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SMOOTH FUZZY CLOSURE AND TOPOLOGICAL SPACES

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ABSTRACT. We will define a smooth fuzzy closure space and a subspace of it. We will investigate relationships between smooth fuzzy closure spaces and smooth fuzzy topological spaces. In particular, we will show that a subspace of a smooth fuzzy topological space can be obtained by the subspace of the smooth fuzzy closure space induced by it.

1. Introduction and preliminaries

R.N. Hazra et al. [6] introduced the concept of gradations of openness as an extension of Chang's fuzzy topology [1]. It has been developed in many directions [2,3,4,5,9].

In this paper, we will define a smooth fuzzy closure space and a subspace of it in view of K.C. Chattopadhyay et al. [3] as an extension of the definition of A.S. Masshour et al.[8]. Also, we will study relationships between smooth fuzzy closure spaces and smooth fuzzy topological spaces. In particular, we will show that a subspace of a smooth fuzzy topological space can be obtained by the subspace of the smooth fuzzy closure space induced by it.

In this paper, let X be a nonempty set, $I = [0,1]$ and $I_0 = (0,1]$.

A *fuzzy set* in X is a function $\mu : X \to I$ and I^X will denote the family of all fuzzy sets in X.

For a family of fuzzy sets $\{\mu_i \mid i \in \Lambda\}$ in X, we define

$$(\bigvee_{i\in\Lambda} \mu_i)(x) = sup_{i\in\Lambda}\mu_i(x), \quad (\bigwedge_{i\in\Lambda} \mu_i)(x) = inf_{i\in\Lambda}\mu_i(x).$$

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For $\mu, \nu \in I^X$, we define $\mu \leq \nu$ iff $\mu(x) \leq \nu(x)$ for all $x \in X$.

A *fuzzy point* x_t, $t \in I_0$, is an element of I^X such that

$$x_t(y) = \begin{cases} t, & \text{if } y = x, \\ 0, & \text{if } y \neq x. \end{cases}$$

A fuzzy point $x_t \in \lambda$ iff $t \leq \lambda(x)$.

If $A \subset X$, we define the *characteristic function* χ_A on X by

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

Let $f : X \to Y$ be a function, $\mu \in I^X$ and $\nu \in I^Y$. We define

$$f(\mu)(y) = \begin{cases} sup\{\mu(x) \mid x \in f^{-1}(\{y\})\}, & \text{if } f^{-1}(\{y\}) \neq \emptyset, \\ 0, & \text{if } f^{-1}(\{y\}) = \emptyset, \end{cases}$$

and $f^{-1}(\nu)(x) = \nu(f(x))$ for all $x \in X$.

We denote $\tilde{0}(x) = 0, \tilde{1}(x) = 1$ for all $x \in X$.

Lemma 1.1. [5] *If $f : X \to Y$, then we have the following properties for direct and inverse image of fuzzy sets under mappings: for $\mu, \mu_i \in I^X$ and $\nu, \nu_i \in I^Y$,*

(1) $\nu \geq f(f^{-1}(\nu))$ with equality if f is surjective,
(2) $\mu \leq f^{-1}(f(\mu))$ with equality if f is injective,
(3) $f^{-1}(\tilde{1} - \nu) = \tilde{1} - f^{-1}(\nu)$,
(4) $f(\tilde{1} - \mu) = \tilde{1} - f(\mu)$ if f is bijective,
(5) $f^{-1}(\bigvee_{i\in\Lambda} \nu_i) = \bigvee_{i\in\Lambda} f^{-1}(\nu_i)$,
(6) $f^{-1}(\bigwedge_{i\in\Lambda} \nu_i) = \bigwedge_{i\in\Lambda} f^{-1}(\nu_i)$,
(7) $f(\bigvee_{i\in\Lambda} \mu_i) = \bigvee_{i\in\Lambda} f(\mu_i)$,
(8) $f(\bigwedge_{i\in\Lambda} \mu_i) \leq \bigwedge_{i\in\Lambda} f(\mu_i)$ with equality if f is injective.

2. Smooth fuzzy topological spaces and smooth fuzzy closure spaces

Definition 2.1. [6,4] A function $\mathcal{T} : I^X \to I$ is called a *smooth fuzzy topology* on X if it satisfies the following conditions:

(O1) $\mathcal{T}(\tilde{0}) = \mathcal{T}(\tilde{1}) = 1$,
(O2) $\mathcal{T}(\mu_1 \wedge \mu_2) \geq \mathcal{T}(\mu_1) \wedge \mathcal{T}(\mu_2)$, for any $\mu_1, \mu_2 \in I^X$,
(O3) $\mathcal{T}(\bigvee_{i\in\Lambda} \mu_i) \geq \bigwedge_{i\in\Lambda} \mathcal{T}(\mu_i)$, for any family $\{\mu_i \mid i \in \Lambda\} \subseteq I^X$.
The pair $(X, \mathcal{T})$ is called a *smooth fuzzy topological space.*

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be smooth fuzzy topologies on X, We say that $\mathcal{T}_1$ is *finer* than $\mathcal{T}_2$ ($\mathcal{T}_2$ is *coarser* than $\mathcal{T}_2$) iff $\mathcal{T}_2(\lambda) \leq \mathcal{T}_1(\lambda)$, for all $\lambda \in I^X$.

