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INVARIANTS WITH RESPECT TO ALL ADMISSIBLE POLAR TOPOLOGIES

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Abstract. Let X and Y be topological vector spaces. For a sequence $\{T_j\}$ of bounded operators from X into Y the c_0-multiplier convergence of $\sum T_j$ is an invariant on topologies which are stronger (need not strictly) than the topology of pointwise convergence on X but are weaker (need not strictly) than the topology of uniform convergence on bounded subsets of X.

Let X be a topological vector space and λ a family of scalar sequences. A series $\sum x_j$ on X is said to be $\lambda - $*multiplier convergent* or, simply, $\lambda - mc$ if $\sum_{j=1}^{\infty} t_j x_j$ converges for each $\{t_j\} \in \lambda$. $c_0 - mc$, $\{0,1\}^{\mathbb{N}} - mc$, $l^p - mc$ $(p > 0)$ and $l^{\infty} - mc$ are important for functional analysis and vector measure theory, e.g., a sequentially complete locally convex space X contains no copy of $(c_0, \|\cdot\|_{\infty})$ if and only if for series on X the $c_0 - mc$, $\{0,1\}^{\mathbb{N}} - mc$ and $l^{\infty} - mc$ are equivalent ([1], Th. 4). Note that $\{0,1\}^{\mathbb{N}} - mc$ is just the subseries convergence.

Recently, Li Ronglu, Cui Chengri and Min-Hyung Cho [2] gave a nice result as follows.

Theorem. *([2], Theorem 3.1) Let X be a Hausdorff locally convex space with the dual X'. For a series $\sum x_j$ on X, the $c_0 - mc$ and the $l^p - mc$ $(p \geq 1)$ are invariants on all (X, X')-admissible topologies, i.e., letting $\lambda = c_0$ or l^p $(p \geq 1)$, if for every $\{t_j\} \in \lambda$ the series $\sum_{j=1}^{\infty} t_j x_j$ converges weakly, then for every $\{t_j\} \in \lambda$ the series $\sum_{j=1}^{\infty} t_j x_j$ converges in the strongest (X, X')-admissible topology $\beta(X, X')$.*

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In this note we would like to establish a similar result for a duality pair consisting of a barrelled space X and the operator space $L(X,Y)$.

THEOREM 1. *Let X be a barrelled space and $L(X,Y)$ the space of continuous linear operators from X into a locally convex space Y. For a sequence $\{T_j\} \subseteq L(X,Y)$, the following (1) and (2) are equivalent.*

(1) *For every $\{t_j\} \in c_0$ the series $\sum_{j=1}^{\infty} t_j T_j$ converges in $L(X,Y)$ with the topology of pointwise convergence on X, i.e., for every $\{t_j\} \in c_0$ and $x \in X$ the series $\sum_{j=1}^{\infty} t_j T_j(x)$ converges.*

(2) *For every $\{t_j\} \in c_0$ the series $\sum_{j=1}^{\infty} t_j T_j(x)$ converges in $L(X,Y)$ with the topology of uniform convergence on bounded subsets of X, i.e., for every $\{t_j\} \in c_0$ and bounded $B \subseteq X$ the series $\sum_{j=1}^{\infty} t_j T_j(x)$ converges uniformly for $x \in B$.*

Proof. (2)$\Rightarrow$(1) is trivial.

