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WEAK COMPACTNESS AND EXTREMAL STRUCTURE IN $L^P(\mu, X)$

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ABSTRACT. We characterize the compactness, weak precompactness and weak compactness in $L^p(\mu, X)$ and in more general space $P_c(\mu, X)$. Moreover, we present this characterization in terms of extremal structure in X.

1. Introduction

Let (Ω, Σ, μ) be a finite measure space, X a real Banach space, X^* the dual space of X and B_X the unit ball of X.

Denote by $L^p(\mu, X)(1 \leq p < \infty)$ the Banach space of all equivalence classes of X-valued Bochner integrable functions f defined on Ω with $\int_\Omega \|f\|^P d\mu < \infty$. The norm $\|\cdot\|_p$ is defined by

$$\|f\|_p = (\int_\Omega \|f\|^p d\mu)^{\frac{1}{p}}, f \in L^p(\mu, X)$$

Denote by $\mathcal{L}^1(\mu, X)$ (resp. $P_c(\mu, X)$) the space of all strongly measurable Pettis integrable (resp. Pettis integrable)functions $f : \Omega \to X$ (resp. having an indefinite integral with relatively compact range)with the Pettis norm $\|f\|_{p_1} = \sup_{x^* \in B_{X^*}} \int_\Omega |x^* f| d\mu$.

Denote by $K(\mu, X)$ the space of all μ -continuous vector measures $G : \Sigma \to X$ whose range is relatively compact with the semivariation norm. Notice that $L_1(\mu, X) \subseteq \mathcal{L}^1(\mu, X) \subseteq P_c(\mu, X) \subseteq K(\mu, X)$. Diestel, Ruess and Schactermayer [5] and Diaz [2] presented characterizations of weakly compact subsets of $L^1(\mu, X)$. Brooks and Dinculeanu

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[1] and Emmanuele [6] characterized weak compactness (resp. precompactness) of subsets of $\mathcal{L}^1(\mu, X)$(resp. $P_c(\mu, X)$).

THEOREM 1.1([7] COROLLARY 3.4). *A sequence $(f_n) \subset \mathcal{L}^1(\mu, X)$ converges weakly to zero if and only if $\int_A f_n d\mu \to 0$ weakly for each $A \in \Sigma$.*

If $\pi = (A_i)_{i\in I}$ be a finite partition of Ω , we define the conditional expectation $E_\pi f$ of f by

$$E_\pi f = \sum_{i\in I} [\mu(A_i)]^{-1}\left(\int_{A_i} f d\mu\right)\chi_{A_i}.$$

It is well known that the family of finite partitions is directed by refinement, that $\|E_\pi f\| \leq \|f\|$ and that $\|E_\pi f - f\| = 0$ for all $f \in P_c(\mu, X)$.

THEOREM 1.2([6] THEOREM 1). *Let H be a bounded subset of $P_c(\mu, X)$. Then the following facts are equivalent :*

(1) *H is precompact,*
(2) *(i) $\{\int_A f d\mu : f \in H\}$ is relatively compact in X for all $A \in \Sigma$,*
(ii) $\lim_\pi E_\pi f = f$ uniformly on $f \in H$.

We shall denote by extB_{X^*} the set of all extreme points of the dull ball B_{X^*} and we shall denote by σ_e the weak topology on X generated by ext B_{X^*}. Observe that the σ_e -topology on X is Hausdorff and that the closed unit ball B_X is also closed in the σ_e -topology. Moors [8] presented a characterization of weak compactness in Banach spaces in terms of σ_e -topology.

THEOREM 1.3([3]. *Let X be a Banach space and (x_n) be a bounded sequence in X. Then (x_n) converges weakly to $x \in X$ if and only if (x_n) converges to x in the σ_e -topology.*

In this paper, we characterize compactness of subsets of $L^1(\mu, X)$ and weak compactness of subsets of $P_c(\mu, X)$ in terms of conditional expectation and σ_e -topology.

2. Results

Notice that a subset K of a Banach space is relatively norm compact if and only if it is totally norm-bounded.

