Kangweon-Kyungki Math. Jour. **7** (1999), No. 1, pp. 85–101

THE THEORY AND APPLICATIONS OF SECOND-ORDER DIFFERENTIAL SUBORDINATIONS

JUN RAK LEE

ABSTRACT. Let p be analytic in the unit disc U and let q be univalent in U. In addition, let Ω be a set in C and let $\psi : C^3 \times U \to C$. The author determines conditions on ψ so that

$$\{\psi(p(z), zp'(z), z^2p''(z); z) | z \in U\} \subset \Omega \Rightarrow p(U) \subset q(U).$$

Applications of this result to differential inequalities, differential subordinations and integral inequalities are presented.

1. Introduction

In the field of differential equations of real-valued functions there are many examples of differential inequalities that have important applications in the general theory. As a very simple example, consider a function f which is twice continuously differentiable on $I = (-1, 1)$ and suppose that the differential operator $D[f](t) = [t^2 f(t) + t^3]'' = t^2 f''(t) + 4tf'(t) + 2f(t) + 6t$ satisfies

$$0 < D[f](t) < 2, \ for \ t \in I. \tag{1}$$

It is easy to show that $-1 < f(t) < 2$, for $t \in I$. This result can be rewritten as

$$D[f](I) \subset (0, 2) \Rightarrow f(I) \subset (-1, 2). \tag{2}$$

Two articles in 1978 [3] and 1981 [4] extended these ideas involving differential inequalities for real-valued functions to complex-valued functions. In this article we will describe some of these new results and their

Received November 23, 1998.
1991 Mathematics Subject Classification: 30C45.
Key words and phrases: admissible function,differential subordination.

applications. A differential inequality of the form (1) does not have a direct analog for complex-valued functions, i.e. we cannot merely replace the real-valued function $f(t)$ in (1) with a complex-valued function $f(z)$. However, the first inclusion relation of (2) does have a natural complex analog such as

$$D[f](U) \subset \Omega,$$

with $D[f](z) = z^2 f''(z) + 4zf'(z) + 2f(z) + 6z$, where $U,\ \Omega \subset C$ and U is the unit disk. If $f : U \to C$ satisfies this inclusions, then analogously to (2) we can ask if there is a "smallest" set $\triangle \subset C$ such that

$$D[f](U) \subset \Omega \Rightarrow f(U) \subset \triangle. \tag{3}$$

There are other problems that are associated with (3). Given Ω and $\triangle$, does there exist a class of functions satisfying (3). And secondly, given f and $\triangle$, does there exist a "largest" set Ω satisfying (3). These problems will be described in this article. Let Ω and $\triangle$ be any sets in C, let p be analytic in the unit disk U, with $p(0) = a$, and let $\psi(r, s, t; z) : C^3 \times U \to C$. The heart of this article deals with the following implication

$$\{\psi(p(z), zp'(z), z^2p''(z); z) | z \in U\} \subset \Omega \Rightarrow p(U) \subset \triangle. \tag{4}$$

Note that (3) is of this form with $\psi(r, s, t; z) = t + 4s + 2r + 6z$.

If either Ω or $\triangle$ in (4) is a simply connected domain then (4) can be rewritten in terms of subordination. Recall that if f and F are analytic in U and F is univalent in U then f is subordinate to F, written $f(z) \prec F(z)$ or $f \prec F$, if $f(0) = F(0)$ and $f(U) \subset F(U)$.

If $\triangle$ is a simply connected domain containing the point a and $\triangle(\neq C)$, then there is a conformal mapping q of U onto $\triangle$ such that $q(0) = a$. In this case (4) can be rewritten as

$$\{\psi(p(z), zp'(z), z^2p''(z); z) | z \in U\} \subset \Omega \Rightarrow p(z) \prec q(z).$$

If Ω is also a simply connected domain and $\Omega(\neq C)$, then there is a conformal mapping h of U onto Ω such that $h(0) = \psi(a, 0, 0; 0)$. If in addition $\psi(p(z), zp'(z), z^2p''(z); z)$ is analytic in U then (4) can be rewritten as

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z) \Rightarrow p(z) \prec q(z). \tag{5}$$

This last result leads us to some of the important definitions that will be used throughout this article.

DEFINITION 1.1. Let $\psi : C^3 \times U \to C$ and let h be univalent in U. If p is analytic in U and satisfies the *(second-order) differential subordination*

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z), \tag{6}$$

then p is called a *solution of the differential subordination*. The univalent function q is called a *dominant of the solutions of the differential subordination*, or more simply a *dominant of the differential subordination* if $p \prec q$ for all p satisfying (6). A dominant $\tilde{q}$ which satisfies $\tilde{q} \prec q$ for all dominants q of (6) is said to be the *best dominant* of (6). (Note that the best dominant is unique up to a rotation of U).

Let Ω be a set in C and suppose (6) is replaced by

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega, \ \ for \ z \in U. \tag{6'}$$

Although this is a "differential inclusion" and $\psi(p(z), zp'(z), z^2p''(z); z)$ may not be analytic in U, we shall also refer to (6′) as a *(second-order) differential subordination*, and use the same definitions of solution, dominant and best dominant as given in Definition 1.1.

