 402-751, Korea
*E-mail*: kgjo@math.inha.ac.kr

Department of Mathematics
Inha University
Inchon 402-751, Korea