(1)$\Rightarrow$(2). Suppose that $\{t_j\} \in c_0$ and B is a bounded subset of X such that the convergence of $\sum_{j=1}^{\infty} t_j T_j(x)$ is not uniform with respect to $x \in B$, i.e., there exists a neighborhood U of $0 \in Y$ for which the following holds :

$\forall n_0 \in \mathbb{N} \quad \exists n > n_0$ and $x \in B$ such that $\sum_{j=n}^{\infty} t_j T_j(x) \notin U$. Pick a neighborhood V of $0 \in Y$ with $V + V \subseteq U$. There is an $n_1 > 1$ and $x_1 \in B$ such that $\sum_{j=n_1}^{\infty} t_j T_j(x_1) \notin U$ and, hence, $\sum_{j=n_1}^{m_1} t_j T_j(x_1) \notin V$ for some $m_1 > n_1$. Similarly, there is an $n_2 > m_1$ and $x_2 \in B$ such that $\sum_{j=n_2}^{\infty} t_j T_j(x_2) \notin U$ and, hence, $\sum_{j=n_2}^{m_2} t_j T_j(x_2) \notin V$ for some $m_2 > n_2$. In this way, we have an integer sequence $n_1 < m_1 < n_2 < m_2 < n_3 < m_3 < \cdots$ and a sequence $\{x_i\} \subseteq B$ such that

$$(*) \qquad \sum_{j=n_i}^{m_i} t_j T_j(x_i) \notin V, \quad i = 1, 2, 3, \cdots .$$

Since $t_j \neq 0$ for infinitely many j, letting $\alpha_k = \sup_{j \geqslant k} \sqrt{|t_j|}$ for each $k \in \mathbb{N}$, $\alpha_k \neq 0$ ($\forall k \in \mathbb{N}$) and $\alpha_k \to 0$. Now consider the matrix

$$[\alpha_{n_i} \sum_{j=n_k}^{m_k} (\frac{t_j}{a_{n_k}}) T_j(x_i)]_{i,k}.$$

Observing $T_j(B)$ is bounded for each j and $\alpha_{n_i} \to 0$,

$$\lim_i \alpha_{n_i} \sum_{j=n_k}^{m_k} (\frac{t_j}{\alpha_{n_k}}) T_j(x_i) = \sum_{j=n_k}^{m_k} (\frac{t_j}{\alpha_{n_k}}) \lim_i \alpha_{n_i} T_j(x_i) = 0$$

for each k. Let $\{k_p\}_{p=1}^{\infty}$ be a strictly increasing sequence in $\mathbb{N}$. For each j, let

$$\gamma_j = \begin{cases} 0, & \text{if} \quad j < n_{k_1} \quad \text{or} \quad m_{k_p} < j < n_{k_{p+1}} \quad \text{for some} \quad p \in \mathbb{N}; \\ \dfrac{t_j}{\alpha_{n_{k_p}}}, & \text{if} \quad n_{k_p} \le j \le m_{k_p} \quad \text{for some} \quad p \in \mathbb{N}. \end{cases}$$

Then $|\gamma_j| = 0$ or $|\gamma_j| = \frac{|t_j|}{\alpha_{n_{k_p}}} = \frac{\sqrt{|t_j|}\sqrt{|t_j|}}{\sup_{i \ge n_{k_p}} \sqrt{|t_i|}} \le \sqrt{|t_j|}$ whenever $n_{k_p} \le j \le m_{k_p}$ and, hence, $\gamma_j \to 0$. By the hypothesis, for each i the series

$$\sum_{p=1}^{\infty} [\alpha_{n_i} \sum_{j=n_{k_p}}^{m_{k_p}} (\frac{t_j}{\alpha_{n_{k_p}}}) T_j(x_i)] = \alpha_{n_i} \sum_{j=1}^{\infty} \gamma_j T_j(x_i)$$

converges and, by the Banach-Steinhaus theorem ([3], p.137),

$$\lim_n \sum_{j=1}^{n} \gamma_j T_j(x) = \sum_{j=1}^{\infty} \gamma_j T_j(x) \qquad (\forall x \in X)$$

shows that $\sum_{j=1}^{\infty} \gamma_j T_j(\cdot) : X \to Y$ is continuous and hence,