2. Preliminary lemmas

In this section we list the main lemmas that will be needed to prove the theorems of the next section. Proofs will be omitted, but references will be indicated. For $z_0 = r_0e^{i\theta_0}$ with $0 < r_0 < 1$, we let $U_{r_0} = \{z : |z| < r_0\}$.

LEMMA 2.1 [3,P.290]. *Let $f(z) = a_nz^n + a_{n+1}z^{n+1} + \cdots$ be continuous on $\bar{U}_{r_0}$ and analytic on $U_{r_0} \cup \{z_0\}$ with $f(z) \not\equiv 0$ and $n \geq 1$. If $|f(z_0)| = max\{|f(z)| : z \in \bar{U}_{r_0}\}$ then there exists an $m \geq n$ such that*
(1) $z_0f'(z_0)/f(z_0) = m$, and
(2) $\mathrm{Re}[z_0f''(z_0)/f'(z_0)] + 1 \geq m$.

DEFINITION 2.2. We denote by Q the set of functions q that are analytic and injective on $\bar{U} \backslash E(q)$, where

$$E(q) = \{\zeta \in \partial U : \lim_{z \to \zeta} q(z) = \infty\},$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial U \backslash E(q)$.

LEMMA 2.3. *[4,p.158] Let $q \in Q$ with $q(0) = a$, and let $p(z) = a + p_n z^n + \cdots$ be analytic in U with $p(z) \not\equiv a$ and $n \geq 1$. If there exist points $z_0 \in U$ and $\zeta_0 \in \partial U \backslash E(q)$ such that $p(z_0) = q(\zeta_0)$ and $p(U_{r_0}) \subset q(U)$, where $r_0 = |z_0|$, then there exists an $m \geq n$ such that*
(1) $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$, and
(2) $\mathrm{Re}[z_0 p''(z_0)/p'(z_0) + 1] \geq m\,\mathrm{Re}[\zeta_0 q''(\zeta_0)/q'(\zeta_0) + 1]$.

We next discuss two important cases of Lemma 2.3 corresponding to $q(U)$ being a disk, and $q(U)$ being a half-plane.

Case 1. The Disk $= \{\omega : |\omega| < M\}$. If we let

$$q(z) = M(Mz + a)/(M + \bar{a}z),$$

with $M > 0$, and $|a| < M$ then $q(U) = \triangle = U_M$, $q(0) = a$, $E(q)$ is empty and $q \in Q$. If there are points $z_0 \in U$, $\zeta_0 \in \partial U$ such that $p(z_0) = q(\zeta_0)$ and $|p(z)| < M$ for $|z| < |z_0|$, then $|p(z_0) = |q(\zeta_0)| = M$,

$$\zeta_0 = q^{-1}(p(z_0)) = M[p(z_0) - a]/[M^2 - \bar{a}p(z_0)],$$

$$\zeta_0 q'(\zeta_0) = [M^2 - \bar{a}p(z_0)][p(z_0) - a]/[M^2 - |a^2|], \tag{7}$$

$$\mathrm{Re}[\zeta_0 q''(\zeta_0)/q'(\zeta_0) + 1] = |p(z_0) - a|^2/[M^2 - |a|^2]. \tag{8}$$

Using these results in Lemma 2.3 we obtain:

LEMMA 2.3′. *Let $p(z) = a + p_n z^n + \cdots$ be analytic in U with $P(z) \not\equiv a$ and $n \geq 1$. If there exists $z_0 \in U$ such that $|p(z_0)| = Max\{|p(z)| : |z| \geq |z_0|\}$ then*
(1)$z_0 p'(z_0)/p(z_0) \geq n|p(z_0) - a|^2/[|p(z_0)|^2 - |a|^2]$ and
(2)$\mathrm{Re}[z_0 p''(z_0)/p'(z_0) + 1] \geq n|p(z_0) - a|^2/[|p(z_0)|^2 - |a|^2]$.

For $a = 0$ this lemma reduces to Lemma 2.1.

Case 2. The Half-Plane $= \{\omega : \mathrm{Re}\,\omega > \alpha,\ \alpha\ real\}$. If we let

$$q(z) = [a - (2\alpha - \bar{a})z]/[1 - z]$$

with $\operatorname{Re} a > \alpha$ then $q(U) = \Delta$, $q(0) = a$, $E(q) = \{1\}$ and $q \in Q$. If there are points $z_0 \in U$ and $\zeta_0 \in \partial U \backslash \{1\}$ such that $p(z_0) = q(\zeta_0)$ and $\operatorname{Re} p(z) > \alpha$ for $|z| < |z_0|$, then $\operatorname{Re} p(z_0) = \alpha$,

$$\zeta_0 = q^{-1}(p(z_0)) = [p(z_0) - a]/[p(z_0) - (2\alpha - \bar{a})],$$

$$\zeta_0 q'(\zeta_0) = -|a - p(z_0)|^2 / 2 \operatorname{Re}[a - p(z_0)] \tag{9}$$

and

$$\operatorname{Re}[\zeta_0 q''(\zeta_0)/q'(\zeta_0) + 1] = 0 \tag{10}$$

Using these results in Lemma 2.3 we obtain :

