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QUASIRETRACT TOPOLOGICAL SEMIGROUPS

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ABSTRACT. In this paper, we introduce the concepts of quasiretract ideals and quasiretract topological semigroups which are weaker than those of retract ideals and retract topological semigroups, respectively. We prove that every n-th power ideal of a commutative power cancellative power ideal topological semigroup is a quasiretract ideal.

1. Introduction and preliminaries

A *semigroup* is a nonempty set S together with an associative multiplication $(x, y) \to xy$ from $S \times S$ into S. If S has a Hausdorff topology such that multiplication is continuous, with the product topology on $S \times S$, then S is called a *topological semigroup*. An ideal I of a (topological) semigroup S is called a *retract ideal* [7] if there exists a (continuous) homomorphism $h : S \longrightarrow I$ such that $h(x) = x$, for each $x \in I$. The (continuous) homomorphism h is called a *retraction*. A (topological) semigroup S is called a *retract (topological) semigroup* [7] if every (closed) ideal is a retract.

In this paper, we introduce the concepts of quasiretract ideals and quasiretract topological semigroups which are weaker than those of retract ideals and retract topological semigroups, respectively. We prove that every n-th power ideal of a commutative power cancellative power ideal topological semigroup is a quasiretract ideal. For general information about (topological) semigroups, one may consult [2], [3] and [5].

DEFINITION 1.1. [6] A subspace A of a topological space X is a *quasiretract* if there exists a continuous function from X into A which is injective on A.

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DEFINITION 1.2. An ideal I of a (topological) semigroup S is called a *quasiretract ideal* of S if there exists a (continuous) homomorphism ϕ from S into I such that ϕ is injective on I. In this case, the (continuous) homomorphism is called a *quasiretraction* of S into the ideal I. The (topological) semigroup S is called a *quasiretract (topological) semigroup* if every ideal of S is a quasiretract ideal.

Every retract ideal is a closed subset of the topological semigroup S. But a quasiretract ideal is not necessarily closed, as in the case of Example 2.1.

Clearly, every retract ideal is quasiretract and each retract (topological) semigroup is also a quasiretract (topological) semigroup, but neither implication is reversible, in general.

EXAMPLE 1.3. Let N denote the additive semigroup of positive integers and let $I = N - \{1\}$. Then the ideal I is a quasiretract ideal but not retract ideal. The semigroup N is, in fact, a quasiretract semigroup.

DEFINITION 1.4. [4] Let A be a subsemigroup of a topological semigroup S. Let f be a continuous homomorphism from A into a topological semigroup T. An *extension* of f over A relative to T is a continuous homomorphism ϕ from S into T satisfying $\phi(x) = f(x)$, for each $x \in A$. For convenience, ϕ is called an extension of f. If ϕ is an extension of f, then f is called extendable to ϕ.

The proofs of the following two propositions are straightforward.

PROPOSITION 1.5. *An ideal I of a topological semigroup S is a quasiretract ideal of S if a continuous injective homomorphism $f : I \longrightarrow I$ is extendable to a continuous homomorphism $\overline{f} : S \longrightarrow I$.*

PROPOSITION 1.6. *An ideal I of a topological semigroup S is a quasiretract ideal of S if and only if for any topological semigroup T, each continuous injective homomorphism $f : I \longrightarrow T$ is extendable over S.*

2. Main results

If S is a finite semigroup, then every quasiretract ideal is a retract ideal. Thus any finite quasiretract semigroup is a retract semigroup.

It is natural to ask that if S is a compact semigroup and if I is a quasiretract ideal of S, is it also a retract ideal of S ? As in the following example, the answer to this question is negative.

EXAMPLE 2.1. Let S denote the min interval I_m and let $I = [0,1)$. Define a function $f : S \longrightarrow [0,1)$ by $f(x) = \frac{1}{2} \cdot x$ for each $x \in S$, where $\frac{1}{2} \cdot x$ is the usual multiplication. Then f is a quasiretraction and hence I is a quasiretract ideal of S, but it is not retract ideal of S.

THEOREM 2.2. *Let I and J be two quasiretract ideals of a topological semigroup S. Then the intersection $I \cap J$ of I and J is also a quasiretract ideal of S.*

Proof. Let I and J be two quasiretract ideals of a topological semigroup S. Then there exist continuous homomorphisms f from S into I and g from S into J such that f and g are injective on I and J, respectively. We now have $I \cap J$ is an ideal of S. Define $\phi : S \longrightarrow I \cap J$ by $\phi(x) = g(f(x))$ for all x in S. Then the map ϕ is a continuous homomorphism which is injective on $I \cap J$. This proves the theorem.□

THEOREM 2.3. *If I and J are quasiretract ideals of topological semigroups S and T, respectively, then $I \times J$ is a quasiretract ideal of $S \times T$.*

Proof. Let I and J be quasiretract ideals of topological semigroups S and T, respectively. Then there exist continuous homomorphisms ϕ_I of S into I and ϕ_J of T into J which are injective on I and J, respectively. Define a map Φ from $S \times T$ into $I \times J$ by $\Phi(x, y) = (\phi_I(x), \phi_J(y))$, for each $(x, y) \in S \times T$. It is easy to see that the map Φ is the required quasiretraction. This completes the proof. □

Corollary 2.4. *If S and T are quasiretract topological semigroups, then so is $S \times T$.*

Theorem 2.5. *If $\phi : S \longrightarrow T$ is a topological isomorphism from a topological semigroup S onto a topological semigroup T, then ϕ preserves quasiretract ideals.*

Proof. Let I denote a quasiretract ideal of S. Then there exists a quasiretraction f of S into I. Define $g : T \longrightarrow \phi(I)$ by $g(t) = \phi(f(\phi^{-1}(t)))$, for each $t \in T$. It is easy to prove that g is a well–defined continuous homomorphism from T into the ideal $\phi(I)$.

