Kangweon-Kyungki Math. Jour. **7** (1999), No. 1, pp. 79–84

HOLOMORPHIC PRINCIPLE LINE BUNDLES OVER COMPLEX GROUPS

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1. Introduction

Let F be a complex line bundles over a complex manifold M. In [7], we investigated the properties of holomorphic line bundles F of cohomology groups for a complex torus. And also we know that the group of holomorphic line bundles on a q-dimensional complex torus with the first Chern class zero is a family of weakly pseudoconvex manifolds. K. H. Shon and H. R. Cho [6] obtained some properties of a family of weakly pseudoconvex manifolds. H. Kazamà and K. H. Shon [2,3] solved the $\bar{\partial}$-problem on a family of weakly pseudoconvex manifolds. T. Ueda [8] investigated some properties of a family of a compact complex curve with topologically trivial normal bundle. Recently H. Kazama, T. Ohta and K. H. Shon [1] obtained (non)vanishing and imbedding theorem on weakly complex spaces. In this paper, we obtain some properties of topological holomorphic line bundles with respect to a complex torus.

2. Topological principle line bundles

DEFINITION 2.1. Let M be a topological space. M is said to have

Received November 16, 1998.

1991 Mathematics Subject Classification: 32C10, 32F05, 32F15.

Key words and phrases: principle line bundle, weakly pseudoconvex manifold, exact sequence, complex torus, pseudoconvex domain, cohomology group, Chern class, topological trivial.

The Present Studies were Supported by the Matching Fund Programs of Research Institute for Basic Sciences, Pusan National University, Korea, Project No. RIBS-PNU-98-101

the structure of an n-dimensional complex manifold if there exists an atlas $\mathcal{A} = \{(U_i, \phi_i) : i \in I\}$ of charts on M such that

(1) ϕ_i is a homeomorphism of U_i onto the open subset $\phi_i(U_i)$ of $\mathbf{C}^n$ for all $i \in I$.

(2) For all $i, j \in I$, $\phi_i \phi_j^{-1}$ is a biholomorphic map of $\phi_j(U_{ij})$ onto $\phi_i(U_{ij})$, where $U_{ij} = U_i \cap U_j$.

DEFINITION 2.2. Let M be a differentiable manifold. A differentiable manifold F is called a (complex) line bundle over M if it satisfies the following conditions:

(1) A C^∞ map $\pi : F \to M$ of F onto M is given.

(2) For every $p \in M, \pi^{-1}(p)$ is an n-dimensional $\mathbf{C}$ -vector space : $\pi^{-1}(p) \cong \mathbf{C}^n$, where n is independent of p

(3) For every $q \in M$, there exists a neighborhood U, $q \in U \subset M$, such that $\pi^{-1}(U) = U \times \mathbf{C}^n$, and that for any $p \in U$, $p \times \mathbf{C}^n$ is isomorphic to $\pi^{-1}(p)$ as an $\mathbf{C}$-vector space : $\pi^{-1}(p) \cong \{p\} \times \mathbf{C}^n$.

We denote the line bundle by $\pi : F \to M$ or just F. Let $\pi : M \times \mathbf{C}^n \to M$ denote projection on the first factor. Then $\pi : M \times \mathbf{C}^n \to M$ is a line bundle over M called the trivial line bundle over M. Now consider line bundles over a complex manifold. Let F be a line bundle over a complex manifold M. If the transitive functions $f_{jk}(p), j, k = 1, 2, \cdots,$ are all holomorphic, F is called a holomorphic line bundle. Here by saying that $f_{jk}(p) = (f^\alpha_{jk\beta}(p))$ is holomorphic, we mean that each component $f^\alpha_{jk\beta}(p)$ is a holomorphic function of $p \in U_{jk}$. Then F is obtained by glueing up $U_k \times \mathbf{C}^n$ by identifying $(p, \zeta_j) \in U_j \times \mathbf{C}^n$ with

$$(p, \zeta_j) = (p, f_{jk}(p)\zeta_k) \in U_j \times \mathbf{C}^n \ ; \ F = \cup_j U_j \times \mathbf{C}^n.$$