LEMMA 2.3. *Let* $p(z) = a + p_n z^n + \cdots$ *be analytic in* U *with* $p(z) \not\equiv a$ *and* $n \geq 1$. *If there exists* $z_0 \in U$ *such that* $\operatorname{Re} p(z_0) = Min\{\operatorname{Re} p(z) : |z| \geq |z_0|\}$ *then*

(1) $z_0 p'(z_0) \leq -n|a - p(z_0)|^2 / 2 \operatorname{Re}[a - p(z_0)]$ *and*

(2) $\operatorname{Re} z_0 p''(z_0)/p'(z_0) + 1 \geq 0$.

REMARK. 1. Since $z_0 p'(z_0)$ is real and negative, the inequality (2) can be replaced by

(2′) $\operatorname{Re}[{z_0}^2 p''(z_0)] + z_0 p'(z_0) \leq 0$.

2. If $\alpha = 0$ and $a = 1$ the inequality (1) becomes

(1′) $z_0 p'(z_0) \leq -n(1 + |p(z_0)|^2)/2 \leq -n/2$

LEMMA 2.4. *Let* $q \in Q$, *with* $q(0) = a$, *and let* $p(z) = a + p_n z^n + \cdots$ *be analytic in* U *with* $p(z) \not\equiv a$ *and* $n \geq 1$. *If* $p \not\prec q$, *then there exist points* $z_0 = r_0 e^{i\theta_0} \in U$ *and* $\zeta_0 \in \partial U \backslash E(q)$ *and an* $m \geq n$ *for which*

(1) $p(U_{r_0}) \subset q(U)$

(2) $p(z_0) = q(\zeta_0)$,

(3) $z_0 p'(z_0) = m \zeta_0 q'(\zeta_0)$, *and*

(4) $\operatorname{Re}[z_0 p''(z_0)/p'(z_0) + 1] \geq m \operatorname{Re}[\zeta_0 q''(\zeta_0)/q'(\zeta_0) + 1]$.

Proof. Since $p(0) = q(0)$, and p and q are analytic on U, we can define

$$r_0 = sup\{r : p(U_r) \subset q(U)\}.$$

Since $p \not\prec q$ we have $p(U) \not\subset q(U)$. Thus for $0 < r_0 < 1$ we get $p(U_{r_0}) \subset q(U)$ and $p(\bar{U}_{r_0}) \not\subset q(U)$. Since $p(\bar{U}_{r_0}) \subset q(\bar{U})$ there exists $z_0 \in \partial U_{r_0}$ such that $p(z_0) \in \partial q(U)$. This implies there exists $\zeta_0 \in \partial U \backslash E(q)$ such that $p(z_0) = q(\zeta_0)$. The conclusions of this lemma now follow by applying Lemma 2.3. □

3. Admissible functions and fundamental theorems

In this section we define the class of functions ψ for which we intend to prove (4).

DEFINITION 3.1. Let Ω be a set in C, $q \in Q$, and n be a positive integer. We define the *class of admissible functions* $\Psi_n[\Omega, q]$ to be those functions $\psi : C^3 \times U \to C$ that satisfy the following *admissibility condition* :

$$\psi(r, s, t;) \notin \Omega \; when \; r = q(\zeta), \;\; s = m\zeta q'(\zeta),$$

(11) $$\mathrm{Re}[t/s + 1] \geq m\,\mathrm{Re}[\zeta q''(\zeta)/q'(\zeta) + 1] \; and \; z \in U,$$

$$for \; \zeta \in \partial U \backslash E(q) \; and \; m \geq n.$$

We write $\Psi_1[\Omega, q]$ as $\Psi[\Omega, q]$.

In the special case when $\Omega(\neq C)$ is a simply connected domain and h is a conformal mapping of U onto Ω we denote the class by $\Psi_n[h, q]$.

THEOREM 3.2. *Let* $\psi \in \Psi_n[\Omega, q]$ *with* $q(0) = a$. *If* $p(z) = a + p_n z^n + \cdots$ *is analytic in* U *and satisfies*

(12) $$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega, \;\; z \in U$$

then $p \prec q$.

Proof. Assume that $p \not\prec q$. By Lemma 2.4 there exist points $z_0 \in U$ and $\zeta_0 \in \partial U \backslash E(q)$ and an $m \geq n$ that satisfy (1)-(4) of Lemma 2.4. Using these conditions with $r = p(z_0)$, $s = z_0 p'(z_0)$, $t = z_0^2 p''(z_0)$ and $z = z_0$ in Definition 3.1 we obtain

$$\psi(p(z_0), zp'(z_0), z_0^2 p''(z_0); z_0) \notin \Omega.$$

Since this contradicts (12) we must have $p \prec q$. □

REMARK. 1. The conclusion of Theorem 3.2 also holds if (12) is replaced by

$$(12') \qquad \psi(p(z), zp'(z), z^2 p''(z); \omega(z)) \in \Omega, \ z \in U,$$

for any function $\omega(z)$ mapping U into U.

