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ON BIPARTITE TOURNAMENT MATRICES

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Abstract. We find bounds of eigenvalues of bipartite tournament matrices. We see when bipartite matrices exist and how players and teams of the matrices are evenly ranked. Also, we show that a bipartite tournament matrix can be both regular and normal when and only when it has the same team size.

1. Introduction

Let $p_1, \dots, p_d$ be positive integers. A digraph obtained by orienting each edge of the complete d-partite graph $K_{p_1,\dots,p_d}$ is called a d-partite tournament and the associated adjacency matrix is called a d-partite tournament matrix. This can be interpreted as the result of a round-robin competition among d teams in which each player competes every other player belonging to different teams from his own [6].

A 2-partite tournament is called a bipartite tournament. We assume that the first team and the second team consist of the players in the set $\{1, \dots, p\}$ and $\{p+1, \dots, p+q\}$, respectively, where $p + q = n$. So a bipartite tournament matrix M of degree n can be written $M = \begin{bmatrix} O_p & A \\ B & O_q \end{bmatrix}$ for some $p \times q$ $(0,1)$-matrix A, where O_p is the zero matrix of degree p and $B = J_{q,p} - A^t$ where $J_{q,p}$ is the $q \times p$ all one's matrix, and then M satisfies the equation

$$M + M^t = J_n - \mathrm{Join}(J_p, I_2, J_q) = J_n - \begin{bmatrix} J_p & O_{p,q} \\ O_{q,p} & J_q \end{bmatrix} = \begin{bmatrix} O_p & J_{p,q} \\ J_{q,p} & O_q \end{bmatrix}.$$

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The bounds of eigenvalues λ of tournament matrices were found by Brauer and Gentry [1,2]. There have been many developments on spectral properties of tournament matrices [3,5,6,7,8,10] since then. Further, to study the ranking scheme of the players has been an interesting problem [5,6,9,10]. We here find the bounds of eigenvalues of bipartite tournament matrices. We study the existence of bipartite matrices and the cases of players and teams possibly-evenly ranked. Also, we show that a bipartite tournament matrix can be both regular and normal when and only when it has the same team size.

2. Bipartite tournament matrices

Let M be a bipartite tournament matrix and let λ be an eigenvalue of M and v be the corresponding eigenvector. Applying the Schwarz inequality and assuming $p \le q$, we obtain the following:

$$
\begin{aligned}
(2\,\mathrm{Re}\lambda)\, v^*v &= v^*(M+M^t)v \\
&= v^*1\,1^tv - [\overline{v}_1, \ldots, \overline{v}_n] \begin{bmatrix} J_p & O_{p,q} \\ O_{q,p} & J_q \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \\
&= |v^*1|^2 - \left[\sum_{i=1}^{p} \overline{v}_i, \ldots, \sum_{i=1}^{p} \overline{v}_i, \sum_{j=p+1}^{n} \overline{v}_j, \ldots, \sum_{j=p+1}^{n} \overline{v}_j \right] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \\
&= |v^*1|^2 - \sum_{i=1}^{p} \overline{v}_i \cdot \sum_{i=1}^{p} v_i - \sum_{j=p+1}^{n} \overline{v}_j \cdot \sum_{j=p+1}^{n} v_j \\
&\ge |v^*1|^2 - p\,(|v_1|^2 + \cdots + |v_p|^2) - q\,(|v_{p+1}|^2 + \cdots + |v_n|^2) \\
&\ge |v^*1|^2 - q\,v^*v,
\end{aligned}
$$

where 1 is the column vector of n coordinates of 1's so that $1\,1^t = J_n$, $O_{p,q}$ is the $p \times q$ zero matrix and $J_n = J_{n,n}$. So we have $(2\,\mathrm{Re}\,\lambda + q)v^*v \ge |v^*1|^2$, that is, $\mathrm{Re}\lambda \ge -q/2$. Therefore we have the following theorem.

THEOREM 1. *Let M be a bipartite tournament matrix with team size p and q. Every eigenvalue λ of M satisfies* $\mathrm{Re}\,\lambda \ge -q/2$. *Fur-*

thermore, $\operatorname{Re}\lambda = -q/2$ *if and only if the corresponding eigenvector* $v = k(1, \ldots, 1, -1, \ldots, -1)$ *consisting* p 1*'s and* p (-1)*'s for* $k \in \mathbb{C}$.

