

SPARSE ORTHOGONAL MATRICES BY WEAVING

GI-SANG CHEON

ABSTRACT. We determine sparse orthogonal matrices of order n which is fully indecomposable by weaving.

1. Introduction

By a *pattern* we simply mean the arrangement of zero and nonzero (denoted $*$) entries in a matrix. An $n \times n$ pattern P is called *orthogonal* if there is a (real) orthogonal matrix U whose pattern is P . By $\#(U)$ or $\#(P)$ we mean the number of nonzero entries in the matrix U or pattern P . An $n \times n$ pattern (or matrix) P is called *fully indecomposable* if it has no $r \times q$ zero submatrix such that $r + q = n$; equivalently, there do not exist permutation matrices Q_1 and Q_2 such that

$$Q_1 P Q_2 = \begin{bmatrix} P_{11} & O \\ P_{21} & P_{22} \end{bmatrix},$$

in which P_{11} and P_{22} are square and nonempty (or, equivalently the bipartite graph of P is connected). If P were an orthogonal pattern and there were such reducing blocks, then an elementary calculation shows that $P_{21} = O$ also.

In 1991, M. Fiedler conjectured that for $n \geq 2$ an $n \times n$ orthogonal matrix which is fully indecomposable has at least $4n - 4$ nonzero entries. In [BBS], this conjecture was shown in the affirmative, and also, the zero patterns of the $n \times n$ orthogonal matrices with exactly $4n - 4$ nonzero entries were determined. B. L. Shader [S] gave a simpler proof of this result, and recently this result was extended in [CS1, CS2].

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First, we describe a few results from [BBS]. Recursively define a family of $(0, 1)$ -matrices of order $n \geq 2$ as follows. Let

$$\mathcal{B}_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

If n is odd, define

$$(1) \quad \mathcal{B}_n = \left[\begin{array}{cccc|c} & & & & 0 \\ & & & & \vdots \\ & & & & 0 \\ & & \mathcal{B}_{n-1} & & 1 \\ & & & & 1 \\ \hline 0 & \dots & 0 & 1 & 1 \end{array} \right].$$

If $n \geq 4$ is even, define

$$(2) \quad \mathcal{B}_n = \left[\begin{array}{ccccc|c} & & & & & 0 \\ & & & & & \vdots \\ & & & & & 0 \\ & & \mathcal{B}_{n-1} & & & 1 \\ & & & & & 1 \\ \hline 0 & \dots & 0 & 1 & 1 & 1 \end{array} \right].$$

For example,

$$\mathcal{B}_5 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad \mathcal{B}_6 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

As noted in [BBS], each of the matrices \mathcal{B}_n ($n \geq 2$) is the zero pattern of an $n \times n$ orthogonal matrix which is fully indecomposable and has exactly $4n - 4$ nonzero entries.

By the *sparse orthogonal matrix* of order n we mean the orthogonal matrix with the same zero pattern as \mathcal{B}_n .

We can ask “what’s the $n \times n$ sparse orthogonal matrix?”. In this paper, we construct such matrices. This construction is also true for complex unitary matrices.

2. Weaving and sparse orthogonal matrices

To construct $n \times n$ fully indecomposable orthogonal matrices (or patterns) with exactly $4n - 4$ nonzero entries, we apply a method called *weaving* described in [C]. It produces a large matrix from a list of row matrices R_i ’s and column matrices C_j ’s with appropriate sizes, via a $(0, 1)$ -matrix.

To every $m \times n$ $(0, 1)$ -matrix $A = [a_{ij}]$ having row sums r_1, r_2, \dots, r_m and column sums c_1, c_2, \dots, c_n , we associate a *woven product* of matrices as follows. We adopt the following notations.

$s = s(i, j) :=$ the number of nonzero positions in row i of A up to column j ,

$t = t(i, j) :=$ the number of nonzero positions in column j of A up to row i .