2. On checking the definitions of Q and $\Psi_n[\Omega, q]$ we see that the hypothesis of Theorem 3.2 requires that q behave very nicely on its boundary. If this is not the case or if the behavior of q on its boundary is not known, it may still be possible to prove that $p \prec q$ by the following limiting procedure.

COROLLARY 3.3. *Let $\Omega \subset C$ and let q be univalent in U. Let $\psi \in \Psi_n[\Omega, q_\rho]$, for some $\rho \in (0,1)$, where $q_\rho(z) = q(\rho z)$. If $p(z) = a + p_n z^n + \cdots$ is analytic in U and $\psi(p(z), zp'(z), z^2 p''(z); z) \in \Omega$ for $z \in U$, then $p \prec q$.*

Proof. The function q_ρ is univalent on $\bar{U}$, and hence $E(q_\rho)$ is empty and $q_\rho \in Q$. The class $\Psi_n[\Omega, q_\rho]$ is an admissible class and from Theorem 3.2 we obtain $p \prec q_\rho$. Since $q_\rho \prec q$ we deduce $p \prec q$. □

We next list the special case when $\Omega (\neq C)$ is a simply connected domain.

THEOREM 3.4. *Let $\psi \in \Psi_n[h, q]$, with $q(0) = a$ and $\psi(a, 0, 0; 0) = h(0)$. If $p(z) = a + p_n z^n + \cdots$ and $\psi(p(z), zp'(z), z^2 p''(z); z)$ are analytic in U, and*

$$(13) \qquad \psi(p(z), zp'(z), z^2 p''(z); z) \prec h(z)$$

then $p \prec q$.

Proof. From (13), $\psi(p(z), zp'(z), z^2p''(z); z) \in h(U) = \Omega$. Therefore, the proof follows immediately from Theorem 3.2. $\square$

An analogue of Corollary 3.3 can be given for $\Psi_n[h, q] = \Psi[h(U), q]$.

COROLLARY 3.5. *Let h and q be univalent in U with $q(0) = a$. Let $\psi : C^3 \times U \to C$, with $\psi(a, 0, 0; 0) = h(0)$, satisfy one of the following conditions:*
(1) $\psi \in \Psi_n[h, q_\rho]$, for some $\rho \in (0, 1)$ or
(2) there exists $\rho_0 \in (0, 1)$ such that $\psi \in \Psi_n[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$, where $q_\rho(z) = q(\rho z)$ and $h_\rho(z) = h(\rho z)$. If $p(z) = a + p_n z^n + \cdots$ and $\psi(p(z), zp'(z), z^2p''(z); z)$ are analytic in U and

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z),$$

then $p \prec q$.

Proof. Case (1). By applying Theorem 3.4 we obtain $p \prec q_\rho$. Since $q_\rho \prec q$ we deduce $p \prec q$.
Case (2). If we let $p_\rho(z) = p(\rho z)$ we have

$$\begin{aligned}\psi(p_\rho(z), zp'_\rho(z), z^2p''_\rho(z); \rho z) \\ = \psi(p(\rho z), \rho z p'(\rho z), \rho^2 z^2 p''(\rho z); \rho z) \in h_\rho(U)\end{aligned}$$

for $z \in U$. By using Theorem 3.2 and Remark 1 following it, with $\omega(z) = \rho z$, we obtain $p_\rho(z) \prec q_\rho(z)$ for $\rho \in (\rho_0, 1)$. By letting $\rho \to 1^-$ we obtain $p \prec q$. $\square$

If $n = 1$ and q is a dominant and solution of (13) then q will be the best dominant. Using this result together with Theorem 3.4 and Corollary 3.5 yields the following theorem.

THEOREM 3.6. *Let h be univalent in U, and let $\psi : C^3 \times U \to C$. Suppose that the differential equation*

$$\psi(q(z), zq'(z), z^2q''(z); z) = h(z)$$

has a solution q and one of the following conditions is satisfied
(1) $q \in Q$ and $\psi \in \Psi[h, q]$,

(2) q is univalent in U and $\psi \in \Psi[h, q_\rho]$ for some $\rho \in (0,1)$, or
(3) q is univalent in U and there exists $\rho_0 \in (0,1)$ such that

$$\psi \in \Psi[h_\rho, q_\rho] \text{ for all } \rho \in (\rho_0, 1).$$

If $p(z) = q(0) + p_1 z + \cdots$ and $\psi(p, zp', z^2p''; z)$ are analytic in U and if p is a solution of (13), then $p \prec q$ and q is the best dominant.

Theorem 3.2 can be used to show that the solutions of certain second order differential equations are contained in a certain set.

THEOREM 3.7. *Let $\psi \in \Psi_n[\Omega, q]$ and let f be an analytic function satisfying $f(U) \subset \Omega$. If the differential equation*

$$\psi(p(z), zp'(z), z^2p''(z); z) = f(z)$$

has a solution $p(z)$ analytic in U with $p(0) = q(0)$, then $p \prec q$.