Proof. If the equality holds, then $v^*1 = 0$, $p = q$, $v_1 = \cdots = v_p$ and $v_{p+1} = \cdots = v_n$ are all satisfied. So $v^{(1)t} = (v_1, \ldots, v_p) = k1_p^t$, and $v^{(2)t} = (v_{p+1}, \ldots, v_n) = \tilde{k}1_q^t$ for some k, $\tilde{k} \in \mathbb{C}$. In fact, $k = -\tilde{k}$ since $v^*1 = 0$. The converse is clear. □

A nonnegative square matrix A of order n is said to be *reducible* if there is a permutation matrix P such that $P^tAP = \begin{bmatrix} A_1 & O \\ * & A_2 \end{bmatrix}$ where A_1, A_2 are non-vacuous square matrices, and called *irreducible* otherwise. It is known that a matrix is irreducible if and only if the associated digraph is strongly connected. It is also known by the Perron-Frobenius theorem [4] that a nonnegative irreducible matrix has its Perron value and corresponding Perron vector whose coordinates are all positive. Note that the Perron value is the spectral radius, that is the maximum modulus of eigenvalues. It suffices to consider irreducible bipartite tournament matrices.

Let M be an irreducible bipartite tournament matrix. Then it has the Perron value ρ with the corresponding Perron vector v whose coordinates are positive real numbers. Let $w_1 = \sum_{i=1}^{p} v_i$, $w_2 = \sum_{j=p+1}^{n} v_j$ and $w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$. We define the variance of a vector $v = (v_1, \ldots, v_n)^t$ by

$$\operatorname{var} v = \sum_{1 \le i < j \le n} (v_i - v_j)^2.$$

As in the above proof, we denote $v^{(1)t} = (v_1, \ldots, v_p)$ and $v^{(2)t} =$

$(v_{p+1}, \ldots, v_n)$. Since

$$\begin{aligned}
v^*(M+M^t)v &= [\overline{v}_1, \ldots, \overline{v}_n] \begin{bmatrix} O_p & J_{p,q} \\ J_{q,p} & O_q \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \\
&= \left[\sum_{j=p+1}^{n} \overline{v}_j, \ldots, \sum_{j=p+1}^{n} \overline{v}_j, \sum_{i=1}^{p} \overline{v}_i, \ldots, \sum_{i=1}^{p} \overline{v}_i \right] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \\
&= \sum_{j=p+1}^{n} \overline{v}_j \cdot \sum_{i=1}^{p} v_i + \sum_{i=1}^{p} \overline{v}_i \cdot \sum_{j=p+1}^{n} v_j \\
&= \overline{w}_2 w_1 + \overline{w}_1 w_2 \\
&= w^* w - |w_1 - w_2|^2
\end{aligned}$$

and

$$\begin{aligned}
w^* w &= (v_1 + \cdots + v_p)^2 + (v_{p+1} + \cdots + v_n)^2 \\
&= p\,(v_1^2 + \cdots + v_p^2) - \sum_{1 \le i < j \le p} (v_i - v_j)^2 \\
&\qquad + q\,(v_{p+1}^2 + \cdots + v_n^2) - \sum_{p+1 \le i < j \le n} (v_i - v_j)^2 \\
&= p\, v^{(1)*} v^{(1)} + q\, v^{(2)*} v^{(2)} - \operatorname{var} v^{(1)} - \operatorname{var} v^{(2)},
\end{aligned}$$

we obtain

$$\begin{aligned}
2\rho\, v^* v &= v^*(M + M^t) v \\
&= w^* w - \operatorname{var} w \\
&= p\, v^{(1)*} v^{(1)} + q\, v^{(2)*} v^{(2)} - \operatorname{var} v^{(1)} - \operatorname{var} v^{(2)} - \operatorname{var} w,
\end{aligned}$$

or

$$\operatorname{var} v^{(1)} + \operatorname{var} v^{(2)} + \operatorname{var} w = 2(p/2 - \rho) v^{(1)*} v^{(1)} + 2(q/2 - \rho) v^{(2)*} v^{(2)}.$$

The left hand side is 0 if and only if $\rho = p/2 = q/2$ $(\rho \le p/2 \le q/2)$. The closer ρ is to $p/2$, the more evenly ranked the teams and the players in a team would be.