THEOREM 2.1. *Let K be a bounded subset of $L^1(\mu, X)$. Then K is relatively $L^1(\mu, X)$ -norm compact in $L^1(\mu, X)$ if and only if*

(1) $\{\int_A f d\mu : f \in K\}$ *is relatively norm compact in X for all $A \in \Sigma$,*

(2) *for each $\epsilon > 0$, there is a finite partition π of Ω such that $\|E_\pi f - f\|_1 < f$ uniformly on $f \in K$.*

Proof. Suppose that K is relatively $L^1(\mu, K)$ -norm compact in $L^1(\mu, X)$. Then (1) follows from the continuity of the mapping $f \to \int_A f d\mu$ of $L^1(\mu, X)$ into X. Let $\epsilon > 0$ be given. Then since K is totally bounded in $L^1(\mu, X)$ there are $f_1, f_2, \cdots, f_n \in K$ such that $K \subset \bigcup_{i=1}^{n} N_{\frac{\epsilon}{3}}(f_i)$, where $N_\epsilon(f_i)$ is a ϵ- neighborhood of f_i .

Notice that $\lim_\pi \|E_\pi f - f\|_1 = 0$ for each $f \in L^1(\mu, X)$, and so we can find a finite partition π of Ω such that $\|E_\pi f_i - f_i\|_1 < \frac{\epsilon}{3}$ for $i = 1, 2, \cdots, n$.

Fix $f \in K$. Then there is $i (1 \le i \le n)$ such that $f \in N_{\frac{\epsilon}{3}}(f_i)$. Hence we have

$$\begin{aligned}\|E_\pi f - f\|_1 &\le \|E_\pi f - E_\pi f_i\|_1 + \|E_\pi f_i - f_i\|_1 + \|f_i - f\|_1 \\ &\le 2\|f - f_i\|_1 + \|E_\pi f_i - f_i\|_1 < \epsilon.\end{aligned}$$

Conversely, suppose that the conditions (1) and (2) hold. Let $\epsilon > 0$ be given. Then there is a finite partition $\pi = (A_i)_{1 \le i \le k}$ of Ω such that $\|E_\pi f - f\|_1 < \frac{\epsilon}{3}$ uniformly on $f \in K$.

We will first show that $\{E_\pi f : f \in K\}$ is relatively $L^1(\mu, X)$ - norm compact in $L^1(\mu, X)$. Consider a sequence $(E_\pi f_n)$ in $\{E_\pi f : f \in K\}$. From (1) we can obtain a subsequence (f_{n_j}) of (f_n) and $x_{A_1}, \cdots, x_{A_k} \in X$ such that $\lim_{j\to\infty} \| \int_{A_i} f_{n_j} d\mu - x_{A_i}\| = 0$ for $i = 1, 2, \cdots, k$. So we have

$$\begin{aligned}\|\sum_{i=1}^{k} \frac{x_{A_i}}{\mu(A_i)} \chi_{A_i} - E_\pi f_{n_j}\|_1 &= \sum_{i=1}^{k} \int_{A_i} \|\frac{x_{A_i}}{\mu(A_i)} - \frac{\int_{A_i} f_{n_j} d\mu}{\mu(A_i)}\| d\mu \\ &= \sum_{i=1}^{k} \|x_{A_i} - \int_{A_i} f_{n_j} d\mu\| \to 0 \text{ as } j \to \infty.\end{aligned}$$

This implies that $\{E_\pi f : f \in K\}$ is relatively $L^1(\mu, X)$ -norm compact in $L^1(\mu, X)$. Hence there are $f_1, \cdots, f_n \in K$ such that $\{E_\pi f : f \in K\} \subset \bigcup_{i=1}^{n} N_{\frac{\epsilon}{3}}(E_\pi f_i)$. Now fix $f \in K$. Then there is an $i (1 \le i \le n)$ such that $E_\pi f \in N_{\frac{\epsilon}{3}}(E_\pi f_i)$. We have

$$\|f - f_i\|_1 \le \|f - E_\pi f\|_1 + \|E_\pi f - E_\pi f_i\|_1 + \|E_\pi f_i - f_i\|_1 < \epsilon.$$

Thus $K \subset \bigcup_{i=1}^{n} N_\epsilon(f_i)$. Hence K is totally bounded in $L^1(\mu, X)$. This completes the proof. □

The next theorem is a generalization of Theorem 1.2

THEOREM 2.2. *Let K be a bounded subset of $P_c(\mu, X)$. Then K is weakly precompact in $P_c(\mu, X)$ if and only if*

(1) *$\{\int_A f d\mu : f \in K\}$ is weakly precompact in X for all $A \in \Sigma$,*
(2) *$\lim_\pi E_\pi f = f$ weakly in $P_c(\mu, X)$ uniformly on $f \in K$.*

Proof. Assume that K is weakly precompact in $P_c(\mu, X)$. Condition (1) follows from the fact that the mapping $f \to \int_A f d\mu$ of $(P_c(\mu, X), \Im^\omega)$ into $(X, \Im^\omega)$ is continuous.