Proof. By hypothesis, $\psi(p(z), zp'(z), z^2p''(z); z) = f(z) \in f(U) \subset \Omega$. Therefore by Theorem 3.2, the theorem is proved. □

4. Special cases: the disc and half-plane

In this section we will apply the theorems of the last section to the particular cases corresponding to $q(U)$ being a disk and $q(U)$ being a half-plane. Some preliminary results for these two cases have been presented in Section 2.

Case 1. The Disk $\Delta = \{\omega : |\omega| < M\}$. The function

$$q(z) = M(Mz + a)/(M + \bar{a}z),$$

with $M > 0$ and $|a| < M$ satisfies $q(U) = \Delta$, $q(0) = a$ and $q \in Q$. We first determine the class of admissible functions, as defined in Definition 3.1, for this particular q. We set $\Psi_n[\Omega, M, a] = \Psi_n[\Omega, q]$ and in the special case when $\Omega = \Delta$ we denote the class by $\Psi_n[M, a]$. Since $q(\zeta) = Me^{i\theta}$ with $\theta \in R$ when $|\zeta| = 1$, by using (7) and (8) the condition of admissibility (11) becomes

$$\psi(r, s, t; z) \notin \Omega \text{ when } r = Me^{i\theta},$$

$$s = mMe^{i\theta}|M - \bar{a}e^{i\theta}|^2/(M^2 - |a|^2), \tag{14}$$

$$\mathrm{Re}(t/s+1) \geq m|M - ae^{-i\theta}|^2/(M^2 - |a|^2),$$

$$for\ z \in U,\ \theta \in R\ and\ m \geq n.$$

If $a = 0$ then (14) simplifies to

$$\psi(Me^{i\theta}, Ke^{i\theta}, L; z) \notin \Omega\ when\ K \geq nM,$$

$$\mathrm{Re}(Le^{-i\theta}) \geq (n-1)K,\ z \in U,\ and\ \theta \in R, \tag{14'}$$

THEOREM 4.1. *Let $p(z) = a + p_n z^n + \cdots$ be analytic in U. If $\psi \in \Psi_n[\Omega, M, a]$ then*

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega \Rightarrow |p(z)| < M.$$

If $\psi \in \Psi_n[M, a]$ then

$$|\psi(p(z), zp'(z), z^2p''(z); z)| < M \Rightarrow |p(z)| < M.$$

The proof of this theorem follows immediately by applying Theorem 3.2.

EXAMPLE 4.2. Let $a = 0$, $n = 1$, $\Omega = h(U)$ where $h(z) = 2Mz$, and $\psi(r, s, t; z) = r + s + t$. We first show that $\psi \in \Psi[h(U), M, 0]$, that is, that admissibility condition (14′) is satisfied. This follows since

$$\begin{aligned}|\psi(Me^{i\theta}, Ke^{i\theta}, L; z)| = |M + K + Le^{-i\theta} &\geq M + K + \mathrm{Re}(Le^{-i\theta}) \\ &\geq M + Mn + (n-1)nM = M(1+n^2) = 2M\end{aligned}$$

when $K \geq nM$ and $\mathrm{Re}(Le^{-i\theta}) \geq (n-1)K$. By Theorem 4.1 we deduce the following result. If $p(z)$ is analytic in U with $p(0) = 0$ then

$$|p(z) + zp'(z) + z^2p''(z)| < 2M \Rightarrow |p(z)| < M.$$

We can use Theorem 3.6 to present a different proof of this result, and to also show that this result is sharp. The differential equation

$$q(z) + zq'(z) + z^2q''(z) = 2Mz,$$

has the univalent solution $q(z) = Mz$. In order to use Theorem 3.6 we need to show that $\psi \in \Psi[2Mz, Mz]$. For $r = M\zeta$, $s = mM\zeta$ and $\operatorname{Re}[t/s + 1] \geq m$, for $|\zeta| = 1$ and $m \geq 1$ we have

$$|\psi(r, s, t)| = |M\zeta + Mm\zeta + t| = M|1 + m + mt/s|$$

$$\geq M(1 + m + m^2 - m) = M(1 + m^2) \geq 2M.$$

Hence $\psi \in \Psi[2Mz, Mz]$, and by Theorem 3.6

$$p(z) + zp'(z) + z^2p''(z) \prec 2Mz \Rightarrow p(z) \prec Mz,$$

and $q(z) = Mz$ is the best dominant.