It remains to show that g is injective on $\phi(I)$. For $t, t' \in \phi(I)$, let $g(t) = g(t')$. Then we have $\phi(f(\phi^{-1}(t))) = \phi(f(\phi^{-1}(t')))$. Since ϕ is injective, $f(\phi^{-1}(t)) = f(\phi^{-1}(t'))$, and hence $\phi^{-1}(t) = \phi^{-1}(t')$ since f is one to one on I. So, $t = t'$. Hence g is one to one on $\phi(I)$. Thus g is the required quasiretraction, completing the proof. □

REMARK 2.6. In the above Theorem 2.5, ϕ does not, in general, preserve quasiretract ideals if it is not a topological isomorphism.

EXAMPLE 2.7. Let S be a topological semigroup and let x be an element of S such that $\theta(x) = \{x, x^2\} \cup M_x$, where $M_x = \{x^3, x^4, x^5\}$ denotes the minimal ideal of $\theta(x)$. Define a map ϕ from the additive topological semigroup N of positive integers onto $\theta(x)$ by $\phi(n) = x^n$, for each $n \in N$. Then we have ϕ is a continuous surjective homomorphism. Let $I = \{n \in N \mid n > 1\}$. Then I is a quasiretract ideal of N, but $\phi(I) = \{x^2\} \cup M_x$ is not a quasiretract ideal of $\theta(x)$.

DEFINITION 2.8. [2] A semigroup S is said to be *power cancellative* if $x^n = y^n$ for $x, y \in S$ and $n \in N$ implies that $x = y$.

DEFINITION 2.9. [2] A semigroup S is called a *power ideal semigroup* if for each $n \in N$, the set $\{x^n \mid x \in S\}$ is an ideal of S. The ideal $\{x^n \mid x \in S\}$ is called an *n-th power ideal* of S and is denoted by S_n.

THEOREM 2.10. *Let I be a quasiretract ideal of a topological semigroup S. If I is commutative power cancellative, then I^n is a quasiretract ideal of S, for each $n \in N$.*

Proof. Let I be a commutative power cancellative quasiretract ideal of S. Then there exists a continuous homomorphism ϕ from S into I such that ϕ is injective on I. For each $n \in N$, I^n is clearly an ideal of S. Define $f_n : S \longrightarrow I^n$ by $f_n(x) = \phi(x)^n$ for each $x \in S$. It is easy to show that f_n is a continuous homomorphism, for every $n \in N$.

It remains to prove that f_n is injective on I^n, for every $n \in N$. Fix $n \in N$. Suppose that $f_n(x) = f_n(y)$, for $x, y \in I^n$. Then we have $x, y \in I$ and $\phi(x)^n = \phi(y)^n$. Since I is power cancellative, $\phi(x) = \phi(y)$ and hence $x = y$, since ϕ is one to one on I. Thus, f_n is injective on I^n. This completes the proof. □

THEOREM 2.11. *If S is a commutative power cancellative power ideal topological semigroup, then every n-th power ideal S_n of S is a quasiretract ideal of S.*

Proof. For each $n \in N$, let S_n be the n-th power ideal of S, that is, $S_n = \{x^n \mid x \in S\}$. The map $\phi : S \longrightarrow S_n$ defined by $\phi(x) = x^n$, for each $x \in S$, is a continuous homomorphism. Since S is power cancellative, ϕ is injective on S_n. This completes the proof. □

REMARK 2.12. In the above Theorem 2.11, not every closed ideal is quasiretract as is shown by the following example.

EXAMPLE 2.13. Let $I_u = [0, 1]$ be the real unit with usual topology and usual multiplication. Then I_u satisfies the hypotheses of Theorem 2.11. The ideal $I = [0, \frac{1}{2}]$ is not a quasiretract ideal of I_u.

THEOREM 2.14. *Let $I \subset J$ be two ideals of a topological semigroup S. If J is a quasiretract ideal of S and if I is a quasiretract ideal of J, then I is a quasiretract ideal of S.*

Proof. If $f : S \longrightarrow J$ and $g : J \longrightarrow I$ are quasiretractions , then $g \circ f$ is a quasiretraction from S into I. □

Recall that an element x of a semigroup S has *of finite order* if there exists $k \in N$ such that $x^k = x^{n+1}$ for some $n \in N$ with $k \leq n$. The least such k is called the *index* of x and is denoted $k(x)$. If each element of S has finite index, then $k = \max\{k(x) \mid x \in S\}$ is called the *index* of S.

Let $\theta(x)$ be the cyclic subsemigroup generated by an element x in a semigroup S. If x has index i, then we write $\theta(x) = \{x, x^2, \cdots, x^{i-1},\} \cup M_x$, where $M_x = \{x^i, \cdots, x^n\}$ is a cyclic group which is the minimal ideal of $\theta(x)$.

Moreover, we have the following.

THEOREM 2.15. *Let S be a topological semigroup and let x be an arbitrary element of S. Then*

(1) *If x has infinite order, then $\theta(x)$ is a quasiretract semigroup.*
(2) *If x has finite index, then the minimal ideal M_x of $\theta(x)$ is a retract ideal of $\theta(x)$.*

Proof. (1) It follows from Theorem 2.5 that $\theta(x)$ is a quasiretract semigroup.

(2) is trivial. □

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