From K. Kodaira [4] and A. Morrow and K. Kodaira [5], if $f_{jk}(p)$ is holomorphic, then the map $(p, \zeta_k) \to (p, \zeta_j)$ is biholomorphic. Let $e_1^*, e_2^*, \cdots, e_{q+1}^*$ be $q+1$ unit vectors of $\mathbf{C}^{q+1}$ and

$$v_i = (v_{i1}, v_{i2}, \cdots, v_{iq}) \in \mathbf{C}^q.$$

For any $v_{1\ q+1}, v_{2\ q+1}, \cdots, v_{q\ q+1} \in \mathbf{C}$, we let

$$\begin{aligned} v_1^* &:= (v_{11}, v_{12}, \cdots, v_{1q}, v_{1\ q+1}) \in \mathbf{C}^{q+1}, \\ v_2^* &:= (v_{21}, v_{22}, \cdots, v_{2q}, v_{2\ q+1}) \in \mathbf{C}^{q+1}, \\ &\cdots \\ v_q^* &:= (v_{q1}, v_{q2}, \cdots, v_{qq}, v_{q\ q+1}) \in \mathbf{C}^{q+1}. \end{aligned}$$

Then

$$\begin{aligned} \Gamma^* &:= Ze_1^* + \cdots + Ze_{q+1}^* + \cdots + Zv_1^* + \cdots + Zv_q^* \\ &= \{m_1 e_1^* + \cdots + m_{q+1} e_{q+1}^* + n_1 v_1^* + n_q v_q^* \ : \ m_i, n_i \in Z\} \end{aligned}$$

is a additive discrete subgroup of $\mathbf{C}^{q+1}$.

LEMMA 2.3. *The quotient group $\mathbf{C}^{q+1}/\Gamma^*$ is a non- compact abelian Lie group.*

Proof. Let $e_1^*, e_2^*, \cdots, e_{q+1}^*, v_1^*, v_2^*, \cdots, v_q^* \in \mathbf{C}^{q+1} \cong \mathbf{R}^{2q+2}$. Then there exists v_{q+1}^* in $\mathbf{C}^{q+1}$ such that

$$\{\mu v_{q+1}^*\} \subset \mathbf{C}^{q+1}, \mu = 1, 2, \cdots$$

and

$$\mu v_{q+1}^* + \Gamma^* \in \mathbf{C}/\Gamma^*.$$

Hence the set $\{\mu v_{q+1}^* + \Gamma^*\}$ is discrete. That is, there exist no convergent subsequence of the set. Thus, we complete the proof. □

Consider the projection

$$p : \mathbf{C}^{q+1} \to \mathbf{C}^q$$

satisfying $(z_1, z_2, \cdots, z_{q+1}) \mapsto (z_1, z_2, \cdots, z_q)$ and

$$p^* : \mathbf{C}^{q+1}/\Gamma^* \to \mathbf{C}^q/p(\Gamma^*).$$

Then

$$p(\Gamma^*) = ze_1 + \cdots + ze_q + zv_1 + \cdots + zv_q = \Gamma.$$

Thus, we have

$$\mathbf{C}^q/p(\Gamma^*) = \mathbf{C}^q/\Gamma = \mathbf{T}^q$$

where $\mathbf{T}^q$ is a complex torus (see [7]). Therefore from Lemma 2.3 ,

$$p^* : \mathbf{C}^{q+1}/\Gamma^* \longrightarrow \mathbf{T}^q$$

have a structure of principle line bundles. Let

$$\tilde{U}_i := \{z_i + \Gamma \in \mathbf{C}/\Gamma : \forall z_i \in U_i\}$$

is an open subset of $\mathbf{T}^1 = \mathbf{C}/\Gamma$. In the case of $\mathbf{C}^2$,

$$p^* : \mathbf{C}^2/\Gamma^* \longrightarrow \mathbf{T}^1,$$
$$\Gamma^* = \mathbf{Z}e_1^* + \mathbf{Z}e_2^* + \mathbf{Z}v_1^*,$$
$$p^{*-1}(\tilde{U}_i) = \{z + \Gamma^* : \forall z = (z_1, z_2) \in U_i \times \mathbf{C}\}.$$