Case 2. The Half-Plane $\Delta = \{\omega : \operatorname{Re}\omega > 0\}$. The function

$$q(z) = (a + \bar{a}z)/(1 - z)$$

with $\operatorname{Re} a > 0$ satisfies $q(U) = \Delta$, $q(0) = a$, $E(q) = \{1\}$, and $q \in Q$. We first determine the class of admissible functions, as defined in Definition 3.1, for this particular q. We set $\Psi_n\{\Omega, a\} = \Psi_n[\Omega, q]$ and in the special case when $\Omega = \Delta$ we denote the class by $\Psi_n\{a\}$. Since $\operatorname{Re} q(\zeta) = 0$ when $\zeta \in \partial U \backslash \{1\}$, by using (9) and (10) the condition of admissibility (11) becomes

$$\psi(i\sigma, \tau, \mu + i\eta; z) \notin \Omega, \ for\ z \in U\ and\ for\ real\ \sigma, \tau, \mu, \eta$$

$$\text{(15)} \qquad satisfying\ \tau \leq -n|a - i\sigma|^2/2\operatorname{Re} a\ and\ \tau + \mu \leq 0.$$

If $a = 1$ then (15) simplifies to

$$\text{(15')} \qquad \psi(i\sigma, \tau, \mu + i\eta;) \notin \Omega;\ for\ z \in U,\ and\ for\ real\ \sigma, \tau, \mu, \eta$$

$$satisfying\ \tau \leq -n(1 + \sigma^2)/2\ and\ \tau + \mu \leq 0.$$

The proof of the following theorem follows immediately by applying Theorem 3.2.

THEOREM 4.3. *Let* $p(z) = a + p_n z^n + \cdots$ *be analytic in* U. *If* $\psi \in \Psi_n\{\Omega, a\}$ *then*

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega \Rightarrow \operatorname{Re} p(z) > 0.$$

If $\psi \in \Psi_n\{a\}$ *then*

$$\operatorname{Re} \psi(p(z), zp'(z), z^2p''(z); z) > 0 \Rightarrow \operatorname{Re} p(z) > 0.$$

EXAMPLE 4.4. a) A simple check of (15′) shows that $\psi(r, s, t; z) = r + s + t \in \Psi\{1\}$. Thus if $p(z) = 1 + p_1 z + \cdots$ is analytic in U then

$$\operatorname{Re}[p(z) + zp'(z) + z^2p''(z)] > 0 \Rightarrow \operatorname{Re} p(z) > 0.$$

b) Let $\psi(r, s, t; z) = r + B(z)s$, where $B : U \to C$ and $\operatorname{Re} B(z) > 0$. A simple check of (15) shows that $\psi \in \Psi\{a\}$ with $\operatorname{Re} a > 0$. Thus if $p(z) = a + p_1 z + \cdots$ is analytic in U then

$$\operatorname{Re}[p(z) + B(z)zp'(z)] > 0 \Rightarrow \operatorname{Re} p(z) > 0.$$

c) Let $\psi(r, s, t; z) = t + 3s - r^2 + 1$. A simple check of (15′) shows that $\psi \notin \Psi_1\{1\}$, but $\psi \in \Psi_2\{1\}$. Thus if $p(z) = 1 + p_2 z^2 + \cdots$ is analytic in U then

$$\operatorname{Re}[z^2p''(z0 + 3zp'(z) - p^2(z) + 1] > 0 \Rightarrow \operatorname{Re} P(z) > 0.$$

Other examples of similar differential inequalities may be found in [1], [2], [3] and [5].

5. Differential and integral operators preserving functions with positive real part

In this section we will be interested in determining dominants of the *second-order linear differential subordination*

$$A(z)z^2p''(z) + B(z)zp'(z) + C(z)p(z) + D(z) \in \Omega \tag{16}$$

for $z \in U$, where $\Omega \subset C$, and A, B, C and D are complex-valued functions defined on U. In this section we let Ω be a set in $\{\omega | \operatorname{Re} \omega > 0\}$ and let $q(z) = (1+z)/(1-z)$ be a dominant of (16). We will determine conditions on A, B, C and D corresponding to this particular Ω and q.

THEOREM 5.1. *Let* $A(z) = A \geq 0$ *and suppose that* $B, C, D : U \to C$ *and satisfy*

(17)

$\operatorname{Re} B(z) \geq A$ *and* $[\operatorname{Im} C(z)]^2 \leq [\operatorname{Re} B(z) - A] \cdot \operatorname{Re}[B(z) - A - 2D(z)]$.

If p *is analytic in* U *with* $p(0) = 1$, *and if*

$$\operatorname{Re}[Az^2p''(z) + B(z)zp'(z) + C(z)p(z) + D(z)] > 0, \tag{18}$$

then $\operatorname{Re} p(z) > 0$.

Proof. If we let $\psi(r, s, t; z) = At + B(z)s + C(z)r + D(z)$, then the conclusion will follow from Theorem 3.1 if we show that $\psi \in \Psi[\Omega, q]$, where $\Omega = \{\omega | \operatorname{Re} \omega > 0\}$ and $q = (1 + z)/(1 - z)$. This follows from (17), and Definition 3.1 or (15') since

$$\begin{aligned}
\operatorname{Re}\psi(i\sigma, \tau, \mu + i\eta; z) &= A\mu + \tau \operatorname{Re} B(z) - \sigma \operatorname{Im} C(z) + \operatorname{Re} D(z)\\
&\leq \tau[\operatorname{Re} B(z) - A] - \sigma \operatorname{Im} C(z) + \operatorname{Re} D(z)\\
&\leq [-(1 + \sigma^2)/2][\operatorname{Re} B(z) - A] - \sigma \operatorname{Im} C(z) + \operatorname{Re} D(z)\\
&= -\{[\operatorname{Re} B(z) - A]\sigma^2 + 2[\operatorname{Im} C(z)]\sigma + \operatorname{Re}[B(z) - A - 2D(z)]\}/2\\
&\leq 0,
\end{aligned}$$

for $z \in U$, and for real σ, τ, μ, η satisfying $\tau \leq -(1+\sigma^2)/2$ and $\tau + \mu \leq 0$. Hence $\psi \in \Psi[\Omega, q]$, $p \prec q$ and $\operatorname{Re} p(z) > 0$. □

If $A = 0$ and $D(z) \equiv 0$ then Theorem 5.1 reduces to the following first order result [6, Theorem 8].