Now let R_i ($i = 1, 2, \dots, m$) have r_i columns \mathbf{u}_s , and C_j ($j = 1, 2, \dots, n$) have c_j rows \mathbf{v}_t^T using the indices s and t introduced above. We define the *woven product*

$$M(A) = (R_1 \cdots R_m) \otimes (C_1 \cdots C_n) = [M_{ij}]$$

block entrywise by

$$M_{ij} = \begin{cases} \mathbf{u}_s \mathbf{v}_t^T & \text{if } a_{ij} = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The matrix A is called the *lattice* of the weaving, and the matrix $M(A)$ obtained by weaving from the lattice A is called the *woven matrix*. Note that the resulting woven matrix depends on the lattice and matrices R_i ’s and C_j ’s.

LEMMA 2.1. *Let A be an $m \times n$ $(0,1)$ -matrix whose bipartite graph is connected. If the R_i 's and C_j 's corresponding to the lattice A are all square and fully indecomposable, then the woven product $M(A) = (R_1 \cdots R_m) \otimes (C_1 \cdots C_n)$ is square and fully indecomposable as well.*

Proof. It is clear from the definition of woven product that if the R_i 's and C_j 's are square then the woven product $M(A) = (R_1 \cdots R_m) \otimes (C_1 \cdots C_n)$ is also a square matrix. Since the R_i 's and C_j 's are also all fully indecomposable, each of their bipartite graphs is connected. Since A has a connected bipartite graph as well, any vertex in the bipartite graph of $M(A)$ may be reached from any other via a sequence of paths guaranteed either by the connectivity of the bipartite graphs of the R_i 's and C_j 's or of A . Thus, the bipartite graph of $M(A)$ is connected, and hence $M(A)$ is fully indecomposable. \square

There is a canonical way of factorizing woven matrices based on the lattice A . The following lemma is due to R. Craigen [C].

LEMMA 2.2. *Let A be an $m \times n$ $(0,1)$ -matrix having row sums r_1, r_2, \dots, r_m and column sums c_1, c_2, \dots, c_n . If R_i ($i = 1, 2, \dots, m$) is an $r_i \times r_i$ matrix and C_j ($j = 1, 2, \dots, n$) is a $c_j \times c_j$ matrix, then there exists a permutation matrix P_A such that*

$$(R_1 \cdots R_m) \otimes (C_1 \cdots C_n) = [R_1 \oplus \cdots \oplus R_m] P_A [C_1 \oplus \cdots \oplus C_n],$$

where \oplus denotes a direct sum.

Throughout the rest of this paper, we let N denote the number of nonzero entries in a given lattice.

THEOREM 2.3. *Let A be an $m \times n$ $(0,1)$ -matrix having row sums r_1, r_2, \dots, r_m and column sums c_1, c_2, \dots, c_n . If R_i ($i = 1, 2, \dots, m$) is an $r_i \times r_i$ orthogonal matrix and C_j ($j = 1, 2, \dots, n$) is a $c_j \times c_j$ orthogonal matrix, then the woven product $M(A) = (R_1 \cdots R_m) \otimes (C_1 \cdots C_n)$ is an $N \times N$ orthogonal matrix.*

Proof. It is clear from Lemma 2.1 that the woven product $M(A)$ is square. By Lemma 2.2, there exists a permutation matrix P_A such that

$$\begin{aligned} M(A)(M(A))^T &= [R_1 \oplus \cdots \oplus R_m] P_A [C_1 \oplus \cdots \oplus C_n] \cdot \\ &\quad [C_1 \oplus \cdots \oplus C_n]^T P_A^T [R_1 \oplus \cdots \oplus R_m]^T. \end{aligned}$$

Since $r_1 + \dots + r_m = c_1 + \dots + c_n = N$, it follows that $M(A)(M(A))^T = I_N$. Thus $M(A)$ is an $N \times N$ orthogonal matrix, which completes the proof. \square

Now, we are ready to construct the sparse orthogonal matrix by weaving. The pattern of a woven product is the woven product of the patterns of the R_i 's and C_j 's. It follows from Lemma 2.1 and Theorem 2.3 that the woven product of fully indecomposable orthogonal patterns through a lattice whose bipartite graph is connected will be a (larger) fully indecomposable orthogonal pattern.