DEFINITION 2.2. [4] Let $\mathcal{T}$ be a smooth fuzzy topology on X and a function $\mathcal{F} : I^X \to I$ defined by $\mathcal{F}(\mu) = \mathcal{T}(\tilde{1} - \mu)$ for all $\mu \in I^X$. Then $\mathcal{F}$ is called a *smooth fuzzy cotopology* on X.

DEFINITION 2.3. A function $C : I^X \times I_0 \to I^X$ is called a *smooth fuzzy closure operator* on X if it satisfies the following conditions: for each $\lambda, \mu \in I^X$, $r, s \in I_0$,
(C1) $C(\tilde{0}, r) = \tilde{0}, C(\tilde{1}, r) = \tilde{1}$,
(C2) $\lambda \leq C(\lambda, r)$,
(C3) if $\lambda \leq \mu$, then $C(\lambda, r) \leq C(\mu, r)$,
(C4) $C(\lambda \vee \mu, r) = C(\lambda, r) \vee C(\mu, r)$,
(C5) if $r \leq s$, then $C(\lambda, r) \leq C(\lambda, s)$.
The pair (X, C) is called *a smooth fuzzy closure space.*

A smooth fuzzy closure space (X, C) is called *topological* if
(C6) $C(C(\lambda, r), r) = C(\lambda, r)$, for all $\lambda \in I^X$, $r \in I_0$.
Let C_1 and C_2 be smooth fuzzy closure operators on X, We say that C_1 is *finer* than C_2 (C_2 is *coarser* than C_1) iff $C_1(\lambda, r) \leq C_2(\lambda, r)$, for all $\lambda \in I^X$, $r \in I_0$.

EXAMPLE 1. Define $C^0, C^1 : I^X \times I_0 \to I^X$ as follows:

$$C^0(\lambda, r) = \begin{cases} \tilde{0}, & \text{if } \lambda = \tilde{0},\ r \in I_0, \\ \tilde{1}, & \text{otherwise,} \end{cases}$$

and $C^1(\lambda, r) = \lambda$, for all $\lambda \in I^X$, $r \in I_0$. Then C^0 is the coarsest smooth fuzzy closure operator on X and C^1 is the finest one on X. □

THEOREM 2.4. [3] *Let $(X, \mathcal{T})$ be a smooth fuzzy topological space. For each $r \in I_0, \lambda \in I^X$, we define an operator $C_{\mathcal{T}} : I^X \times I_0 \to I^X$ as follows:*

$$C_{\mathcal{T}}(\lambda, r) = \bigwedge\{\mu \mid \mu \geq \lambda, \mathcal{F}(\mu) \geq r\}.$$

Then it satisfies the following properties: for each $\lambda, \mu \in I^X$, $r, s \in I_0$,

(1) $C_{\mathcal{T}}(\tilde{0}, r) = \tilde{0}, C_{\mathcal{T}}(\tilde{1}, r) = \tilde{1}$,

(2) $\lambda \leq C_{\mathcal{T}}(\lambda, r)$,

(3) *if* $\lambda \leq \mu$, *then* $C_{\mathcal{T}}(\lambda, r) \leq C_{\mathcal{T}}(\mu, r)$,

(4) $C_{\mathcal{T}}(\lambda \vee \mu, r) = C_{\mathcal{T}}(\lambda, r) \vee C_{\mathcal{T}}(\mu, r)$,

(5) *if* $r \leq s$, *then* $C_{\mathcal{T}}(\lambda, r) \leq C_{\mathcal{T}}(\lambda, s)$,

(6) $C_{\mathcal{T}}(C_{\mathcal{T}}(\lambda, r), r) = C_{\mathcal{T}}(\lambda, r)$,

(7) *if* $r = \bigvee\{s \in I_0 \mid C_{\mathcal{T}}(\lambda, s) = \lambda\}$, *then* $C_{\mathcal{T}}(\lambda, r) = \lambda$.

THEOREM 2.5. [3] *Let* (X, C) *be a smooth fuzzy closure space. Define the function* $\mathcal{F}_C : I^X \to I$ *on* X *by*

$$\mathcal{F}_C(\lambda) = \bigvee\{r \in I_0 \mid C(\lambda, r) = \lambda\}.$$

Then:

(1) $\mathcal{F}_C$ *is a smooth fuzzy cotopology on* X.

(2) *We have* $C = C_{\mathcal{T}_C}$ *iff* (X, C) *satisfies the following conditions:*

(a) *It is a topological smooth fuzzy closure space.*

(b) *If* $r = \bigvee\{s \in I_0 \mid C(\lambda, s) = \lambda\}$, *then* $C(\lambda, r) = \lambda$.