$$\{\sum_{j=1}^{\infty} \gamma_j T_j(x) : x \in B\}$$

is bounded. Therefore,

$$\lim_i \sum_{p=1}^{\infty} [\alpha_{n_i} \sum_{j=n_{k_p}}^{m_{k_p}} (\frac{t_j}{\alpha_{n_{k_p}}}) T_j(x_i)] = \lim_i \alpha_{n_i} \sum_{j=1}^{\infty} \gamma_j T_j(x) = 0$$

because $\{x_i\} \subseteq B$ and $\alpha_{n_i} \to 0$. Thus, by the Antosik-Mikusinski matrix theorem ([4],[5]),

$$\lim_i \sum_{j=n_i}^{m_i} t_j T_j(x_i) = \lim_i \alpha_{n_i} \sum_{j=n_i}^{m_i} (\frac{t_j}{\alpha_{n_i}}) T_j(x_i) = 0$$

and hence, $\sum_{j=n_i}^{m_i} t_j T_j(x_i) \in V$ eventually. This contradicts $(*)$. □

COROLLARY 2. *Let X be a Banach space and Y a normed space. If $\{T_j\} \subseteq L(X,Y)$ and for every $\{t_j\} \in c_0$ the series $\sum_{j=1}^{\infty} t_jT_j(x)$ converges at each $x \in X$, then for every $\{t_j\} \in c_0$ the series $\sum_{j=1}^{\infty} t_jT_j$ converges in the operator norm, i.e., $\sum_{j=1}^{\infty} t_jT_j(\cdot) \in L(X,Y)$ and*

$$\lim_n \|\sum_{j=n}^{\infty} t_jT_j(\cdot)\| = \lim_n \sup_{\|x\|\leq 1} \|\sum_{j=n}^{\infty} t_jT_j(x)\| = 0.$$

In fact, $B = \{x \in X : \|x\| \leq 1\}$ is bounded and, by Theorem 1, for every $\epsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that if $n > n_0$, then

$$\|\sum_{j=n}^{\infty} t_jT_j(x)\| < \epsilon, \quad \forall x \in B,$$

i.e.,

$$\sup_{x\in B} \|\sum_{j=n}^{\infty} t_jT_j(x)\| \leq \epsilon.$$

It is easy to see that the same argument as in the proof of Theorem 1 yields a generalization of Theorem 1 as follows.

THEOREM 3. *Let X, Y be topological vector spaces. If $\{T_j\}$ is a sequence of bounded operators from X into Y (i.e., each T_j sends bounded sets to bounded sets) such that for every $\{t_j\} \in c_0$ and $x \in X$ the series $\sum_{j=1}^{\infty} t_jT_j(x)$ converges and $\sum_{j=1}^{\infty} t_jT_j(\cdot)$ is bounded, then for every $\{t_j\} \in c_0$ and bounded $B \subseteq X$, the series $\sum_{j=1}^{\infty} t_jT_j(x)$ converges uniformly for $x \in B$.*

A topological vector space X is said to be a κ-space if $x_j \to 0$ in X, then there is an increasing $\{j_k\} \subseteq \mathbb{N}$ such that the series $\sum_{k=1}^{\infty} x_{j_k}$ converges in X. κ-spaces make a large family containing complete metric linear spaces, some non-complete metric linear spaces and some locally convex spaces. Especially, κ-spaces have been shown to enjoy many nice properties ([4],[5],[6],[7]). Letting

$$X^b = \{f \in \mathbb{C}^X : f \text{ is linear and } f(B) \text{ is bounded for every bounded } B \subseteq X\},$$

if X is a locally convex κ-space, then (X, X^b) is a Banach-Mackey pair ([8], Theorem 2). Using this result, we have the following

THEOREM 4. *Let X be a locally convex κ-space and Y an arbitrary locally convex space. If $\{T_j\}$ is a sequence of bounded linear operators from X into Y such that $\lim_j T_j(x) = T(x)$ exists at each $x \in X$, then the limit operator $T : X \to Y$ is also bounded.*

Proof. By Theorem 2 of [8], (X, X^b) is a Banach-Mackey pair, i.e., $(X, \sigma(X, X^b))$ is a Banach-Mackey space. Thus, by Theorem 8 of [9], $(X^b, \sigma(X^b, X))$ is sequentially complete.