THEOREM 2. *For any eigenvalue λ of M, it holds that* $-q/2 \leq \operatorname{Re}\lambda \leq p/2$.

We say that a bipartite tournament matrix M with $p = q$, i.e., of same team size, is regular if the row sums of M are constant. This implies that the column sums of M are also constant and equal to a row sum. In the associated digraph each vertex has constant indegree and outdegree, which is interpreted that every player in a regular tournament wins and loses the same number of times. In this paper, we simply adopt the name a *regular bipartite tournament matrix* for the matrix M whose row sums are constant, or which satisfies $M1 = t1$ for some positive integer t.

A bipartite tournament matrix $M = \begin{bmatrix} O_p & A \\ J_{q,p} - A^t & O_q \end{bmatrix}$ is regular if and only if the row sums of A are constant. If $M1 = t1$, then from $M + M^t = \begin{bmatrix} O_p & J_{p,q} \\ J_{q,p} & O_q \end{bmatrix}$ we find that the number of 1's in M is pq and so $t = pq/n$.

A row sum of $A = t$ implies that a column sum of $B=$ a row sum of $B^t = q - t$, where $B = J_{p,q} - A^t$. Similarly a row sum of $B = t$ implies that a column sum of $A = p - t$. So we have $1^t M = (q-t, \ldots, q-t, p-t, \ldots, p-t)$. Notice that 1 is not a left eigenvector of M if $p \neq q$.

EXAMPLES. We have seen that if a bipartite tournament matrix M satisfies $M1 = t1$ then $t = pq/n$. We show here that if n is divisible by a square then there are p's and q's such that $p + q = n$ and n is a divisor of pq so that we have regular bipartite tournament matrices. We describes the cases and present some examples.

EXAMPLE 1. Suppose that k^2 be a divisor of n. Let $p = n/k$, $q = (k-1)n/k$. Then $p + q = n$ and $pq = (k-1)n^2/k^2$. So $t = pq/n = (k-1)n/k^2$. Therefore, there is a regular bipartite tournament matrix with all the row sums t. For example, take $n = 9$, $p = 3$, $q = 6$ so that $pq = 18$ and $t = pq/n = 2$. Then, there is a regular bipartite tournament matrix

$$M = \begin{bmatrix} O_3 & A \\ B & O_6 \end{bmatrix}, \text{where } A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

We can easily see that $A + B^t = J_{3,6}$, $M1 = 2 \cdot 1$, and $1^t M = (4,4,4,1,1,1,1,1,1)^t$. Every player of the two teams of 3 and 6 players each wins twice.

When there are 18 players, divide them into two teams, each of whose size is 6 and 12. Then we can have a regular tournament where every player defeat others 4 times exactly.

EXAMPLE 2. When n is a square, we can take $p = \sqrt{n}$ and $q = n - \sqrt{n}$. Then $p + q = n$, $pq = n\sqrt{n} - n$ and $t = pq/n = \sqrt{n} - 1$. For example, when $n = 16$ we may take $p = 4$ and $q = 12$ to obtain a regular bipartite tournament matrix with constant row sum 3.

THEOREM 3. *There are positive integers p and q such that $p+q = n$ and n divides pq if and only if n is divisible by a square.*

Proof. It is enough to show the necessity of the condition. If n is square free, it is factored as $n = p_1 p_2 \ldots p_k$ for distinct primes p_i $(i = 1, \ldots, k)$. Since $n|pq$ and $p + q = n$, we have $n|(np - p^2)$. So $n|p^2$. This implies that each $p_i|p$ so that $n|p$, a contradiction. □