COROLLARY 5.2. *Let* $B(z)$ *and* $C(z)$ *be functions defined on* U, *with*

$$|\operatorname{Im} C(z)| \leq \operatorname{Re} B(z). \tag{19}$$

If p *is analytic in* U *with* $P(0) = 1$, *and if*

$$\operatorname{Re}[B(z) \cdot zp'(z) + C(z) \cdot p(z)] > 0, \tag{20}$$

then $\operatorname{Re} p(z) > 0$.

We can apply Corollary 5.2 to obtain a corresponding result for integrals [6, Theorem 9].

THEOREM 5.3. *Let $\gamma \neq 0$ be a complex number with $\operatorname{Re}\gamma \geq 0$, and let φ and Φ be analytic in U, with $\varphi(z) \cdot \Phi(z) \neq 0$, $\varphi(0) = \Phi(0)$, and*

$$|\operatorname{Im}[(\gamma\Phi(z) + z\Phi'(z))/\gamma\varphi(z)]| \leq \operatorname{Re}[\Phi(z)/\gamma\varphi(z)]. \tag{21}$$

Let f be analytic in U with $f(0) = 1$ and $\operatorname{Re} f(z) > 0$, for $z \in U$. If F is defined

$$F(z) = \gamma z^{-\gamma}\Phi(z)^{-1} \int_0^z f(t)t^{\gamma-1}\varphi(t)\,dt, \tag{22}$$

then F is analytic in U, $F(0) = 1$ and $\operatorname{Re} F(z) > 0$ for $z \in U$.

Proof. If we let $B(z) = \Phi(z)/\gamma\varphi(z)$, $C(z) = [\gamma\Phi(z)+z\Phi'(z)]/\gamma\varphi(z)$, then condition (21) implies condition (19). By differentiating (22) we obtain

$$\operatorname{Re}[B(z) \cdot zF'(z) + C(z) \cdot F(z)] = \operatorname{Re} f(z) > 0.$$

Hence (20) of Corollary 5.2 is satisfied with $p = F$, and we conclude that $\operatorname{Re} F(z) > 0$. □

If we let $\varphi = \Phi$ and $\gamma > 0$ then (21) reduces to

$$|\operatorname{Im} z\varphi'(z)/\varphi(z)| \leq 1. \tag{23}$$

In this case we deduce

$$\operatorname{Re} f(z) > 0 \Rightarrow \operatorname{Re}\left[z^{-\gamma}\varphi(z)^{-1} \int_0^z f(t)t^{\gamma-1}\varphi(t)\,dt\right] > 0.$$

EXAMPLE 5.4. The function $\varphi(z) = e^{\lambda z}$ satisfies (23) for $|\lambda| \leq 1$. In this case we obtain

$$\operatorname{Re} f(z) > 0 \Rightarrow \operatorname{Re}\left[z^{-\gamma}e^{-\lambda z} \int_0^z f(t)t^{\gamma-1}e^{\lambda t}\,dt\right] > 0. \tag{24}$$

Corollary 5.2 involves a first order linear differential subordination and its integral analog is the linear operator given in Theorem 5.3. The second order linear differential subordination given in Theorem 5.1 also has an integral analog. However, in this case the second order differential subordination gives rise to a double integral.

THEOREM 5.5. *Let β and γ be complex numbers with $\beta\gamma > 0$, $\operatorname{Re}\beta \geq 0$, $\operatorname{Re}\gamma \geq 0$, let φ and Φ be analytic in U with $\varphi(z)\cdot\Phi(z) \neq 0$, $\varphi(0) = \Phi(0)$ and let ω be analytic in U with $\omega(0) = 0$. Suppose that (17) holds with*

(25)

$$\begin{aligned}
&A \equiv 1/\beta\gamma,\\
&B(z) \equiv [\beta + \gamma + 1 + z\varphi'(z)/\varphi(z) + z\Phi'(z)/\Phi(z)]/\beta\gamma, \text{ and}\\
&C(z) \equiv [(\beta + z\Phi'(z)/\Phi(z))(\gamma + z\varphi'(z)/\varphi(z)) + z(z\varphi'(z)/\varphi(z))']/\beta\gamma,\\
&D(z) \equiv -\omega(z).
\end{aligned}$$

Let f be analytic in U with $f(0) = 1$ and $\operatorname{Re} f(z) > 0$ for $z \in U$. If F is defined by

$$F(z) = \frac{\beta\gamma}{z^{\gamma}\varphi(z)} \int_0^z \frac{\varphi(t)}{\Phi(t)} t^{\gamma-\beta-1} \int_0^t [f(s) + \omega(s)]\Phi(s)s^{\beta-1}\, ds\, dt, \tag{26}$$

then F is analytic in U, $F(0) = 1$ and $\operatorname{Re} F(z) > 0$ in U.