EXAMPLE 2. Let C^0, C^1 be smooth fuzzy closure operators on X defined by Example 1. Then C^0, C^1 induce the following smooth fuzzy cotopologies on X: for all $\lambda \in I^X$,

$$\mathcal{F}_{C^0}(\lambda) = \begin{cases} 1, & \text{if } \lambda = \tilde{0} \text{ or } \tilde{1}, \\ 0, & \text{otherwise} \end{cases}$$

and $\mathcal{F}_{C^1}(\lambda) = 1$. Therefore $\mathcal{T}_{C^0}$ is the coarsest smooth fuzzy topology on X and $\mathcal{T}_{C^1}$ is the finest one on X. □

The following example is that we have $C \neq C_{\mathcal{T}_C}$ if (X, C) does not satisfy the condition (a) of Theorem 2.5(2).

EXAMPLE 3. Define $C : I^X \times I_0 \to I^X$ as follows:

$$C(\lambda, r) = \begin{cases} \tilde{0}, & \text{if } \lambda = \tilde{0},\ r \in I_0, \\ \chi_{\{x,y\}}, & \text{if } \lambda = x_t \in \chi_{\{x,y\}}, r \leq \frac{1}{2}, \\ \chi_{\{z\}}, & \text{if } \lambda = z_s \in \chi_{\{z\}}, r \leq \frac{1}{2}, \\ \tilde{1}, & \text{otherwise.} \end{cases}$$

Then (X, C) is a smooth fuzzy closure space. Since $C(x_t, \frac{1}{3}) = \chi_{\{x,y\}}$ and $C(\chi_{\{x,y\}}, \frac{1}{3}) = \tilde{1}$, we have

$$C(C(x_t, \frac{1}{3}), \frac{1}{3}) \neq C(x_t, \frac{1}{3}).$$

Hence (X, C) is not a topological smooth fuzzy closure space.

From Theorem 2.5, we can obtain the smooth fuzzy cotopology $\mathcal{F}_C$ on X induced by (X, C) as follows:

$$\mathcal{F}_C(\lambda) = \begin{cases} 1, & \text{if } \lambda = \tilde{0} \text{ or } \tilde{1}, \\ \frac{1}{2}, & \text{if } \lambda = \chi_{\{z\}}, \\ 0, & \text{otherwise.} \end{cases}$$

From Theorem 2.4, we have the smooth fuzzy closure operator $C_{\mathcal{T}_C}$ on X induced by $(X, \mathcal{T}_C)$ as follows:

$$C_{\mathcal{T}_C}(\lambda, r) = \begin{cases} \tilde{0}, & \text{if } \lambda = \tilde{0},\ r \in I_0, \\ \chi_{\{z\}}, & \text{if } \lambda = z_s \in \chi_{\{z\}}, r \leq \frac{1}{2}, \\ \tilde{1}, & \text{otherwise.} \end{cases}$$

Hence $C \neq C_{\mathcal{T}_C}$. □

The following example is that we have $C \neq C_{\mathcal{T}_C}$ if (X, C) does not satisfy the condition (b) of Theorem 2.5(2).

EXAMPLE 4. For $\mu \in I^X$ such that $\mu \neq \tilde{0}, \tilde{1}$, we define a smooth fuzzy closure operator $C : I^X \times I_0 \to I^X$ as follows:

$$C(\lambda, r) = \begin{cases} \tilde{0}, & \text{if } \lambda = \tilde{0},\ r \in I_0, \\ \mu, & \text{if } \tilde{0} \neq \lambda \leq \mu, r < \frac{1}{2}, \\ \tilde{1}, & \text{otherwise.} \end{cases}$$

Then (X, C) is a topological smooth fuzzy closure space.

Hence $\mathcal{F}_C$ is defined by

$$\mathcal{F}_C(\lambda) = \begin{cases} 1, & \text{if } \lambda = \tilde{0} \text{ or } \tilde{1} \\ \frac{1}{2}, & \text{if } \lambda = \mu \\ 0, & \text{otherwise.} \end{cases}$$

Since $C(\mu, r) = \mu$ for all $r < \frac{1}{2}$,

$$\bigvee\{r \in I_0 \mid C(\mu, r) = \mu\} = \frac{1}{2}, \quad \text{but } C(\mu, \frac{1}{2}) = \tilde{1}.$$

On the other hand, we have $C_{\mathcal{T}_C}(\mu, \frac{1}{2}) = \mu$. Hence $C \neq C_{\mathcal{T}_C}$. □

THEOREM 2.6. *Let $(X, \mathcal{T})$ be a smooth fuzzy topological space. Let $(X, C_{\mathcal{T}})$ be a smooth fuzzy closure space. Then $\mathcal{F}_{C_{\mathcal{T}}}$ is a smooth fuzzy cotopology on X such that*

$$\mathcal{F}_{C_{\mathcal{T}}} = \mathcal{F}, \text{ that is }, \mathcal{T}_{C_{\mathcal{T}}} = \mathcal{T}$$

where $\mathcal{F}(\mu) = \mathcal{T}(\tilde{1} - \mu)$, for all $\mu \in I^X$.

Proof. From Theorem 2.5, $\mathcal{F}_{C_{\mathcal{T}}}$ is a smooth fuzzy cotopology on X. We only show that $\mathcal{F}_{C_{\mathcal{T}}} = \mathcal{F}$.