Now we note that K is totally bounded in $(P_c(\mu, X), \Im^\omega)$. For each $g \in P_c(\mu, X)^*, g \circ E_\pi$ is also in $P_c(\mu, X)^*$. Let $\epsilon > 0$ be given. Then the set $N(O; g, g \circ E_\pi; \frac{\epsilon}{3}) = \{f \in P_c(\mu, X) : |g(f)| < \frac{\epsilon}{3}$ and $|(g \circ E_\pi)(f)| < \frac{\epsilon}{3}\}$ is an open neighborhood of O in $(P_c(\mu, X), \Im^\omega)$. Because K is totally bounded in $(P_c(\mu, X), \Im^\omega)$, there are $f_1, f_2, \cdots, f_m \in P_c(\mu, X)$ such that $K \subset \bigcup_{i=1}^{m} [f_i + N(O; g, g \circ E_\pi; \frac{\epsilon}{3})]$. For any $f \in K$, there is an i $(1 \le i \le m)$ such that $f \in f_i + N(O; g, g \circ E_\pi; \frac{\epsilon}{3})$. Since $\lim_\pi E_\pi f = f$ in norm for all $f \in P_c(\mu, X)$, we have $\lim_\pi g(E_\pi f) = g(f)$ for all $f \in P_c(\mu, X)$. Hence there is a π' such that

$$\pi > \pi' \Rightarrow |g(E_\pi f_i) - g(f_i)| < \frac{\epsilon}{3} \text{ for } i = 1, 2, \cdots, m.$$

Hence

$$\pi > \pi' \Rightarrow |g(E_\pi f) - g(f)| \le |g(E_\pi f) - g(E_\pi f_i)| + |g(E_\pi f_i) - g(f_i)| + |g(f_i) - g(f)| < \epsilon.$$

Thus $\lim_{\pi} E_\pi f = f$ weakly in $P_c(\mu, X)$ uniformly on $f \in K$.

Conversely, assume that conditions (1) and (2) hold. Let (f_n) be any sequence in K and let $g \in P_c(\mu, X)^*$ be arbitrary. Then given $\epsilon > 0$, by (2) there is a finite partition $\pi' = (A_i)_{i \in I}$ of Ω such that $|g(E_{\pi'} f) - g(f)| < \frac{\epsilon}{3}$ uniformly on $f \in K$. So $E_{\pi'}(K)$ is contained in the set $\sum_{i \in I} \mu(A_i)^{-1}\{\int_{A_i} f d\mu : f \in H\}\chi_{A_i}$, which is weakly precompact by (1). Hence $(E_{\pi'} f_n)$ has a weak Cauchy subsequence , say $(E_{\pi'} f_{n_k})$. Therefore there is a $N \in \mathbf{N}$ such that

$$k, k' > N \Rightarrow |g(E_{\pi'} f_{n_k}) - g(E_{\pi'} f_{n_{k'}})| < \frac{\epsilon}{3}.$$

Hence we have

$$\begin{aligned} k, k' > N \Rightarrow & |g(f_{n_k}) - g(f_{n_{k'}})| \\ & \leq |g(f_{n_k}) - g(E_{\pi'} f_{n_k})| + |g(E_{\pi'} f_{n_k}) - g(E_{\pi'} f_{n_{k'}})| \\ & + |g(E_{\pi'} f_{n_{k'}}) - g(f_{n_{k'}})| \\ & < \epsilon. \end{aligned}$$

Thus K is weakly precompact in $P_c(\mu, X)$. □

THEOREM 2.3. *Let K be a bounded subset of $P_c(\mu, X)$. Then K is weakly precompact in $P_c(\mu, X)$ if and only if*

(1) *$\{\int_A f d\mu : f \in K\}$ is weakly precompact in X for all $A \in \Sigma$,*
(2) *for any sequence $(f_k) \subset K$ there is a sequence (π_n) of finite partitions, cofinal to the net (π), such that $\lim_n E_{\pi_n} f_k = f_k$ weakly in $P_c(\mu, X)$ uniformly on $k \in \mathbf{N}$.*

Proof. Assume that K is weakly precompact in $P_c(\mu, X)$. Then it is clear that condition (1) holds.