Proof. By differentiating (26) and using (25) we obtain $\operatorname{Re}[Az^2F''(z) + B(z)zF'(z) + C(z)F(z) + D(z)] = \operatorname{Re} f(z) > 0$. Since the conditions of (17) hold, we apply Theorem 5.1 with $p = F$ to conclude that $\operatorname{Re} F(z) > 0$ in U. □

Note that if $D(z) = -\omega(z) \equiv 0$ the second-order linear differential subordination (18) gives rise to a linear (double) integral operator $F = I(f)$ given by (26).

If we let $\varphi(z) \equiv 1$, $\omega(z) \equiv 0$, $\beta > 0$ and $\gamma > 0$ then from (25) we have $D(z) \equiv 0$,

$$B(z) = [\beta + \gamma + 1 + z\Phi'(z)/\Phi(z)]/\beta\gamma \text{ and}$$

$$C(z) = [\beta + z\Phi'(z)/\Phi(z)]/\beta.$$

In this case we obtain the following corollary.

COROLLARY 5.6. *Let $\beta > 0$, $\gamma > 0$, and let Φ be analytic in U with $\Phi(z) \neq 0$. Suppose that*

$$\gamma \left| \operatorname{Im} \frac{z\Phi'(z)}{\Phi(z)} \right| \leq \operatorname{Re} \left[\beta + \gamma + \frac{z\Phi'(z)}{\Phi(z)} \right]. \tag{27}$$

Let f be analytic in U with $f(0) = 1$ and $\operatorname{Re} f(z) > 0$ in U. If F is defined by

$$F(z) = \beta\gamma z^{-\gamma} \int_0^z t^{\gamma-\beta-1}\Phi(t)^{-1} \int_0^t f(s)\Phi(s)s^{\beta-1}\, ds\, dt,$$

then F is analytic in U, $F(0) = 1$ and $\operatorname{Re} F(z) > 0$ in U.

The complicated condition (27) has a simple geometric interpretation. If we let $\omega = z\Phi'(z)/\Phi(z) = u + iv$ then (27) becomes

$$\gamma|v| \leq \beta + \gamma + u.$$

Hence (27) requires that $z\Phi'/\Phi$ lies in the closed sector $S(\beta,\gamma)$ containing the origin and bounded by the lines

$$\gamma|v| = \beta + \gamma + u.$$

If we take $\Phi(z) = e^{\lambda z}$ then $z\Phi'(z)/\Phi(z) = \lambda z$. Since the distance δ from the origin to the boundary of the sector $S(\beta,\gamma)$ is given by

$$\delta = (\beta + \gamma)/(1 + \gamma^2)^{1/2}, \tag{28}$$

we obtain the following example.

EXAMPLE 5.7. If $|\lambda| \leq \delta$ where δ is given by (28) then

$$\operatorname{Re} f(z) > 0 \Rightarrow \operatorname{Re}\left[z^{-\gamma} \int_0^z e^{-\lambda t} t^{\gamma-\beta-1} \int_0^t f(s)e^{\lambda s} s^{\beta-1}\, ds\, dt\right] > 0.$$

In the particular case $\beta = \gamma = 1$, we deduce that for $|\lambda| \leq \sqrt{2}$ we have

$$\operatorname{Re} f(z) > 0 \Rightarrow \operatorname{Re}\left[z^{-1} \int_0^z e^{-\lambda t} t^{-1} \int_0^t f(s)e^{\lambda s}\, ds\, dt\right] > 0. \tag{29}$$

Note that in Example 5.4 we can apply (24) twice with $\gamma = 1$ to obtain (29). However, by using this method (29) will be valid only for $|\lambda| \leq 1$.

References

1. Lewandowski,Z., Miller,S.S., Zlotkiewicz, E., *Generating functions for some classes of univalent functions*, Proc. Amer. Math. Soc. **56** (1976), 111–117.
2. Miller,S.S., *A class of differential inequalities implying boundedness*, Illinois J. Math. **20** (1976), 647–649.
3. Miller,S.S., Mocanu, P.T., *Second order differential inequalities in the complex plane*, J. Math. Anal. Appl. **65** (1978), 289–305.
4. ———, *Differential subordinations and univalent functions*, Michigan Math. J. **28** (1981), 157–171.
5. ———, *On classes of functions subordinate to the Koebe function*, Rev. Roumaine Math. Pures Appl. **25** (1981), 95–99.
6. ———, *Differential subordinations and inequalities in the complex plane*, J. of Diff. Eqns. **67** (1987), 199–211.

SAMCHOK NATIONAL UNIVERSITY
253 Gyodong, Samchok
Kangwondo 245-080, Korea