Suppose that there exists $\lambda \in I^X$ such that

$$\mathcal{F}_{C_{\mathcal{T}}}(\lambda) > \mathcal{F}(\lambda).$$

From the definition of $\mathcal{F}_{C_{\mathcal{T}}}$ there exists $r_0 \in I_0$ with $C_{\mathcal{T}}(\lambda, r_0) = \lambda$ such that

$$\mathcal{F}_{C_{\mathcal{T}}}(\lambda) \geq r_0 > \mathcal{F}(\lambda).$$

On the other hand, since $C_{\mathcal{T}}(\lambda, r_0) = \lambda$, we have $\mathcal{F}(\lambda) \geq r_0$ from the definition of $C_{\mathcal{T}}$. It is a contradiction. Hence $\mathcal{F}_{C_{\mathcal{T}}} \leq \mathcal{F}$.

Suppose that there exists $\rho \in I^X$ such that

$$\mathcal{F}_{C_{\mathcal{T}}}(\rho) < \mathcal{F}(\rho).$$

Then there exists $r_1 \in I_0$ such that

$$\mathcal{F}_{C_{\mathcal{T}}}(\rho) < r_1 \leq \mathcal{F}(\rho).$$

On the other hand, since $\mathcal{F}(\rho) \geq r_1$, by the definition of $C_{\mathcal{T}}$ we have

$$C_{\mathcal{T}}(\rho, r_1) = \bigwedge\{\mu \mid \mu \geq \rho, \mathcal{F}(\mu) \geq r_1\} = \rho.$$

Hence $\mathcal{F}_{C_{\mathcal{T}}}(\rho) \geq r_1$. It is a contradiction. Hence $\mathcal{F}_{C_{\mathcal{T}}} \geq \mathcal{F}$. □

DEFINITION 2.7. Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be smooth fuzzy topological spaces. A function $f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ is called smooth continuous if $\mathcal{T}_2(\mu) \leq \mathcal{T}_1(f^{-1}(\mu))$ for all $\mu \in I^Y$.

REMARK. A function $f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ is smooth continuous if $\mathcal{F}_2(\nu) \leq \mathcal{F}_1(f^{-1}(\nu))$ for all $\nu \in I^Y$ because $f^{-1}(\tilde{1} - \nu) = \tilde{1} - f^{-1}(\nu)$ from Lemma 1.2 (3).

DEFINITION 2.8. Let (X, C_1), (Y, C_2) be two smooth fuzzy closure spaces. A function $f : (X, C_1) \to (Y, C_2)$ is called a *C- map* if for all $\lambda \in I^X$, $r \in I_0$, $f(C_1(\lambda, r)) \leq C_2(f(\lambda), r)$.

THEOREM 2.9. *Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be smooth fuzzy topological spaces. A function $f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ is a smooth continuous map iff $f : (X, C_{\mathcal{T}_1}) \to (Y, C_{\mathcal{T}_2})$ is a C- map.*

Proof. Let f be a smooth continuous map. For all $\lambda \in I^X$, $r \in I_0$, from Lemma 1.1 we have the following:

$$C_{\mathcal{T}_2}(f(\lambda), r) = \bigwedge\{\mu \mid \mu \geq f(\lambda), \mathcal{F}_2(\mu) \geq r\}$$

(Since $f^{-1}(\mu) \geq f^{-1}(f(\lambda)) \geq \lambda, \mathcal{F}_1(f^{-1}(\mu)) \geq \mathcal{F}_2(\mu) \geq r$,)

$$\begin{aligned} &\geq \bigwedge\{f(f^{-1}(\mu)) \mid f^{-1}(\mu) \geq \lambda, \mathcal{F}_2(f^{-1}(\mu)) \geq r\} \\ &\geq f(\bigwedge\{f^{-1}(\mu) \mid f^{-1}(\mu) \geq \lambda, \mathcal{F}_2(f^{-1}(\mu)) \geq r\}) \\ &\geq f(C_{\mathcal{T}_1}(\lambda, r)). \end{aligned}$$

Conversely, we will show that $\mathcal{F}_2(\nu) \leq \mathcal{F}_1(f^{-1}(\nu))$, for all $\nu \in I^Y$.

Suppose that there exists $\nu \in I^Y$ such that

$$\mathcal{F}_2(\nu) > \mathcal{F}_1(f^{-1}(\nu)).$$

Since $\mathcal{F}_{C_{\mathcal{T}_2}} = \mathcal{F}_2$ from Theorem 2.6, by the definition of $\mathcal{F}_{C_{\mathcal{T}_2}}$, there exists $r \in I_0$ with $C_{\mathcal{T}_2}(\nu, r) = \nu$ such that

$$\mathcal{F}_2(\nu) \geq r > \mathcal{F}_1(f^{-1}(\nu)).$$

On the other hand, since f is a C-map, we have

$$f(C_{\mathcal{T}_1}(f^{-1}(\nu),r)) \leq C_{\mathcal{T}_2}(f(f^{-1}(\nu)),r) \leq C_{\mathcal{T}_2}(\nu,r) = \nu.$$

Using Lemma 1.1(2), we have

$$C_{\mathcal{T}_1}(f^{-1}(\nu),r) \leq f^{-1}(f(C_{\mathcal{T}_1}(f^{-1}(\nu),r))) \leq f^{-1}(\nu).$$

Hence $C_{\mathcal{T}_1}(f^{-1}(\nu),r) = f^{-1}(\nu)$ from Definition 2.3(C2). Since $\mathcal{F}_{C_{\mathcal{T}_1}} = \mathcal{F}_1$ from Theorem 2.6, by the definition of $\mathcal{F}_{C_{\mathcal{T}_1}}$, we have $\mathcal{F}_1(f^{-1}(\nu)) \geq r$. It is a contradiction. □

Using Theorem 2.9, we can easily prove the following corollary.