Suppose that

$$\pi_i : \ p^{*-1}(\tilde{U}_i) \longrightarrow \tilde{U}_i \times \mathbf{C}^*$$

satisfying $\pi_i(z + \Gamma^*) = (z_1 + \Gamma, exp\ 2\pi\sqrt{-1}z_2)$ where $z = (z_1, z_2)$ and $z_1 \in U_i$.

LEMMA 2.4. *π_i is a well defined biholomorphic onto mapping.*

Proof. Suppose that

$$z + \Gamma^* = \tilde{z} + \Gamma^* \in p^{*-1}(\tilde{U}_i)$$

where $z = (z_1, z_2), \tilde{z} = (\tilde{z}_1, \tilde{z}_2), z_1, \tilde{z}_1 \in U_i$. Since $z - \tilde{z} \in \Gamma^*$, there exists $m_i \in \mathbf{Z}$ such that

$$z - \tilde{z} = m_1 e_1^* + m_2 e_2^* + m_3 v_1^*.$$

Thus, we have

$$z_1 - \tilde{z}_1 = m_1 + m_3 v_{11},$$
$$z_2 - \tilde{z}_2 = m_2 + m_3 v_{12}.$$

Since $z_1, \tilde{z}_1 \in U_i$, we have

$$m_1 = m_3 = 0.$$

Hence $z_2 - \tilde{z}_2 = m_2$. And

$$(z_1 + \Gamma, exp\ 2\pi\sqrt{-1}z_2)$$
$$=(\tilde{z}_1 + \Gamma, exp\ 2\pi\sqrt{-1}(\tilde{z}_2 + m_2))$$
$$=(\tilde{z}_1 + \Gamma, exp\ 2\pi\sqrt{-1}\tilde{z}_2).$$

□

THEOREM 2.5.

$$p^* : \mathbf{C}^{q+1}/\Gamma^* \longrightarrow \mathbf{T}^q$$

is a topological trivial holomorphic principle line bundle.

Proof. From Lemma 2.4, it is a holomorphic principle line bundle. Now we prove that it is topological trivial. We consider exact sequences:

$$0 \to \mathbf{Z} \to C \xrightarrow{\Phi} C^* \to 0,$$
$$0 \to \mathbf{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0,$$

where $\Phi(\cdot) = exp 2\pi\sqrt{-1}(\cdot)$, C is the sheaf of germs of continuous functions, C^* is the nonzero sheaf, $\mathcal{O}$ is the sheaf of germs of complex - valued C^∞ functions and $\mathcal{O}^*$ is the nonzero sheaf. Hence we have the long exact sequences

$$\cdots \to H^1(\mathbf{T}^q, C) \to H^1(\mathbf{T}^q, C^*) \to H^2(\mathbf{T}^q, \mathbf{Z}) \to \cdots,$$
$$\cdots \to H^1(\mathbf{T}^q, \mathcal{O}) \to H^1(\mathbf{T}^q, \mathcal{O}^*) \to H^2(\mathbf{T}^q, \mathbf{Z}) \to \cdots.$$

Since $H^1(\mathbf{T}^q, \mathcal{O}^*)$ is the group of all holomorphic line bundles on $\mathbf{T}^q$, the Chern class $H^2(\mathbf{T}^q, \mathbf{Z})$ is zero. Therefore we complete the proof.□

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