For $m \geq 2$, we define the $m \times m$ lattice A_o and $m \times (m + 1)$ lattice A_e as follows:

$$(3) \quad A_o = \begin{bmatrix} 1 & 1 & & & & \\ & 1 & 1 & & O & \\ & & \ddots & \ddots & & \\ & O & & 1 & 1 & \\ & & & & & 1 \end{bmatrix},$$

and

$$(4) \quad A_e = \begin{bmatrix} 1 & 1 & & & & \\ & 1 & 1 & & O & \\ & & \ddots & \ddots & & \\ & O & & 1 & 1 & \\ & & & & 1 & 1 \end{bmatrix}.$$

Then $\#(A_o) = 2m - 1$ and $\#(A_e) = 2m$. From Theorem 2.3, the woven products obtained by the lattices A_o and A_e produce $(2m - 1) \times (2m - 1)$ woven matrix and $2m \times 2m$ woven matrix, respectively.

Note that row sums r_i 's and column sums c_j ' of A_o and A_e are 1 or 2. To construct fully indecomposable orthogonal patterns, we shall only use the 1×1 orthogonal pattern $[*]$ and the 2×2 orthogonal pattern $\begin{bmatrix} * & * \\ * & * \end{bmatrix}$ for the R_i 's and C_j 's.

It is clear from the definition of woven product that the orthogonal patterns of a woven product of the orthogonal patterns R_i 's and C_j 's

Then $M(A_o)$ is a $(2m - 1) \times (2m - 1)$ orthogonal pattern which is fully indecomposable, and we get

$$\#(M(A_o)) = 4(2m - 1 - 1) = 4(2m - 1) - 4.$$

Similarly, $M(A_e)$ is a $2m \times 2m$ orthogonal pattern which is fully indecomposable, and we get

$$\#(M(A_e)) = 4(2m - 1) = 4(2m) - 4.$$

Note that if n is an odd then the zero pattern of $M(A_o)$ is precisely coincide with (1), and if n is an even then the zero pattern of $M(A_e)$ is precisely coincide with (2).

It is clear that the 1×1 fully indecomposable orthogonal matrix U whose pattern is $[*]$ is $[1]$, and the 2×2 fully indecomposable orthogonal matrix U whose pattern is $\begin{bmatrix} * & * \\ * & * \end{bmatrix}$ has the form

$$(7) \quad \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

where $0 < \theta < 2\pi$, $\theta \neq \frac{\pi}{2}, \pi, \frac{3}{2}\pi$.

Thus we have the following theorem.

THEOREM 2.4. *Let A_o and A_e be the lattices defined in (3) and (4) respectively. Then an $n \times n$ fully indecomposable sparse orthogonal pattern is*

$$\begin{cases} M(A_o) \text{ in (5)} & \text{if } n = 2m - 1 \text{ for } m \geq 2, \\ M(A_e) \text{ in (6)} & \text{if } n = 2m \text{ for } m \geq 2. \end{cases}$$

Note that we can get many sparse orthogonal matrices with the same pattern $M(A_o)$ or $M(A_e)$, since we can arbitrarily choose θ in (7).

For example, let

$$A_o = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then $\#(A_o) = N = 5$, and $r_1 = r_2 = 2 = c_2 = c_3$, $r_3 = 1 = c_1$. Take

$$R_1 = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}, \quad R_2 = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix},$$

$$C_2 = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, \quad C_3 = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix},$$

and

$$R_3 = C_1 = [1].$$

Then

$$M(A_o) = (R_1 \ R_2 \ R_3) \otimes (C_1 \ C_2 \ C_3)$$

$$= \begin{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{bmatrix} [1] & \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} & O \\ O & \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} & \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \\ O & O & [1] \begin{bmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{4} & \frac{\sqrt{3}}{4} & 0 & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{4} & \frac{3}{4} & 0 & 0 \\ 0 & -\frac{\sqrt{6}}{4} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{\sqrt{6}}{4} \\ 0 & \frac{\sqrt{6}}{4} & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{\sqrt{6}}{4} \\ 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}.$$

Thus $M(A_o)$ is a 5×5 orthogonal matrix with exactly $4n - 4 = 16$ nonzero entries which is fully indecomposable.

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Department of Mathematics
Daejin University
Pocheon 487-711, S. Korea