COROLLARY 2.10. *Let* (X, C_1), (Y, C_2) *be smooth fuzzy closure spaces. If* $f : (X, C_1) \to (Y, C_2)$ *is a C- map, then* $f : (X, \mathcal{T}_{C_1}) \to (Y, \mathcal{T}_{C_2})$ *is a smooth continuous map.*

The following example shows that $f : (X, C_1) \to (Y, C_2)$ is not a C-map but $f : (X, \mathcal{T}_{C_1}) \to (Y, \mathcal{T}_{C_2})$ is a smooth continuous map.

EXAMPLE 5. Let $X = \{x, y, z\}$ be a set.
Define $C_1, C_2 : I^X \times I_0 \to I^X$ as follows:

$$C_1(\lambda, r) = \begin{cases} \tilde{0}, & \text{if } \lambda = \tilde{0},\ r \in I_0, \\ \chi_{\{z\}}, & \text{if } \lambda = z_s, r \leq \frac{1}{2}, \\ \tilde{1}, & \text{otherwise,} \end{cases}$$

and

$$C_2(\lambda, r) = \begin{cases} \tilde{0}, & \text{if } \lambda = \tilde{0},\ r \in I_0, \\ \chi_{\{x,y\}}, & \text{if } \lambda = x_t, r \leq \frac{1}{3}, \\ \chi_{\{z\}}, & \text{if } \lambda = z_s, r \leq \frac{1}{2}, \\ \tilde{1}, & \text{otherwise.} \end{cases}$$

Then the identity function $1 : (X, C_1) \to (Y, C_2)$ is not a C- map because $\tilde{1} = C_1(x_t, \frac{1}{4}) \not\leq C_2(x_t, \frac{1}{4}) = \chi_{\{x,y\}}$.

On the other hand, from Theorem 2.5, we can obtain the smooth fuzzy topology $\mathcal{T}_{C_1} : I^X \to I$ as follows:

$$\mathcal{T}_{C_1}(\lambda) = \begin{cases} 1, & \text{if } \lambda = \tilde{0} \text{ or } \tilde{1}, \\ \frac{1}{2}, & \text{if } \lambda = \chi_{\{x,y\}}, \\ 0, & \text{otherwise.} \end{cases}$$

Also, we have $\mathcal{T}_{C_2} = \mathcal{T}_{C_1}$. Therefore $1 : (X, \mathcal{T}_{C_1}) \to (Y, \mathcal{T}_{C_2})$ is a smooth continuous map.

THEOREM 2.11. *Let X be a set and (Y, C_1) be a smooth fuzzy closure space. Let $f : X \to Y$ be a function. For all $\lambda \in I^X$, $r \in I_0$, we define the function $C^f : I^X \times I_0 \to I^X$ by*

$$C^f(\lambda, r) = f^{-1}(C_1(f(\lambda), r)).$$

Then C^f is the coarsest smooth fuzzy closure operator on X which f is a C-map.

Proof. First, we will show that C^f is the smooth fuzzy closure operator on X

(C1) From the definition of C^f, we easily prove $C^f(\tilde{0}) = \tilde{0}$ and $C^f(\tilde{1}) = \tilde{1}$.

(C2) Since $C^f(\lambda, r) = f^{-1}(C_1(f(\lambda), r)) \geq f^{-1}(f(\lambda)) \geq \lambda$ from Lemma 1.1 (2) and Definition 2.3 (C2), we have $\lambda \leq C^f(\lambda, r)$.

(C3) and (C5) are trivial.

(C4) It is proved from the following:

$$\begin{aligned} C^f(\lambda \vee \mu, r) &= f^{-1}(C_1(f(\lambda \vee \mu), r)) \\ &= f^{-1}(C_1(f(\lambda) \vee f(\mu), r)) && (\text{ Lemma 1.1 (7)}) \\ &= f^{-1}(C_1(f(\lambda), r) \vee C_1(f(\mu), r)) && (\text{ (C4)}) \\ &= f^{-1}(C_1(f(\lambda), r)) \vee f^{-1}(C_1(f(\mu), r)) && (\text{ Lemma 1.1 (5)}) \\ &= C^f(\lambda, r) \vee C^f(\mu, r). \end{aligned}$$

Second, from the definition of C^f, since $C^f(\lambda, r) = f^{-1}(C_1(f(\lambda), r))$, we have

$$f(C^f(\lambda, r)) = f(f^{-1}(C_1(f(\lambda), r)) \leq C_1(f(\lambda), r).$$

Hence $f : (X, C^f) \to (Y, C_1)$ is a C-map.