Now let B be a bounded subset of X. For every continuous linear functional y' on Y, $y' \circ T_j \in X^b$ for each j and

$$\lim_j (y' \circ T_j)(x) = \lim_j y'(T_j x) = y'(Tx) = (y' \circ T)(x)$$

at each $x \in X$, $y' \circ T \in X^b$ because $(X^b, \sigma(X^b, X))$ is sequentially complete. Therefore, $(y' \circ T)(B) = \{y'(Tx) : x \in B\}$ is bounded and, by the Mackey theorem, $T(B) = \{Tx : x \in B\}$ is bounded, i.e., $T : X \to Y$ is a bounded linear operator. □

As an immediate consequence of Theorem 3 and 4, we have the following

COROLLARY 5. *Let X be a locally convex κ-space and Y an arbitrary locally convex space. Then for a sequence $\{T_j\}$ of bounded linear operators from X into Y, the following conditions (a) and (b) are equivalent.*

(a) For every $\{t_j\} \in c_0$ and $x \in X$, $\sum_{j=1}^{\infty} t_j T_j(x)$ converges.

(b) For every $\{t_j\} \in c_0$ and bounded $B \subseteq X$, $\sum_{j=1}^{\infty} t_j T_j(x)$ converges uniformly with respect to $x \in B$.

A topological vector space X is said to be an $\mathcal{A}$-space if for every bounded $\{x_j\} \subseteq X$ and $t_j \to 0$ in $\mathbb{C}$ there exists an increasing $\{j_k\} \subseteq \mathbb{N}$ such that $\sum_{k=1}^{\infty} t_{j_k} x_{j_k}$ converges. κ-spaces are $\mathcal{A}$-spaces but the converse is not true, e.g., (l^p, weak) for $1 < p < +\infty$ and $(l^1, \sigma(l^1, c_0))$ are $\mathcal{A}$-spaces but are not κ-spaces. Sequentially complete locally convex spaces are $\mathcal{A}$-spaces. $\mathcal{A}$-spaces have an important property : If X is an $\mathcal{A}$-space and Y is an arbitrary topological vector space and $\{T_\alpha : \alpha \in I\}$ is a family of sequentially continuous linear operators from X into Y such that $\{T_\alpha x : \alpha \in I\}$ is bounded at each $x \in X$, then $\{T_\alpha : \alpha \in I\}$ is uniformly bounded on each bounded $B \subseteq X$, i.e., $\{T_\alpha x : \alpha \in I, x \in B\}$ is bounded ([5], Corollary 4).

This result and Theorem 3 imply the following

COROLLARY 6. *Let X be an $\mathcal{A}$-space and Y an arbitrary topological vector space. Then for a sequence $\{T_j\}$ of sequentially continuous linear operators from X into Y, the conditions (a) and (b) are equivalent.*

Proof. Let $\{t_j\} \in c_0$. If (a) holds, then $\{\sum_{j=1}^n t_j T_j : n \in \mathbb{N}\}$ is pointwise bounded on X and, hence, for every bounded $B \subseteq X$, $\{\sum_{j=1}^n t_j T_j x : n \in \mathbb{N}, x \in B\}$ is bounded because X is an $\mathcal{A}$-space. Therefore, for every bounded $B \subseteq X$, the condition (a) shows that $\{\sum_{j=1}^\infty t_j T_j x : x \in B\}$ is bounded because the closure

$$\overline{\{\sum_{j=1}^n t_j T_j x : n \in \mathbb{N}, x \in B\}}$$

is bounded, i.e., $\sum_{j=1}^\infty t_j T_j(\cdot)$ is a bounded operator. Thus, (b) follows from Theorem 3. □

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