3. Normal bipartite tournament matrices

A normal matrix N is a matrix satisfying $NN^t = N^t N$ [8]. Now we consider a normal bipartite tournament matrix $M = \begin{bmatrix} O_p & A \\ B & O_q \end{bmatrix}$. Then it satisfies $MM^t = M^t M$, where

$$MM^t = \begin{bmatrix} O_p & A \\ B & O_q \end{bmatrix} \begin{bmatrix} O_p & B^t \\ A^t & O_q \end{bmatrix} = \begin{bmatrix} AA^t & O_{p,q} \\ O_{q,p} & BB^t \end{bmatrix}$$

and

$$M^tM = \begin{bmatrix} O_p & B^t \\ A^t & O_q \end{bmatrix} \begin{bmatrix} O_p & A \\ B & O_q \end{bmatrix} = \begin{bmatrix} B^tB & O_{p,q} \\ O_{q,p} & A^tA \end{bmatrix}.$$

So $MM^t = M^tM$ if and only if $AA^t = B^tB$, equivalently $A^tA = BB^t$. Using $A + B^t = J_{p,q}$ and $B + A^t = J_{q,p}$, we have $AJ_{q,p} - AB = AA^t = B^tB = J_{p,q}B - AB$, that is,

$$\begin{bmatrix} s_1 & s_1 & \cdots & s_1 \\ s_2 & s_2 & \cdots & s_2 \\ \vdots & \vdots & \ddots & \vdots \\ s_p & s_p & \cdots & s_p \end{bmatrix} = AJ_{q,p} = J_{p,q}B = \begin{bmatrix} r_1 & r_2 & \cdots & r_p \\ r_1 & r_2 & \cdots & r_p \\ \vdots & \vdots & \ddots & \vdots \\ r_1 & r_2 & \cdots & r_p \end{bmatrix},$$

where $(s_1, \ldots, s_p)^t$ is the row sum vector of A and $(r_1, \ldots, r_p)^t$ is the column sum vector of B, which is again if and only if $s_1 = \cdots = s_p = r_1 = \cdots = r_p = s$. So we have $(s_1, \ldots, s_p)^t = (r_1, \ldots, r_p)^t = s1_p$. Since

$$M + M^t = \begin{bmatrix} O_p & A \\ B & O_q \end{bmatrix} + \begin{bmatrix} O_p & B^t \\ A^t & O_q \end{bmatrix} = \begin{bmatrix} O_p & J_{p,q} \\ J_{q,p} & O_q \end{bmatrix},$$

ith row sum of A + ith row sum of $B^t = 2s = q$.

Similarly, from the fact that $BB^t = A^tA$, we have $BJ_{p,q} - BA = J_{q,p}A - BA$ and so

$$\begin{bmatrix} l_1 & l_1 & \cdots & l_1 \\ l_2 & l_2 & \cdots & l_2 \\ \vdots & \vdots & \ddots & \vdots \\ l_q & l_q & \cdots & l_q \end{bmatrix} = BJ_{p,q} = J_{q,p}A = \begin{bmatrix} m_1 & m_2 & \cdots & m_q \\ m_1 & m_2 & \cdots & m_q \\ \vdots & \vdots & \ddots & \vdots \\ m_1 & m_2 & \cdots & m_q \end{bmatrix},$$

where $(l_1, \ldots, l_q)^t$ is the row sum vector of B, and $(m_1, \ldots, m_q)^t$ is the column sum vector of A, which is if and only if $l_1 = \cdots = l_q = m_1 = \cdots = m_q = l$. So ith row sum of B + ith row sum of $A^t = 2l = p$.

Hence a bipartite tournament matrix M is normal if and only if A has constant row sum s, constant column sum l, and B has constant row sum l, constant column sum s, where $s = q/2$ and $l = p/2$. In particular, if $p \neq q$ then a normal bipartite tournament matrix M can't be regular. Also a regular bipartite tournament matrix M can't be normal if $p \neq q$.

Summarizing the result as a theorem, we therefore have:

THEOREM 4. *Let M be a bipartite tournament matrix of the form* $\begin{bmatrix} O_p & A \\ J_{q,p} - A^t & O_q \end{bmatrix}$, *for some $p \times q$ $(0,1)$-matrix A. Then the fact that M is regular if and only if M is normal is true if and only if $p = q$.*

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