Finally, if $f : (X, C^*) \to (Y, C_1)$ is a C-map, then we have

$$f(C^*(\lambda, r)) \leq C_1(f(\lambda), r).$$

It follows that $f^{-1}(f(C^*(\lambda, r))) \leq f^{-1}(C_1(f(\lambda), r))$. Hence

$$C^*(\lambda, r) \leq C^f(\lambda, r).$$

□

From Theorem 2.11, we can define a subspace of a smooth fuzzy closure space.

Definition 2.12. Let (X, C) be a smooth fuzzy closure space and A a subset of X. The pair (A, C^i) is said to be a subspace of (X, C) if C^i is the coarsest smooth fuzzy closure operator on X for which the inclusion map $i : A \to X$ is a C-map.

Theorem 2.13. [7] *Let X be a set and $(Y, \mathcal{T})$ be a smooth fuzzy topological space. Let $f : X \to Y$ be a function and $\Gamma_\mu = \{\nu \in I^Y \mid f^{-1}(\nu) = \mu\}$. Define by $\mathcal{T}^f : I^X \to I$ as follows:*

$$\mathcal{T}^f(\mu) = \begin{cases} \sup\{\mathcal{T}(\nu) \mid \nu \in \Gamma_\mu\}, & \text{if } \Gamma_\mu \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\mathcal{T}^f$ is the coarsest smooth fuzzy topology on X for which f is a smooth continuous map.

Definition 2.14. [7] Let $(X, \mathcal{T})$ be a smooth fuzzy topological space and A a subset of X. The pair $(A, \mathcal{T}^i)$ is said to be a subspace of $(X, \mathcal{T})$ if $\mathcal{T}^i$ is the coarsest smooth fuzzy topology on X for which the inclusion map $i : A \to X$ is a smooth continuous map.

Lemma 2.15. *Let $(X, \mathcal{T})$ be a smooth fuzzy topological space. For each $\lambda \in I^X$, $r, s \in I_0$, we have*

$$\bigvee_{s<r} C_{\mathcal{T}}(\lambda, s) = C_{\mathcal{T}}(\lambda, r).$$

Proof. Let $\mathcal{F}_r = \{\mu \in I^X \mid \mathcal{F}(\mu) \geq r\}$ and $\mathcal{F}_{r-} = \bigcap_{s<r}\{\mu \in I^X \mid \mathcal{F}(\mu) \geq s\}$. Since $\bigvee_{s<r} C_{\mathcal{T}}(\lambda, s) = \bigwedge\{\mu \in I^X \mid \mu \geq \lambda,\ \mu \in \mathcal{F}_{r-}\}$ from the definition of $C_{\mathcal{T}}$, we only show that

$$\mathcal{F}_r = \mathcal{F}_{r-}.$$

Since $\mathcal{F}_s \supseteq \mathcal{F}_r$ for $s < r$,

$$\bigcap_{s<r}\{\mu \in I^X \mid \mathcal{F}(\mu) \geq s\} \supseteq \{\mu \in I^X \mid \mathcal{F}(\mu) \geq r\}.$$

Conversely, if $\rho \notin \{\mu \in I^X \mid \mathcal{F}(\mu) \geq r\}$, that is, $\mathcal{F}(\rho) < r$, then there exists $s_0 \in I_0$ such that

$$\mathcal{F}(\rho) < s_0 < r.$$

Hence $\rho \notin \bigcap_{s_0<r}\{\mu \in I^X \mid \mathcal{F}(\mu) \geq s_0\}$. It follows that

$$\bigcap_{s<r}\{\mu \in I^X \mid \mathcal{F}(\mu) \geq s\} \subseteq \{\mu \in I^X \mid \mathcal{F}(\mu) \geq r\}.$$

□

THEOREM 2.16. *Let X be a set and $(Y, \mathcal{T})$ be a smooth fuzzy topological space. Let $f : X \to Y$ be a function. For each $\lambda \in I^X$, $r \in I_0$, we define the function $C^f : I^X \times I_0 \to I^X$ by*

$$C^f(\lambda, r) = f^{-1}(C_{\mathcal{T}}(f(\lambda), r)).$$

Then we have the following statements:

(1) $\mathcal{T}_{C^f} = \mathcal{T}^f$.

(2) For each $\lambda \in I^X$, $r \in I_0$, $C_{\mathcal{T}^f}(\lambda, r) = f^{-1}(C_{\mathcal{T}}(f(\lambda), r))$.

(3) (a) (X, C^f) is a topological smooth fuzzy closure space.

(b) If $r = \bigvee\{s \in I_0 \mid C^f(\lambda, s) = \lambda\}$, then $C^f(\lambda, r) = \lambda$.