Now let (f_k) be any sequence in K. It is well known that there is a sequence (π_n) of finite partitions cofinal to the net (π) such that $\lim_n \|E_{\pi_n} f_k - f_k\|_{p_1} = 0$ for all $k \in \mathbf{N}$. Notice that the set $\{f_k : k \in \mathbf{N}\}$ is totally bounded in $(P_c(\mu, X), \Im^\omega)$. Using the similar method in the proof of Theorem 2.2, we have $\lim_n E_{\pi_n} f_k = f_k$ weakly in $P_c(\mu, X)$ uniformly on $k \in \mathbf{N}$.

Conversely, assume that conditions(1) and (2) hold. Using the similar method in the proof of Theorem 2.2, we can show that K is weakly precompact in $P_c(\mu, X)$. $\square$

Notice that the mapping $T : \mathcal{L}^1(\mu, X) \to K(\mu, X), T(f)(A) = \int_A f d\mu, f \in \mathcal{L}^1(\mu, X), A \in \Sigma$, is linear and isometry [4].

Define $\tilde{T} : P_c(\mu, X) \to K(\mu, X)$ by $\tilde{T}(f)(A) = \int_A f d\mu$ for each $f \in P_c(\mu, X)$ and $A \in \Sigma$. Then $\tilde{T}$ is also well-defined, linear and isometry.

The next lemma is an extended version of Theorem 1.1.

LEMMA 2.4. *Let (f_n) be a sequence in $P_c(\mu, X)$. Then (f_n) converges to 0 weakly in $P_c(\mu, X)$ if and only if $\int_A f_n d\mu \to 0$ weakly in X for each $A \in \Sigma$.*

Proof. Suppose that (f_n) converges to 0 weakly in $P_c(\mu, X)$, and let $A \in \Sigma$. Then $T_A : P_c(\mu, X) \to X, T_A(f) = \int_A f d\mu$, is a bounded linear operator. For each $x^* \in X^*$, we have

$$< x^*, \int_A f_n d\mu >=< x^*, \ T_A(f_n) >= (x^* \circ T_A)(f_n) \to 0 \text{ as } n \to \infty.$$

Hence $\int_A f_n d\mu \to 0$ weakly in X.

Conversely, suppose that $\int_A f_n d\mu \to 0$ weakly in X for each $A \in \Sigma$. Let (f_n) be a sequence in $P_c(\mu, X)$. Then there is a sequence (G_{f_n}) in $K(\mu, X)$ such that $\|f_n\|_{p_1} = \|G_{f_n}\|_{S.V}$ for each $n \in \mathbf{N}$. Because $\mathcal{L}^1(\mu, X)$ is considered as a dense subspace of $K(\mu, X)$[4], for each $n \in \mathbf{N}$ there is $f'_n \in \mathcal{L}^1(\mu, X)$ with $\|G_{f_n'} - G_{f_n}\|_{S.V} < \frac{1}{n}$ where $T(f'_n) = G_{f'_n}$. Hence $\|f'_n - f_n\|_{p_1} = \|G_{f_n'} - G_{f_n}\|_{S.V} < \frac{1}{n}$ for each $n \in \mathbf{N}$. We can show easily that $\int_A f'_n d\mu \to 0$ weakly in X for each $A \in \Sigma$. By Theorem 1.1 , (f'_n) converges to 0 weakly in $\mathcal{L}^1(\mu, X)$.