Proof. (1) Suppose there exists $\lambda \in I^X$ such that

$$\mathcal{T}_{C^f}(\lambda) > \mathcal{T}^f(\lambda).$$

By the definition of $\mathcal{T}_{C^f}$, there exists $r_0 \in I_0$ such that $C^f(\tilde{1} - \lambda, r_0) = \tilde{1} - \lambda$ and

$$\mathcal{T}_{C^f}(\lambda) \geq r_0 > \mathcal{T}^f(\lambda).$$

From the definition of C^f, it implies $f^{-1}(C_{\mathcal{T}}(f(\tilde{1} - \lambda), r_0)) = \tilde{1} - \lambda$. Hence

$$\lambda = \tilde{1} - f^{-1}(C_{\mathcal{T}}(f(\tilde{1} - \lambda), r_0)) = f^{-1}(\tilde{1} - C_{\mathcal{T}}(f(\tilde{1} - \lambda), r_0)).$$

From the definition of $\mathcal{T}^f$, we have

$$\mathcal{T}^f(\lambda) \geq \mathcal{T}(\tilde{1} - C_{\mathcal{T}}(f(\tilde{1} - \lambda), r_0)) \geq r_0,$$

because $C_{\mathcal{T}}$ is a topological smooth fuzzy closure space from Theorem 2.4(6) and $\mathcal{T}_{C_{\mathcal{T}}} = \mathcal{T}$ from Theorem 2.6. It is a contradiction. Therefore we have

$$\mathcal{T}_{C^f} \leq \mathcal{T}^f.$$

Suppose there exists $\mu \in I^X$ such that

$$\mathcal{T}_{C^f}(\mu) < \mathcal{T}^f(\mu).$$

From the definition of $\mathcal{T}^f$, there exists $\nu \in I^Y$ such that $f^{-1}(\nu) = \mu$ and

$$\mathcal{T}_{C^f}(\mu) < \mathcal{T}(\nu) \leq \mathcal{T}^f(\mu).$$

From the definition of $\mathcal{T}$, there exists $r_1 \in I_0$ such that $C_{\mathcal{T}}(\tilde{1}-\nu, r_1) = \tilde{1}-\nu$ and

$$\mathcal{T}_{C^f}(\mu) < r_1 \leq \mathcal{T}(\nu).$$

Hence we have the followings:

$$\begin{aligned} C^f(\tilde{1}-\mu, r_1) &= f^{-1}(C_{\mathcal{T}}(f(\tilde{1}-\mu), r_1)) \\ &= f^{-1}(C_{\mathcal{T}}(f(\tilde{1}-f^{-1}(\nu)), r_1)) \quad (\text{ since } f^{-1}(\nu)=\mu\ ) \\ &= f^{-1}(C_{\mathcal{T}}(f(f^{-1}(\tilde{1}-\nu)), r_1)) \quad (\text{ by Lemma 1.1 }) \\ &\leq f^{-1}(C_{\mathcal{T}}(\tilde{1}-\nu, r_1)) \quad (\text{ by Lemma 1.1 }) \\ &= f^{-1}(\tilde{1}-\nu) \\ &= \tilde{1}-f^{-1}(\nu) = \tilde{1}-\mu. \end{aligned}$$

It follows that $C^f(\tilde{1}-\mu, r_1) = \tilde{1}-\mu$ from Definition 2.2 (C2). Therefore $\mathcal{T}_{C^f}(\mu) \geq r_1$. It is a contradiction.

(2) From Theorem 3.4, we have

$$C_{\mathcal{T}^f}(\lambda, r) = \bigwedge\{\mu \mid \mu \geq \lambda,\ \ \mathcal{T}^f(\mu) \geq r\}$$

and using Lemma 1.1(6),

$$\begin{aligned} f^{-1}(C_{\mathcal{T}}(f(\lambda), r)) &= f^{-1}(\bigwedge\{\rho \mid \rho \geq f(\lambda),\ \ \mathcal{T}(\rho) \geq r\}) \\ &= \bigwedge\{f^{-1}(\rho) \mid \rho \geq f(\lambda),\ \ \mathcal{T}(\rho) \geq r\}. \end{aligned}$$

Since $f^{-1}(\rho) \geq f^{-1}(f(\lambda)) \geq \lambda$ and $\mathcal{T}^f(f^{-1}(\rho)) \geq \mathcal{T}(\rho) \geq r$ from definition of $\mathcal{T}^f$, we have

$$C_{\mathcal{T}^f}(\lambda, r) \leq f^{-1}(C_{\mathcal{T}}(f(\lambda), r)).$$

Suppose there exist $\lambda \in I^X$, $r \in I_0$ such that

$$C_{\mathcal{T}^f}(\lambda, r) \not\geq f^{-1}(C_{\mathcal{T}}(f(\lambda), r)).$$

Then there exists $x_0 \in X$ such that

$$C_{\mathcal{T}^f}(\lambda, r)(x_0) < f^{-1}(C_{\mathcal{T}}(f(\lambda), r))(x_0).$$

From the definition of $C_{\mathcal{T}^f}(\lambda, r)$, there exists $\mu \in I^X$ with $\mu \geq \lambda$ and $\mathcal{T}^f(\tilde{1} - \mu) \geq r$ such that

$$C_{\mathcal{T}^f}(\lambda, r)(x_0) \leq \mu(x_0) < f^{-1}(C_{\mathcal{T}}(f(\lambda), r)(x_0).$$

On the other hand, since $\mathcal{T}^f(\tilde{1} - \mu) \geq r$, for every $s < r$, there exists $\nu_s \in I^Y$ such that

$$\mathcal{T}(\nu_s) \geq s \text{ and } f^{-1}(\nu_s) = \tilde{1} - \mu.$$

Since $f^{-1}(\nu_s) = \tilde{1} - \mu \leq \tilde{1} - \lambda$, we have

$$\lambda \leq \tilde{1} - f^{-1}(\nu_s).$$

It implies

$$f(\lambda) \leq f(\tilde{1} - f^{-1}(\nu_s)) = f(f^{-1}(\tilde{1} - \nu_s)) \leq \tilde{1} - \nu_s.$$

Since $\mathcal{T}(\nu_s) \geq s$, we have $C_{\mathcal{T}}(f(\lambda), s) \leq \tilde{1} - \nu_s$. Hence we obtain

$$f^{-1}(C_{\mathcal{T}}(f(\lambda), s)) \leq f^{-1}(\tilde{1} - \nu_s) = \mu.$$

It follows

$$\begin{aligned} f^{-1}(C_{\mathcal{T}}(f(\lambda), r)) &= f^{-1}(\bigvee_{s<r} C_{\mathcal{T}}(f(\lambda), s)) \text{ (by Lemma 2.15)} \\ &= \bigvee_{s<r} f^{-1}(C_{\mathcal{T}}(f(\lambda), s)) \text{ (by Lemma 1.1 (5))} \\ &\leq \mu. \end{aligned}$$

It is a contradiction.

(3) Since $C_{\mathcal{T}_{C^f}} = C_{\mathcal{T}^f} = C^f$ from (1) and (2), it is easily proved from Theorem 2.5. □

From Theorem 2.16, we can easily obtain the following corollary.

Corollary 2.17. *Let $(X, \mathcal{T})$ be a smooth fuzzy topological space and A a subset of X. Let $i : A \to X$ be a inclusion map. For all λ, $r \in I_0$, define the function $C^i : I^A \times I_0 \to I^A$ by*

$$C^i(\lambda, r) = i^{-1}(C_{\mathcal{T}}(i(\lambda), r)).$$

Then:

(1) The space (A, C^i) is a subspace of a smooth fuzzy closure space $(X, C_{\mathcal{T}})$.

(2) The space $(A, \mathcal{T}_{C^i})$ is a subspace of a smooth fuzzy topological space $(X, \mathcal{T})$ where the smooth fuzzy topology $\mathcal{T}_{C^i}$ induced by C^i.

Example 6. Let $A = \{a, b\}, X = \{a, b, c\}$ be sets.

$$\mu(a) = 0.2, \mu(b) = 0.3, \mu(c) = 0.5$$

$$\text{and } \rho(a) = 0.2, \rho(b) = 0.3, \rho(c) = 0.4.$$

Define $\mathcal{T} : I^X \to I$ as follows:

$$\mathcal{T}(\nu) = \begin{cases} 1, & \text{if } \nu = \tilde{0} \text{ or } \tilde{1}, \\ \frac{2}{3}, & \text{if } \nu = \mu, \\ \frac{1}{2}, & \text{if } \nu = \rho, \\ 0, & \text{otherwise.} \end{cases}$$

Using Theorem 2.4, we can obtain

$$C_{\mathcal{T}}(\nu, r) = \begin{cases} \tilde{0}, & \text{if } \nu = \tilde{0} \text{ or } \tilde{1},\ r \in I_0 \\ \tilde{1} - \mu, & \text{if } \nu \le \tilde{1} - \mu,\ 0 < r \le \frac{2}{3} \\ \tilde{1} - \rho, & \text{if } \tilde{1} - \mu < \nu \le \tilde{1} - \rho,\ 0 < r \le \frac{1}{2} \\ \tilde{1}, & \text{otherwise.} \end{cases}$$

Since $C^i(\lambda, r) = i^{-1}(C_{\mathcal{T}}(i(\lambda), r))$ for $\lambda \in I^A$, $r \in I_0$ from Theorem 2.15 , we have

$$C^i(\lambda, r) = \begin{cases} \tilde{0}, & \text{if } \lambda = \tilde{0} \text{ or } \tilde{1}, r \in I_0 \\ \tilde{1} - i^{-1}(\mu), & \text{if } \lambda \le \tilde{1} - i^{-1}(\mu),\ 0 < r \le \frac{2}{3} \\ \tilde{1}, & \text{otherwise.} \end{cases}$$

From Theorem 2.5, we can obtain the function $\mathcal{T}_{C^i} : I^A \to I$ on A by

$$\mathcal{T}_{C^i}(\lambda) = \begin{cases} 1, & \text{if } \lambda = \tilde{0} \text{ or } \tilde{1}, \\ \frac{2}{3}, & \text{if } \lambda = i^{-1}(\mu), \\ 0 & \text{otherwise.} \end{cases}$$

Let $\lambda_1 = i^{-1}(\mu) = i^{-1}(\rho)$ be given. By the definition of $\mathcal{T}^i$ from Theorem 2.13 , we have

$$\mathcal{T}^i(\lambda) = \begin{cases} 1, & \text{if } \lambda = \tilde{0} \text{ or } \tilde{1}, \\ \frac{2}{3}, & \text{if } \lambda = \lambda_1, \\ 0, & \text{otherwise.} \end{cases}$$

Hence $\mathcal{T}_{C^i}(\lambda) = \mathcal{T}^i(\lambda)$ for all $\lambda \in I^A$.

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