Now let $g \in P_c(\mu, X)^*$. Then $g \in \mathcal{L}^1(\mu, X)^*$. Hence $g(f'_n) \to 0$ weakly as $n \to \infty$. We have $|g(f_n) - g(f'_n)| \le \|g\| \|f_n - f'_n\|_{p_1} \le \|g\| \frac{1}{n} \to 0$ as $n \to \infty$. Thus $\lim_{n\to\infty} g(f_n) = \lim_{n\to\infty} g(f'_n) = 0$. This implies that (f_n) converges to 0 weakly in $P_c(\mu, X)$. $\square$

THEOREM 2.5. *A bounded subset K of $P_c(\mu, X)$ is relatively weakly compact if*

(1) *The set $\{\mu(A)^{-1} \int_A f d\mu : f \in K, A \in \Sigma, \mu(A) > 0\}$ is relatively weakly compact in X,*

(2) $\lim_{\pi} E_\pi f = f$ *weakly in $P_c(\mu, X)$ uniformly on $f \in K$.*

Proof. For every $A \in \Sigma$, we have

$$\left\{\int_A f d\mu : f \in K\right\} \subset \mu(A) \left\{\frac{1}{\mu(B)} \int_B f d\mu : f \in K, B \in \Sigma, \mu(B) > 0\right\}$$

Hence by (1), $\{\int_A f d\mu : f \in K\}$ is relatively weakly compact in X. By Theorem 2.2, H is weakly precompact in $P_c(\mu, X)$.

Now let (f_n) be a sequence in K. Then (f_n) has a weak Cauchy subsequence, say (f_{n_k}). For every $A \in \Sigma$, the sequence $(\int_A f_{n_k} d\mu)$ is also a weak Cauchy sequence in $\{\int_A f d\mu : f \in K\}$. Since $\{\int_A f d\mu : f \in K\}$ is relatively weakly compact in X, there is an $m(A) \in X$ such that $< m(A), x^* >= \lim_{k\to\infty} \int_A < f_{n_k}(s), x^* > d\mu,$ for all $x^* \in X^*$.

The set function $m : \Sigma \to X$ is a μ -continuous vector measure and the average range $\{m(A)/\mu(A) : A \in \Sigma, \mu(A) > 0\}$ is contained in the weak closed convex hull of $\{\mu(A)^{-1} \int_A f d\mu : f \in K, A \in \sum, \mu > 0\}$, which is a weakly compact convex set in X. Hence the average range $\{m(A)/\mu(A) : A \in \Sigma, \mu(A) > 0\}$ is relatively weakly compact in X. Thus there is an $f \in P_c(\mu, X)$ such that $m(A) = \int_A f d\mu$ for all $A \in \Sigma$. Hence $\int_A < f(s), x^* > d\mu = \lim_{k\to\infty} \int_A < f_{n_k}(s), x^* > d\mu$ for all $A \in \Sigma$ and $x^* \in X^*$. By Lemma 2.4, $\lim_{k\to\infty} f_{n_k} = f$ weakly in $P_c(\mu, X)$. Thus K is relatively weakly compact in $P_c(\mu, X)$. □

We obtain the following corollary from Theorem 1.3 and Lemma 2.4.

COROLLARY 2.6. *Let (f_n) be a sequence in $P_c(\mu, X)$. Then (f_n) converges to 0 weakly in $P_c(\mu, X)$ if and only if $\int_A f_n d\mu \to 0$ in the σ_e -topology for each $A \in \Sigma$.*

We obtain the following corollaries from Theorem 2.2, Theorem 2.5 and Corollary 2.6.

COROLLARY 2.7. *Let K be a bounded subset of $P_c(\mu, X)$. Then K is weakly precompact in $P_c(\mu, X)$ if and only if*

(1) $\{\int_A f d\mu : f \in K\}$ *is precompact in the* σ_e *-topology for each* $A \in \Sigma$,

(2) $\lim_{\pi} \int_A E_\pi f d\mu = \int_A f d\mu$ *uniformly on* $f \in K$ *in the* σ_e *-topology for each* $A \in \Sigma$.

COROLLARY 2.8. *Let K be a bounded subset of $P_c(\mu, X)$. Then K is relatively weakly compact if*

(1) $\{\mu(A)^{-1} \int_A f d\mu : f \in K, A \in \Sigma, \mu(A) > 0\}$ *is relatively compact in the* σ_e*-topology,*

(2) $\lim_{\pi} \int_A E_\pi f d\mu = \int_A f d\mu$ *uniformly on* $f \in K$ *in the* σ_e *-topology for each* $A \in \Sigma$.

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